Modulation and Demodulation

Analog modulation
- **AM**
- **PM / FM**

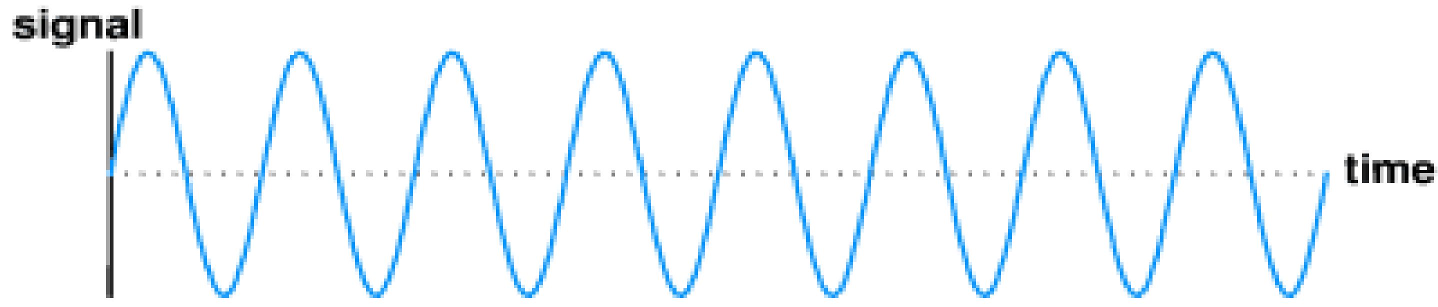
Digital modulation
- **Binary Modulation**
- **Quadrature Modulation**

Power Efficiency

Noncoherent Detection

Modulation

- Important fact: a continuous, oscillating signal will propagate farther than other signals.
- Start with a carrier signal
  - usually a sine wave that oscillates continuously.
  - Frequency of carrier fixed



Carrier Signal

- In analog transmission, the sending device produces a high-frequency signal that acts as a basis for the information signal.
  - This base signal is called the carrier signal or carrier frequency
- The receiving device is tuned to the frequency of the carrier that it expects from the sender.

Modulation

- Signal information is modulated on the carrier signal by modifying one of its characteristics (amplitude, frequency, phase).
- This modification is called *modulation*
- Same idea as in radio, TV transmission
- The information signal is called a *modulating signal*.

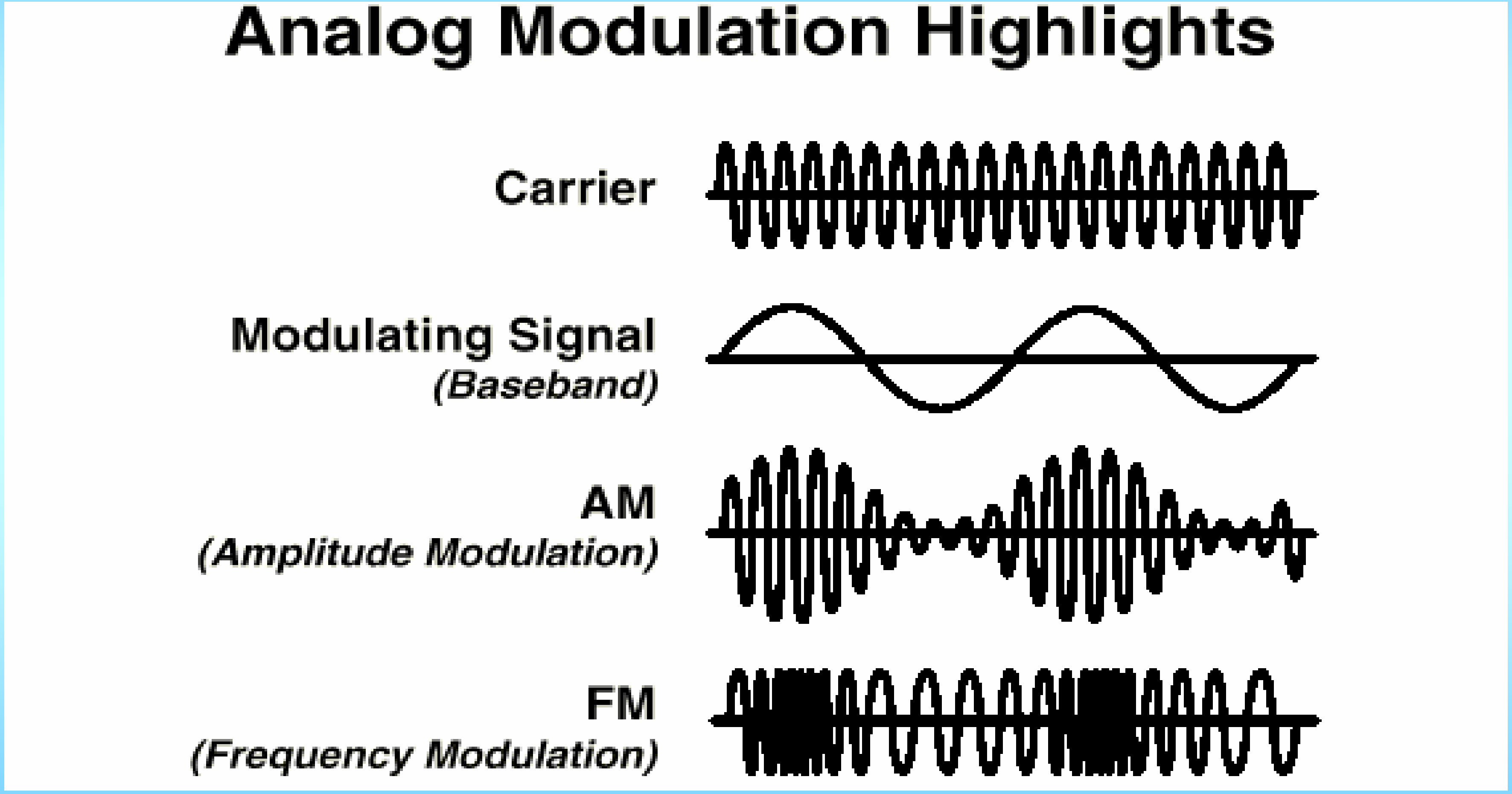
Modulation

- Modulation: process of changing a carrier wave to encode information.
- Modulation used with all types of media
- Why is modulation needed?
  - Allows data to be sent at a frequency which is available
  - Allows a strong carrier signal to carry a weak data signal
  - Reduces effects of noise and interference

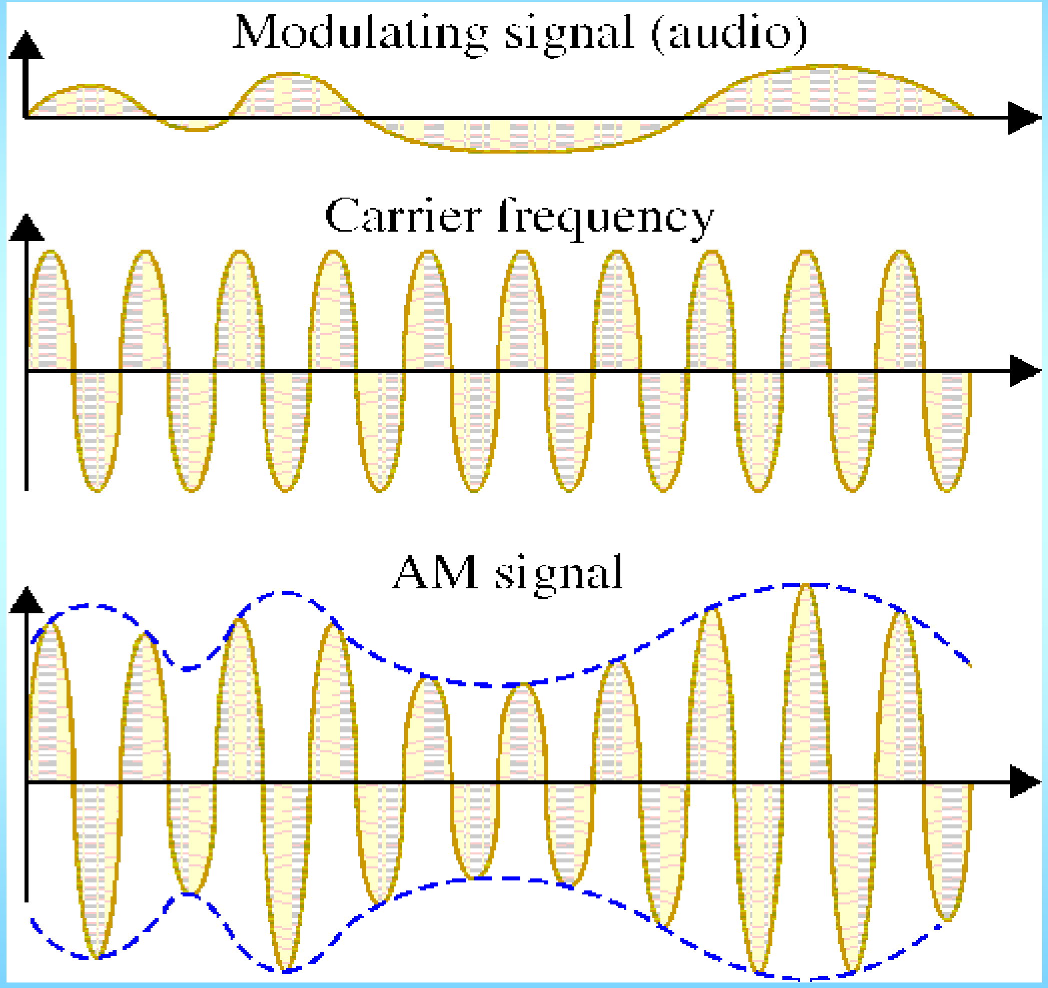
Types of modulation

- Amplitude modulation (used in AM radio) – strength, or amplitude of carrier is modulated to encode data
- Frequency modulation (used in FM radio) – frequency of carrier is modulated to encode data
- Phase shift modulation (used for data) – changes in timing, or phase shifts encode data

Analog modulation

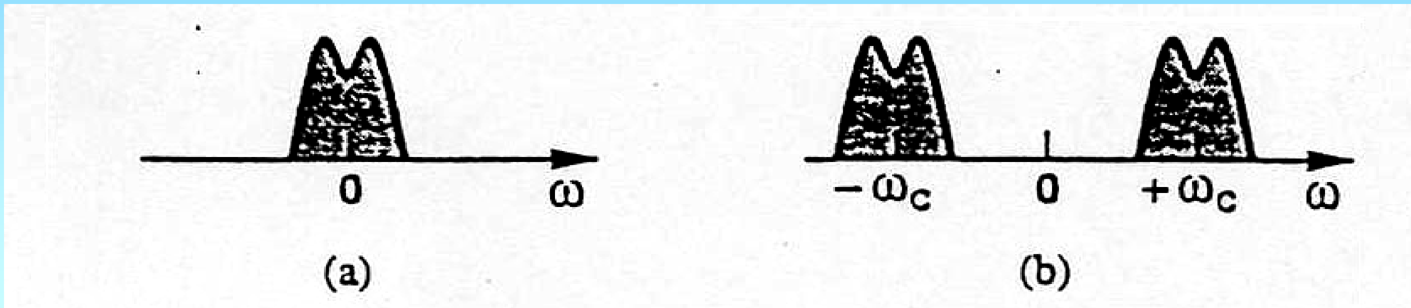


Amplitude Modulation



Amplitude modulation

DSB-SC
AM



Signal: ***m(t)***
Modulated signal: ***A(t)=A$_c$m(t),*** **1-1 correspondence to *m(t)***
Carrier: ***Cos*** w_c***t***

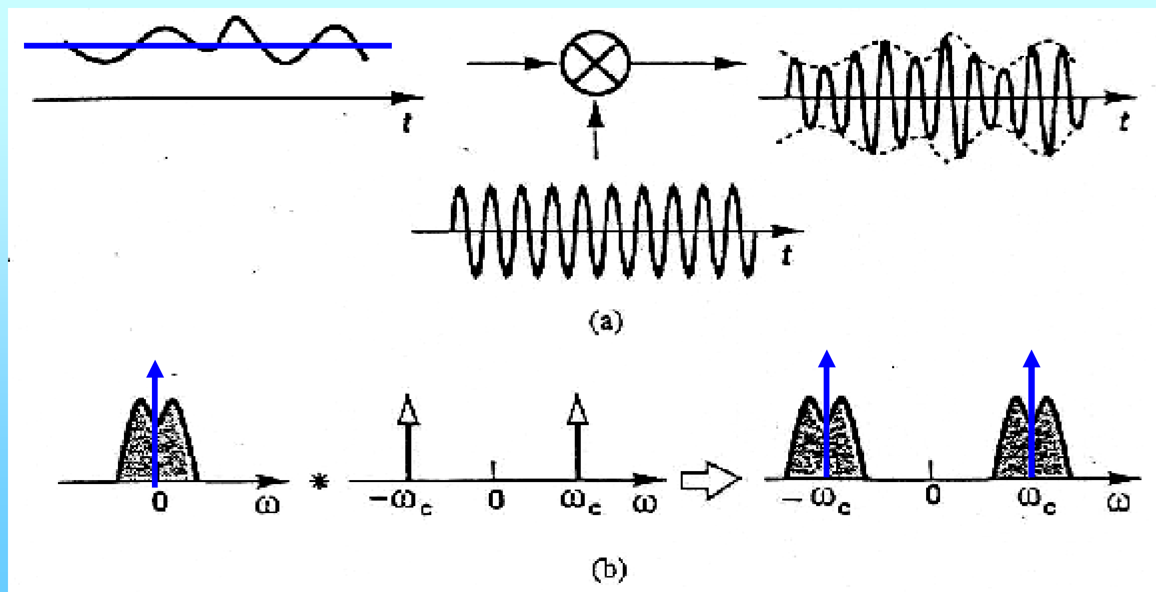
$$\text{General form}: x_c(t) = \underbrace{A(t)}_{\text{modulated signal}} \times \overbrace{\cos \underbrace{w_c}_{\text{fixed}} t}^{\text{carrier}}$$

.

Doubled-Sideband Modulation (Suppressed Carrier): DSB-SC

$$\text{General form}: x_c(t) = \underbrace{A(t)}_{\text{modulated signal}} \times \cos \overbrace{\underbrace{w_c}_{\text{fixed}}\, t}^{\text{carrier}}$$

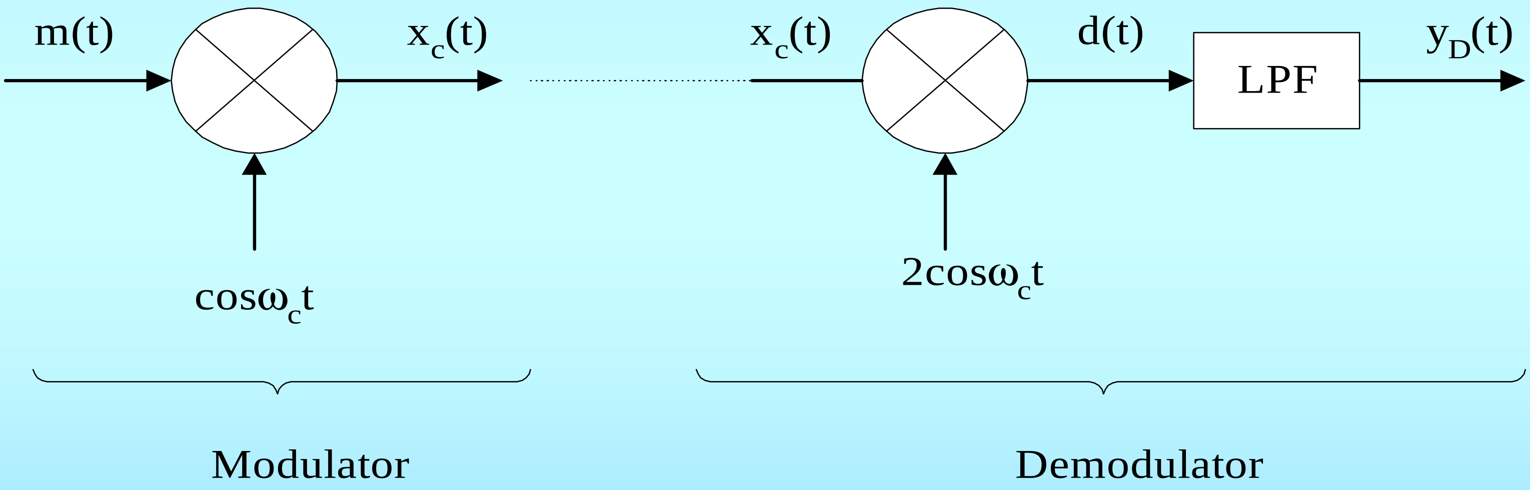
$$X_c(f) = \underbrace{\frac{1}{2} A_c M(f + f_c) + \frac{1}{2} A_c M(f - f_c)}_{\text{translation of } M(f)}, \quad f_c = \frac{w_c}{2p}$$



Demodulation

Coherent (Synchronous) Demodulator (Detector):

The receiver knows exactly the phase and frequency of the received signal.

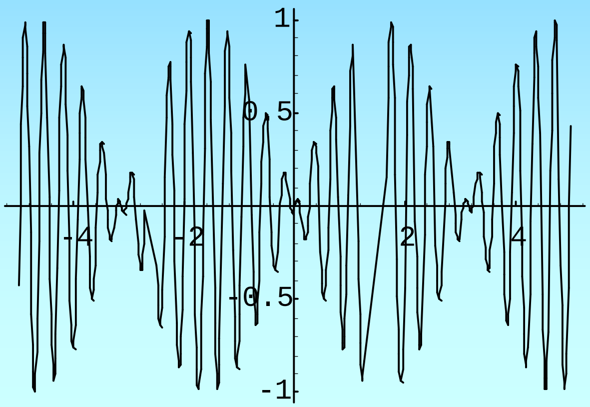


$x_c(t) = A_c m(t)\cos\omega_c t$

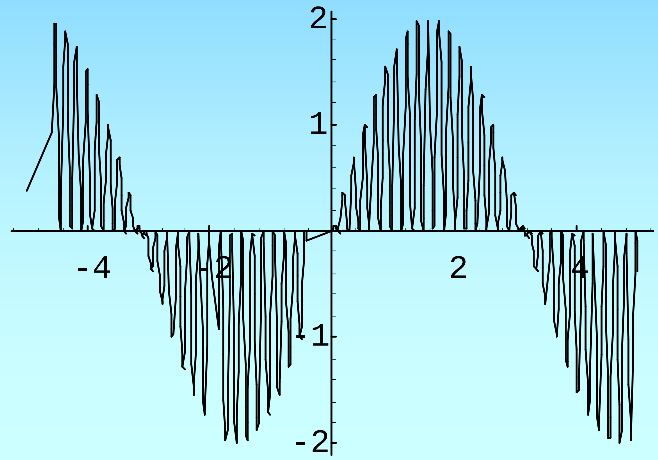
$d(t) = [A_c m(t)\cos\omega_c t]\, 2\cos\omega_c t$

$= A_c m(t)$ *Message we want!* $+ 2A_c\, m(t)\cos\omega_c t$

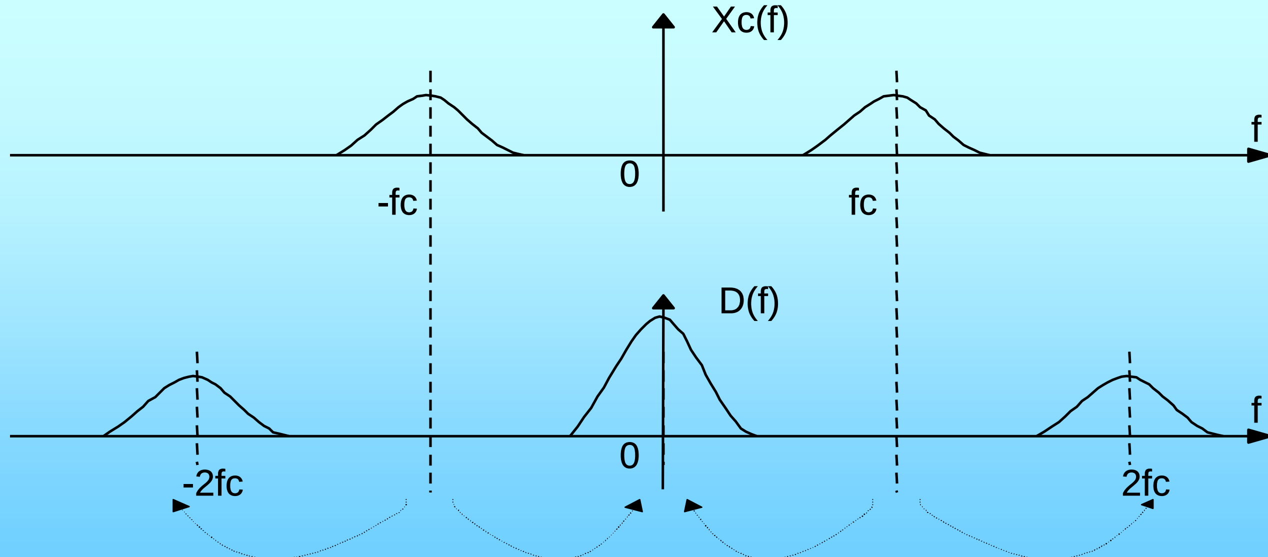
$y_1(t) = A_c m(t)$



X(t)



d(t)



Power Analysis

$\langle x_c^2(t)\rangle = \langle [A+m(t)]^2 (A_c)^2 \cos^2\omega_c t\rangle$

Assume m(t) varies slowly w.r.t. $\cos 2w_c t$

$$= \langle (1/2)(A_c)^2 [A + m(t)]^2 \rangle + \langle (A_c)^2 [A+m(t)]^2 \cos 2\omega_c t\rangle$$

Since $\langle \cos X\rangle = 0$:

$$=(1/2)(A_c)^2 \langle [A^2 + 2A\langle m(t) + \langle m^2(t)\rangle]$$

Assume $\langle m(t)\rangle = 0$:

$$= \underbrace{(1/2)(A_c)^2}_{\text{Carrier power}} \underbrace{[\ A^2}_{\text{dc bias power}} + \langle \underbrace{m(t)^2}_{\text{power of m(t) signal power}} \rangle]$$

Efficiency E: the percentage of total power that conveys information.

$$E = \frac{\text{Signal Power}}{\text{Total Power}}$$

$$E_{AM} = \frac{[1/2(Ac^2)] < m^2(t) >}{[1/2(Ac^2)][A^2 + < m^2(t) >]} (100\,\%)$$

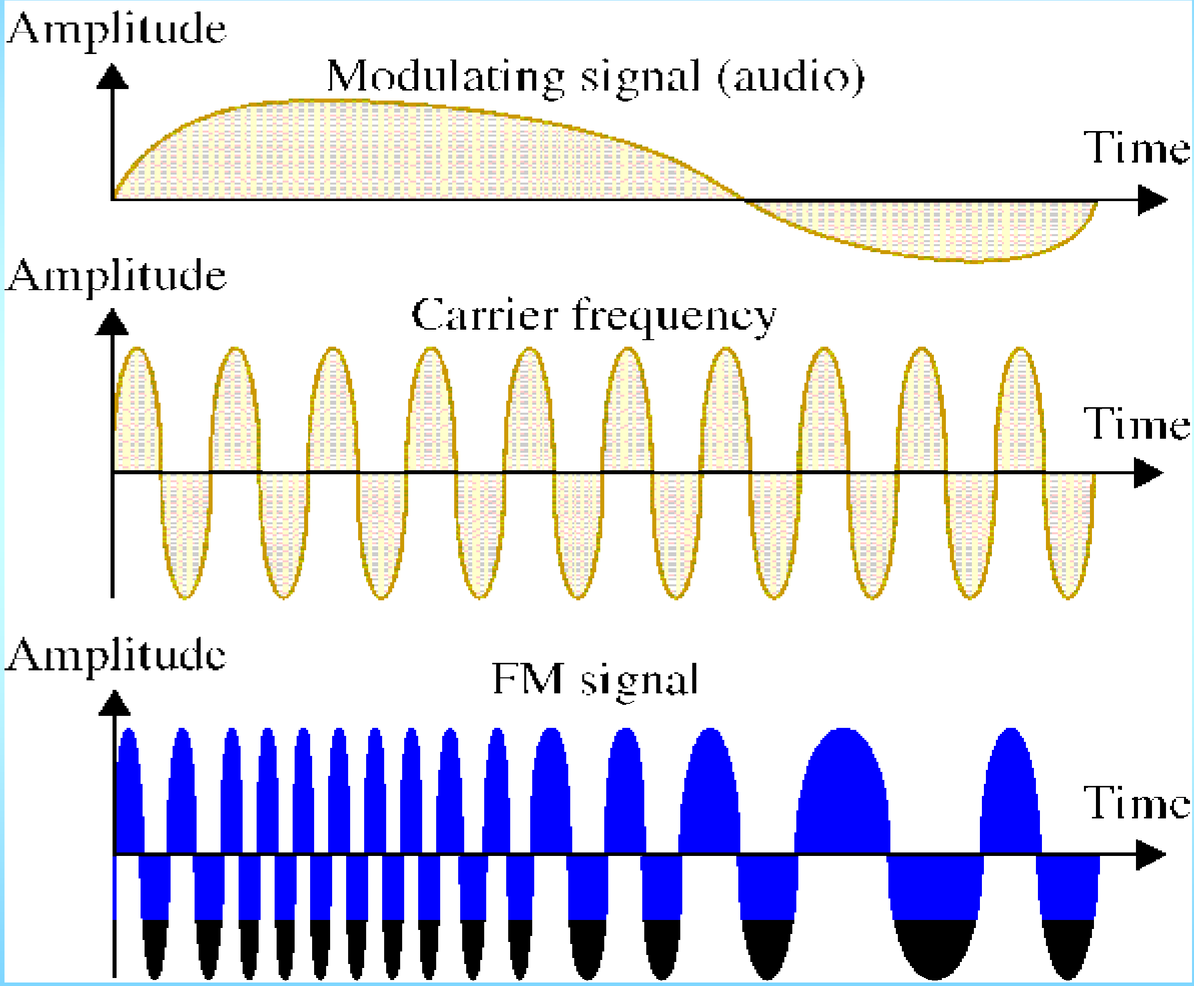
$$= \frac{a^2 < m_n^2(t) >}{1 + a^2 < m_n^2(t) >} (100\%)$$

$$= \frac{< m^2(t) >}{A^2 + < m^2(t) >} \quad (100\,\%)$$

Frequency Modulation

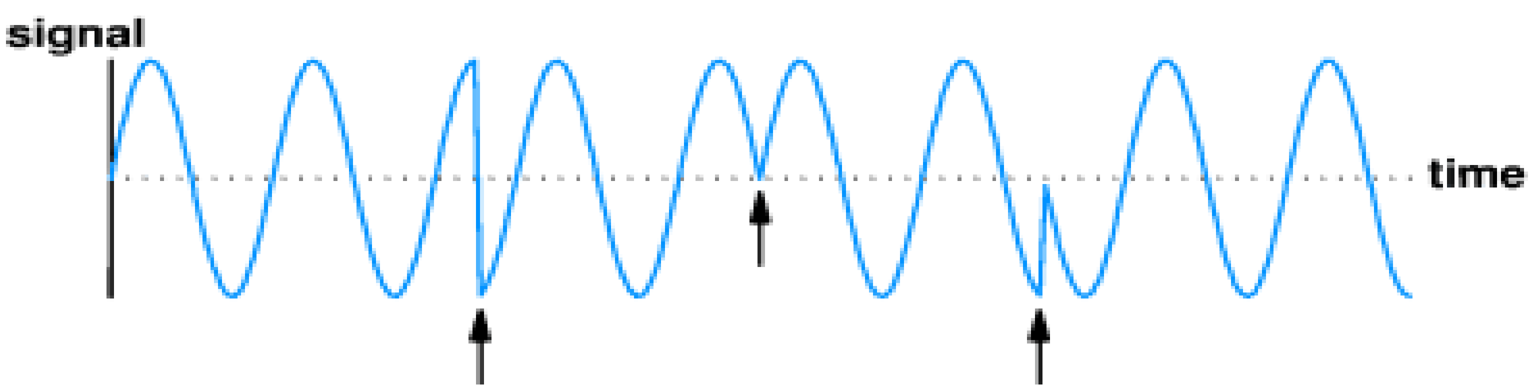
- Frequency of the carrier signal is modulated to follow the changing voltage level (amplitude) of the modulating signal.
- Peak amplitude and phase of the carrier signal remain constant, but as the amplitude of the info signal changes, the frequency of the carrier changes accordingly.

Frequency Modulation



Phase-Shift Modulation

- vary phase of carrier
- may use more than simply 180 degree shift (binary)
- this allows higher bit rate than baud rate
  - Eight angles results in 3 bits per signal element. Or 3 bits per baud!



Angle Modulation: FM and PM

General form : $x_c(t)=A_c\cos[\omega_c t+\underline{\phi(t)}]$

Phase modulation: $\phi(t)=K_p m(t)$, *K_p: deviation constant*

Frequency modulation : $d\,\phi(t)/dt = K_f.m(t)$

K_f: deviation constant;

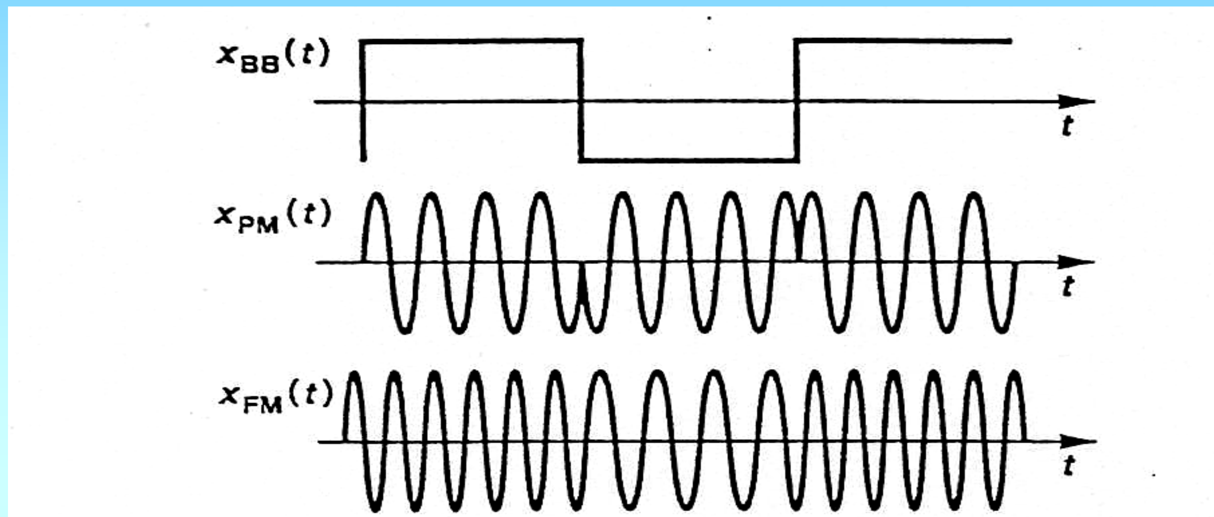
fd: Frequency deviation constant

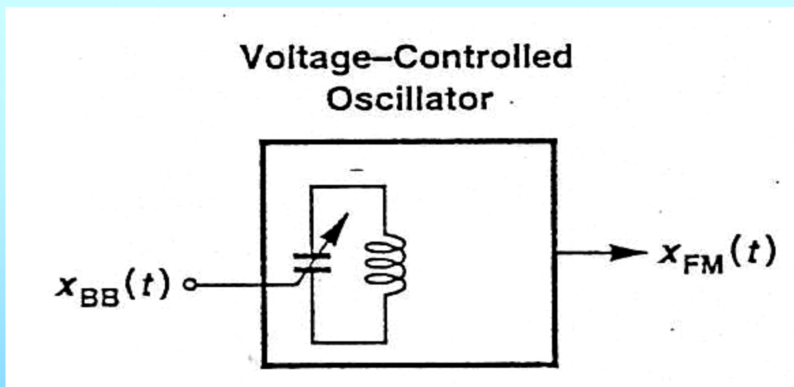
$$\Phi(t) = K_f \int_{t_o}^{t} m(a)\,da + \Phi_o$$

$$= 2p\ f_d \int_{t_o}^{t} m(a)\,da + \Phi_o$$

$$\begin{cases} \text{PM}: \ \underline{x_c(t) = A_c\cos[w_c t + k_p m(t)]} \\ \text{FM}: \ \underline{x_c(t) = A_c\cos[w_c t + 2p f_d \int_{t_o}^{t} m(a)\,da]} \end{cases}$$

FM/PM waveforms



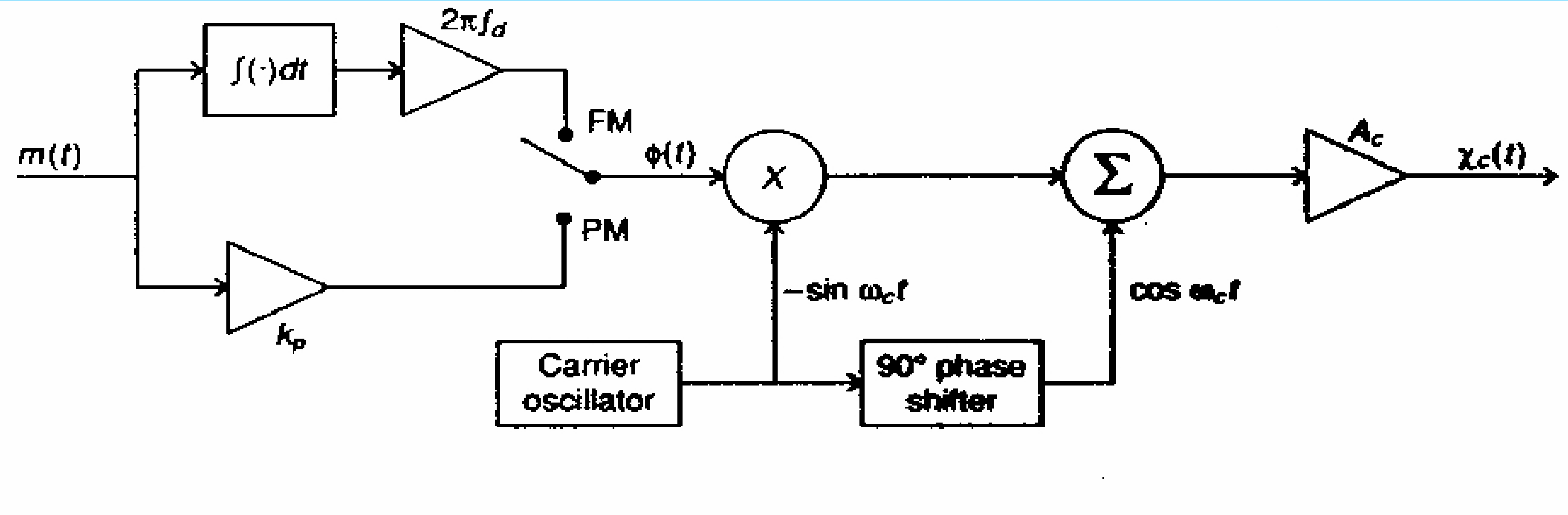


$$f_c = \frac{1}{2p\sqrt{LC}}$$

x_{BB} controls the value of C

Indirect implementation (Armstrong)

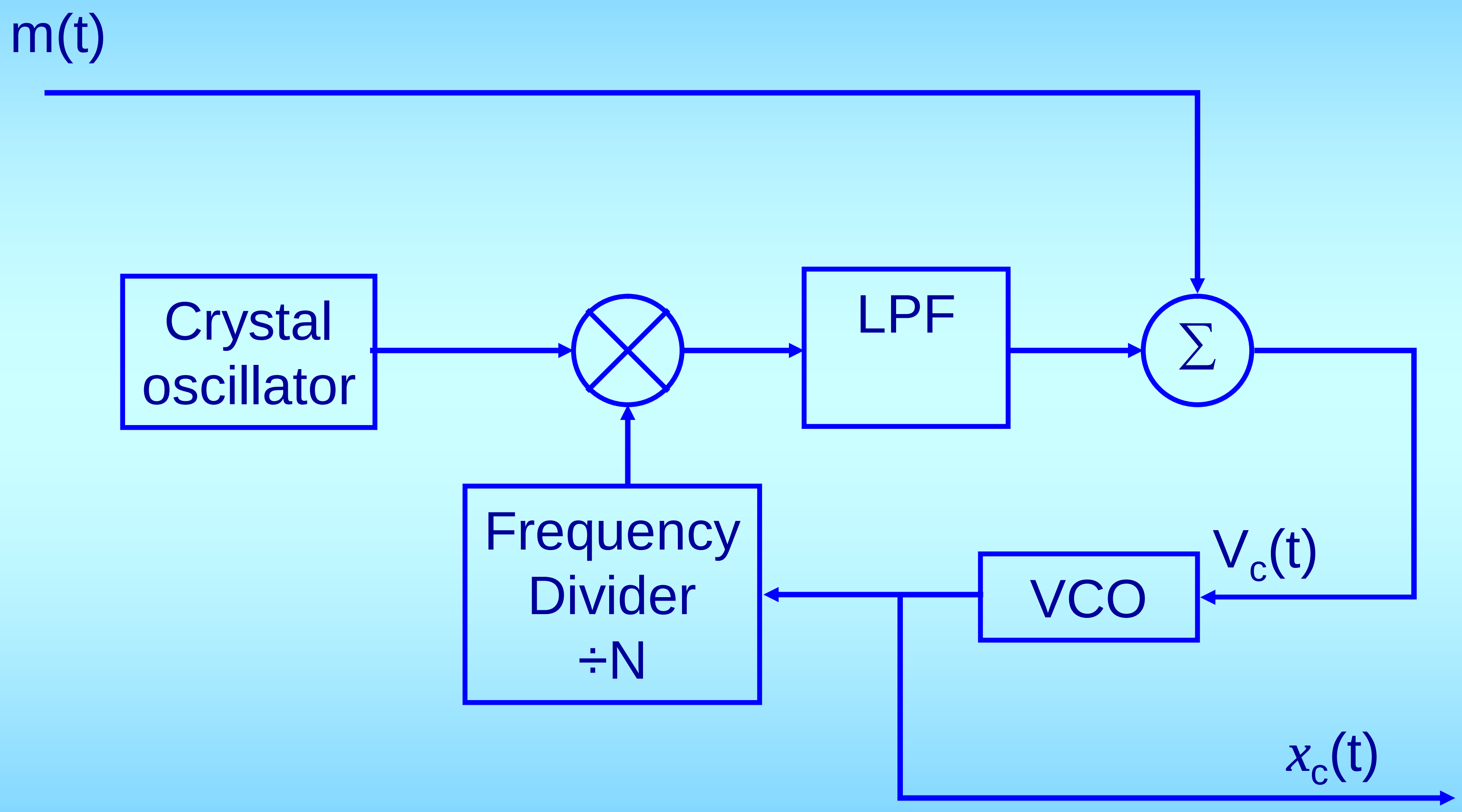
using a mixer and summer



$$x_c(t) = A_c(\cos w_c t - f(t)\sin w_c t)$$

If φ(t) very small: $x_c(t) = A_c \cos(w_c t + f(t))$

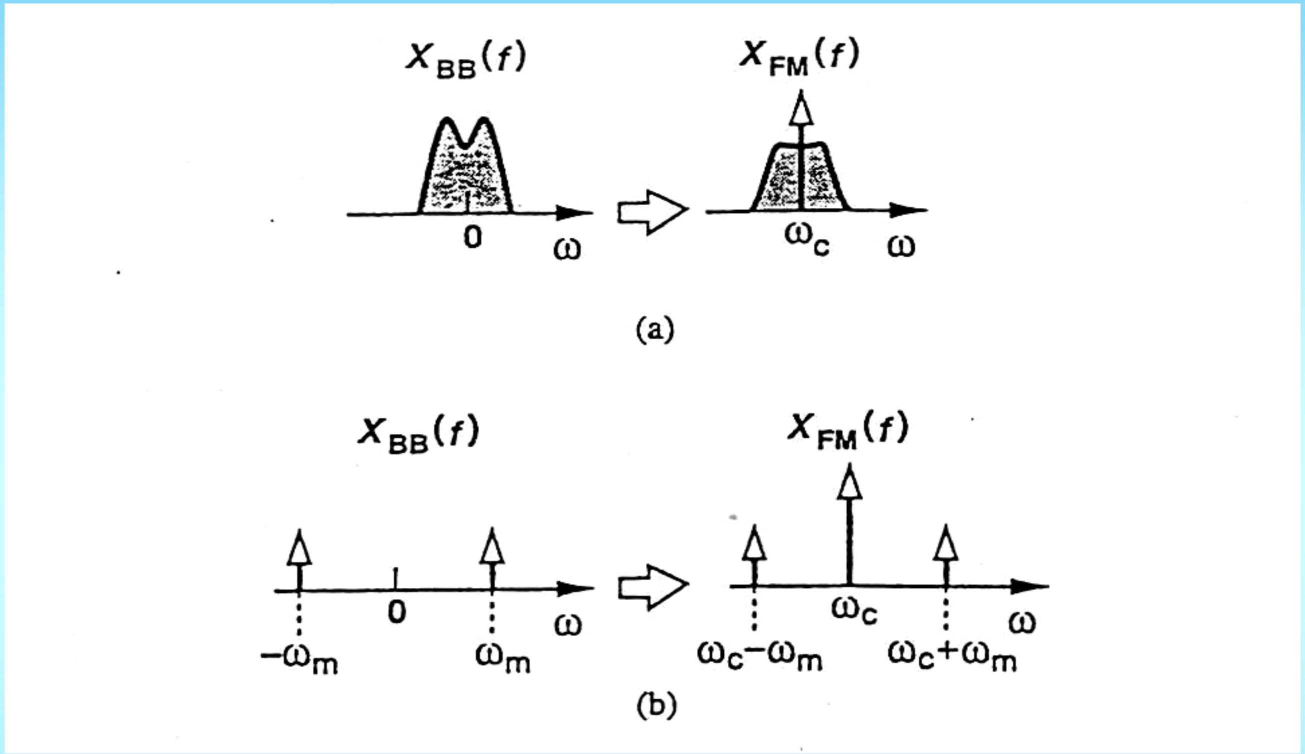
Direct method using PLL



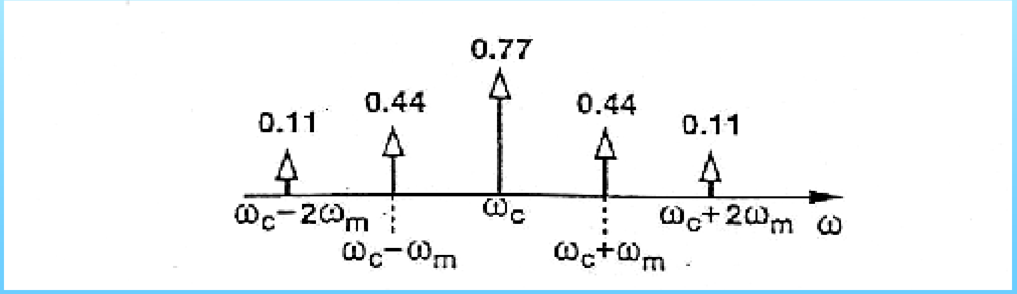
V_c controls frequency, so is m(t)

FM/PM spectrum

Ideally



Reality



Theorem

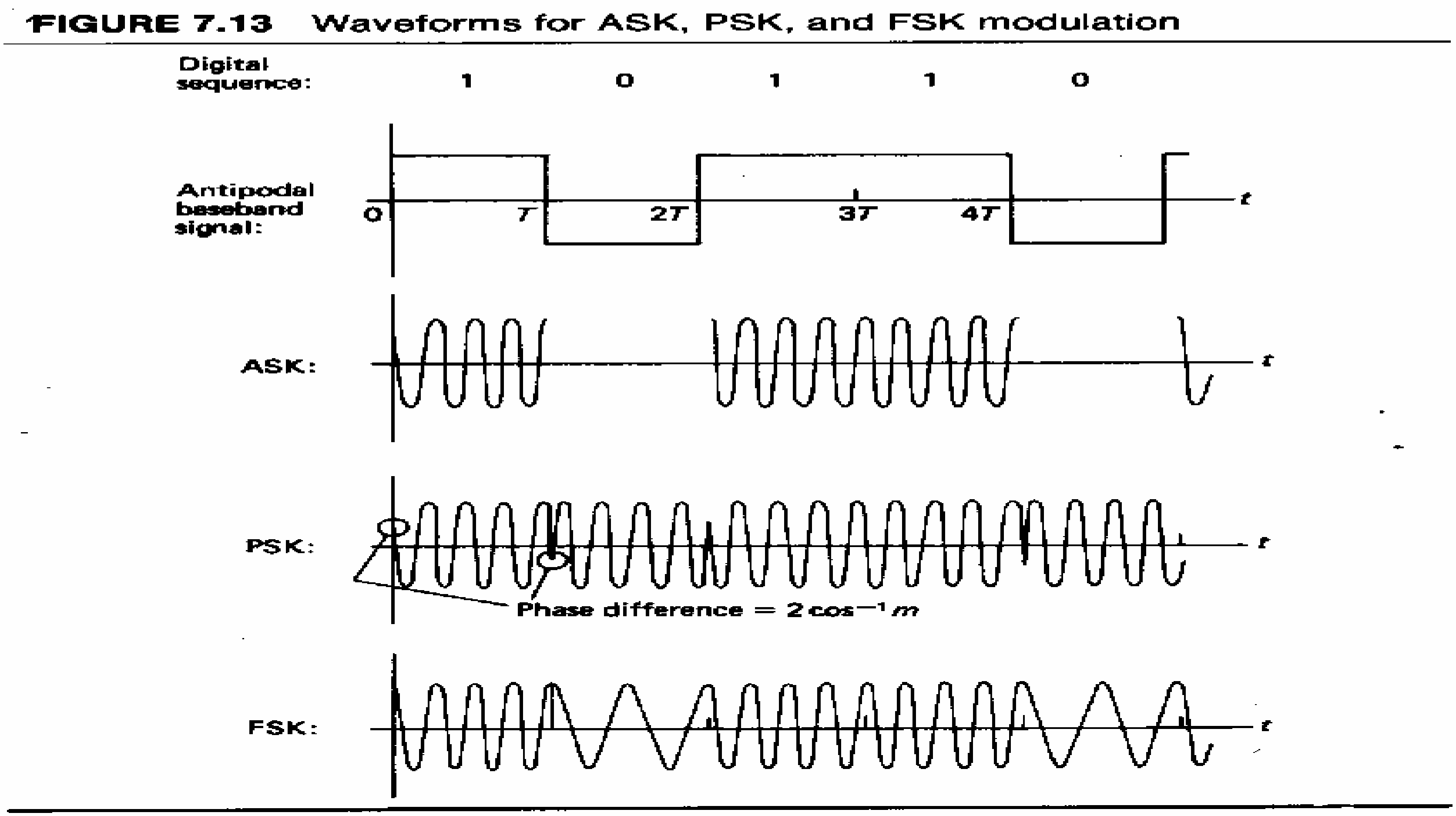
- If $|\Phi(t)|<<1$,

$$X_c(f) = \frac{A_c}{2}\{[d(f-f_c)+d(f+f_c)]+j[\Phi(f-f_c)-\Phi(f+f_c)]\}$$

- Or if $\frac{K_f \max[m(t)]}{2pB} > 1$ where B is the bandwidth of m

$$X_c(f) = \frac{pA_c^2}{2K_f}\left[M(\frac{2p}{K_f}(f-f_c)+M(\frac{2p}{K_f}(-f-f_c)\right]$$

Digital Modulation



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Design Parameters

- **Power efficiency** : describes the ability of a modulation technique to preserve the fidelity of the digital message at low power levels.
  - η_P : E_b/N_0
- **Bandwidth efficiency**: describes the ability of a modulation scheme to accommodate data within a limited bandwidth.
  - η_B : R/B bps/Hz

Channel capacity formula

- Channel capacity formula
  - $R_{max} <= C = Blog2(1+S/N)$
  - $\eta_{Bmax} = C/B = \log_2(1+S/N)$
    - C is the channel capacity (in bps)
    - B is the RF bandwidth
    - S/N is the signal-to-noise ratio

Bandwidth

- Absolute bandwidth : The range of frequencies over which the signal has a 'non-zero' power spectral density.
- Null-to-null bandwidth : equal to the width of main spectral lobe.
- Half-power (3-dB) bandwidth : the interval between frequencies at which the PSD has dropped to half power, or 3dB below the peak value.
- 99 percent bandwidth (by Federal Communication Commission) : occupied 99 percent of signal power.

General Digital Modulation

- **Geometric** Representation of Modulation
  - set of modulation signal :

$$S = \{s_1(t), s_2(t), \ldots\ldots.\, s_M(t)\}$$

  - vector representation with orthogonal basis functions :
  - $$s_i(t) = \sum_{j=1}^{N} s_{ij} \Phi_j(t)$$
  - $$\int_{-\infty}^{\infty} \Phi_i(t) \Phi_j(t) dt = 0 \qquad \mathrm{i} \neq \mathrm{j}$$

$$E = \int_{-\infty}^{\infty} \Phi^2{}_i(t) dt = 1$$

Geometric Representation of Modulation

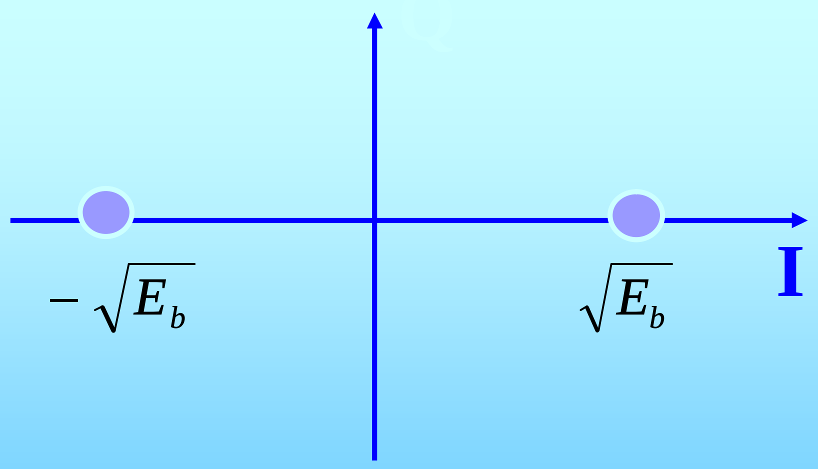
- BPSK:

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2pf_c t) \qquad 0 \le \mathrm{t} \le \mathrm{T_b}$$

$$s_2(t) = -\sqrt{\frac{2E_b}{T_b}}\cos(2pf_c t) \qquad 0 \le \mathrm{t} \le \mathrm{T_b}$$

$$\Phi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2pf_c t) \qquad 0 \le \mathrm{t} \le \mathrm{T_b}$$

$$S_{BPSK} = \left\{\sqrt{E_b}\,\Phi_1(t), -\sqrt{E_b}\,\Phi_1(t)\right\}$$



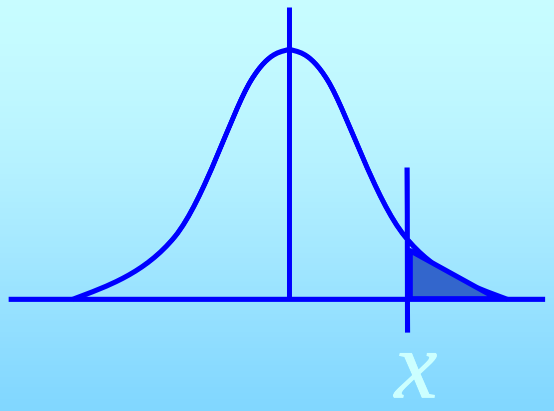
Geometric Representation of Modulation

- Bit error analysis: in AWGN channel with a noise spectral density No.

$$P_s(e \mid s_i) \leq \sum_{j=1,\, j\neq i} Q\left(\frac{d_{ij}}{\sqrt{2N_0}}\right)$$

d_{ij}: distance between the ith and jth signal point in the constellation

$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2p}} \exp(-x^2/2)\,dx$$



$$P_s(e) = P_s(e \mid s_i)P(s_i) = \frac{1}{M}\sum_{i=1}^{M} P_s(e \mid s_i)$$

Linear Modulation Techniques

- Transmitted signal amp. is proportional to modulated signal
- Good spectral efficiency
- Bad power efficiency since linear AMP is needed.
- E.g. QPSK, OQPSK, π/4QPSK…

$$s(t) = \mathrm{Re}[\,Am(t)\exp(j2\pi f_c t)$$
$$= A[m_R(t)\cos(2\pi f_c t) - m_I(t)\sin(2\pi f_c t)]$$

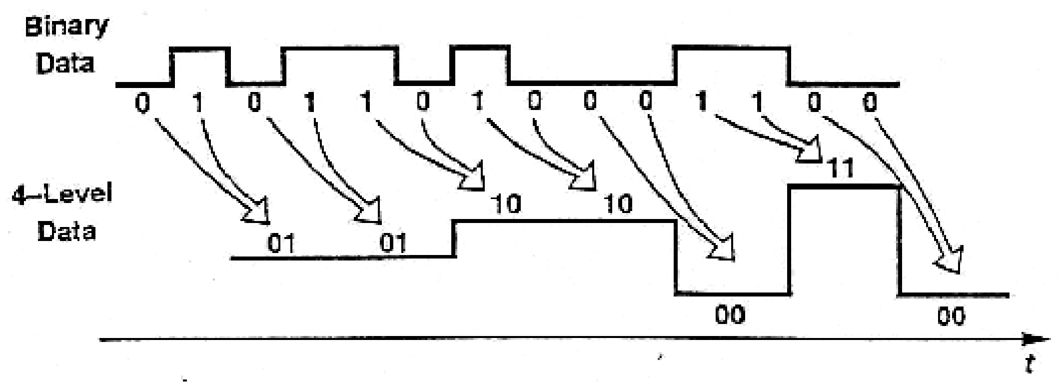
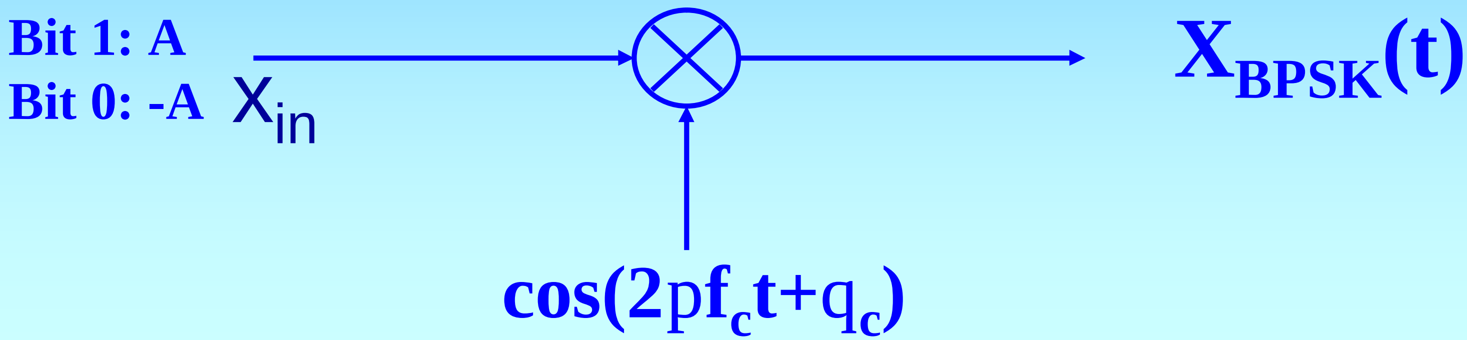


Figure 3.13 Four-level digital representation of a binary data stream.

Linear Modulation Techniques

- BPSK Transmitter

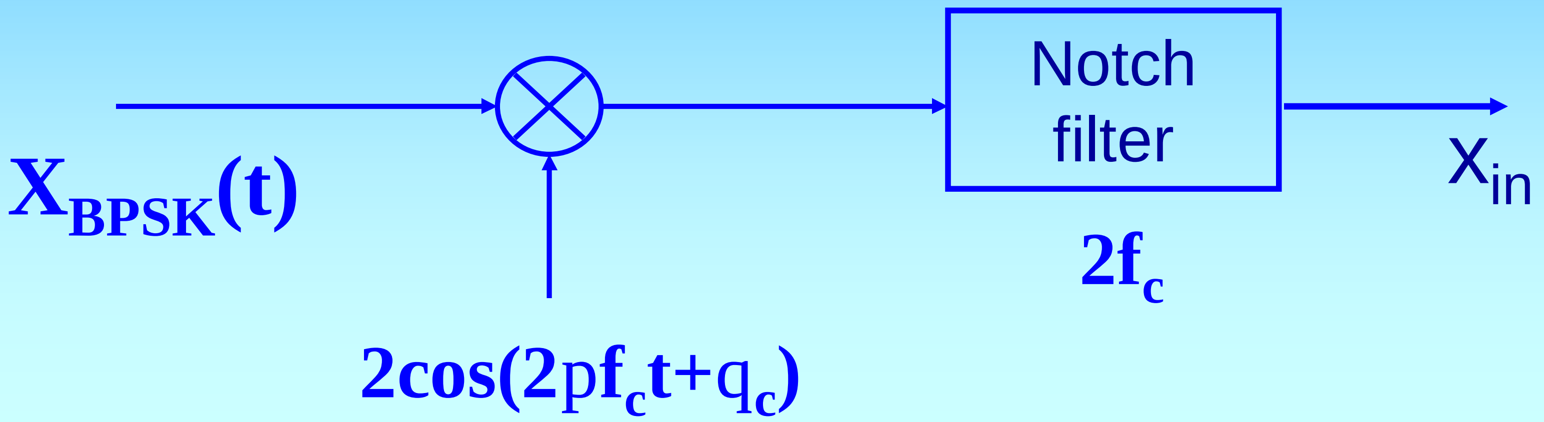


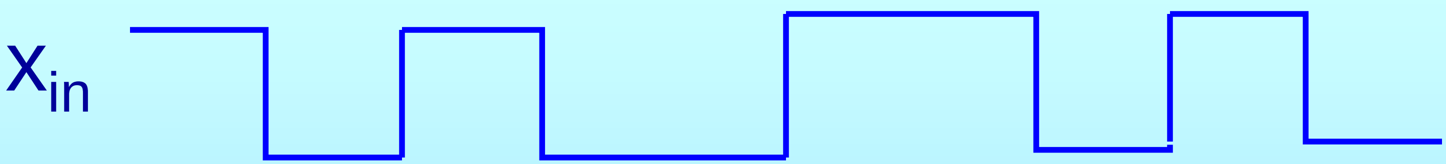
$$s_{BPSK}(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t + \theta_c) \qquad (bit\ 1)$$

$$s_{BPSK}(t) = -\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t + \theta_c) \qquad (bit\ 0)$$

$$P_{\mathrm{BPSK}} = P_p(f+f_c) + P_p(f-f_c), \qquad P_p(f) = A_c^2 T_b \left(\frac{\sin \pi f T_b}{\pi f T_b}\right)^2$$

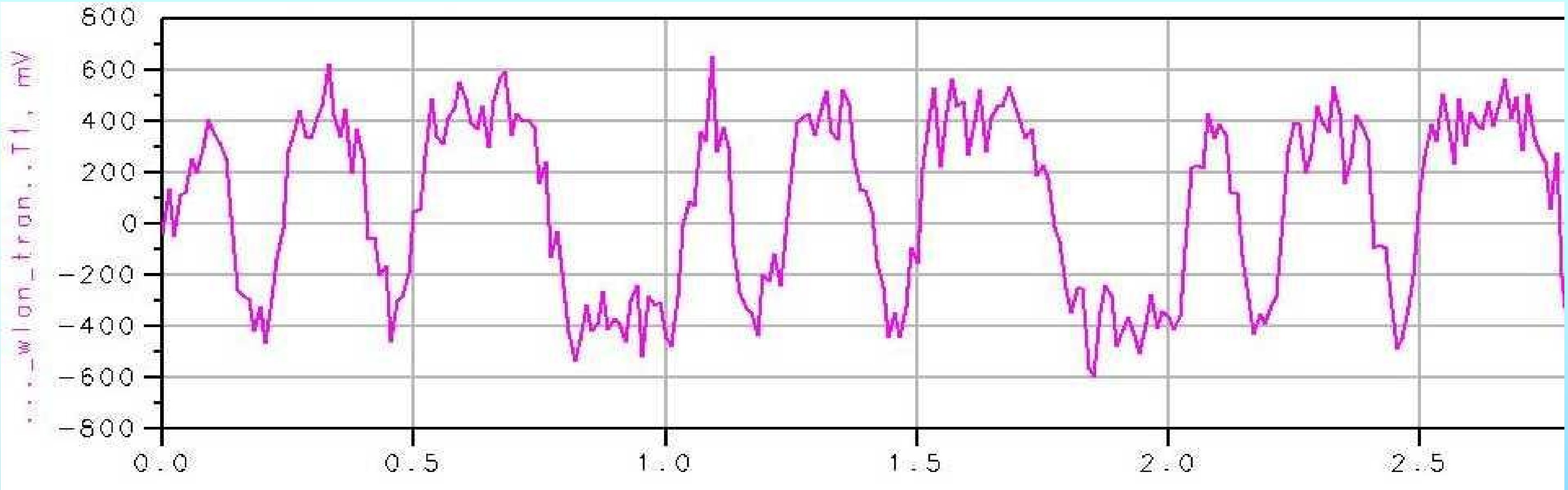
Demodulation Techniques



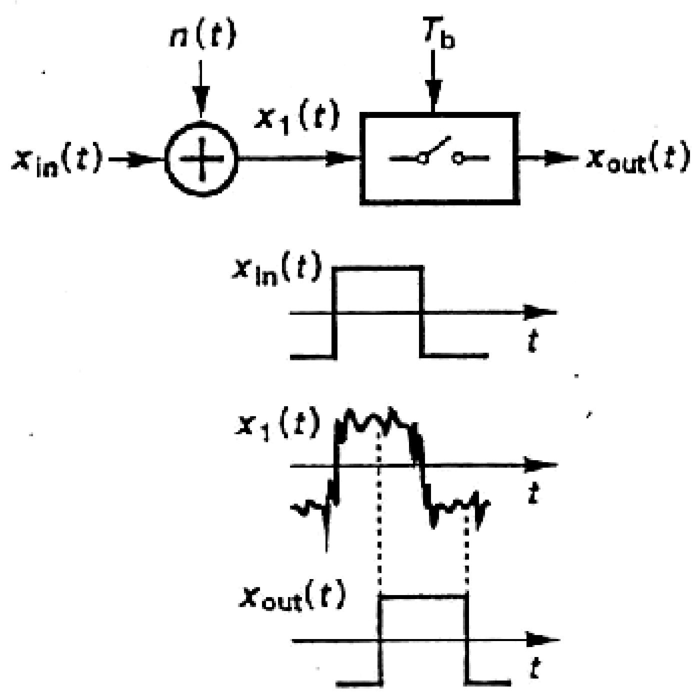


Linear Modulation Techniques

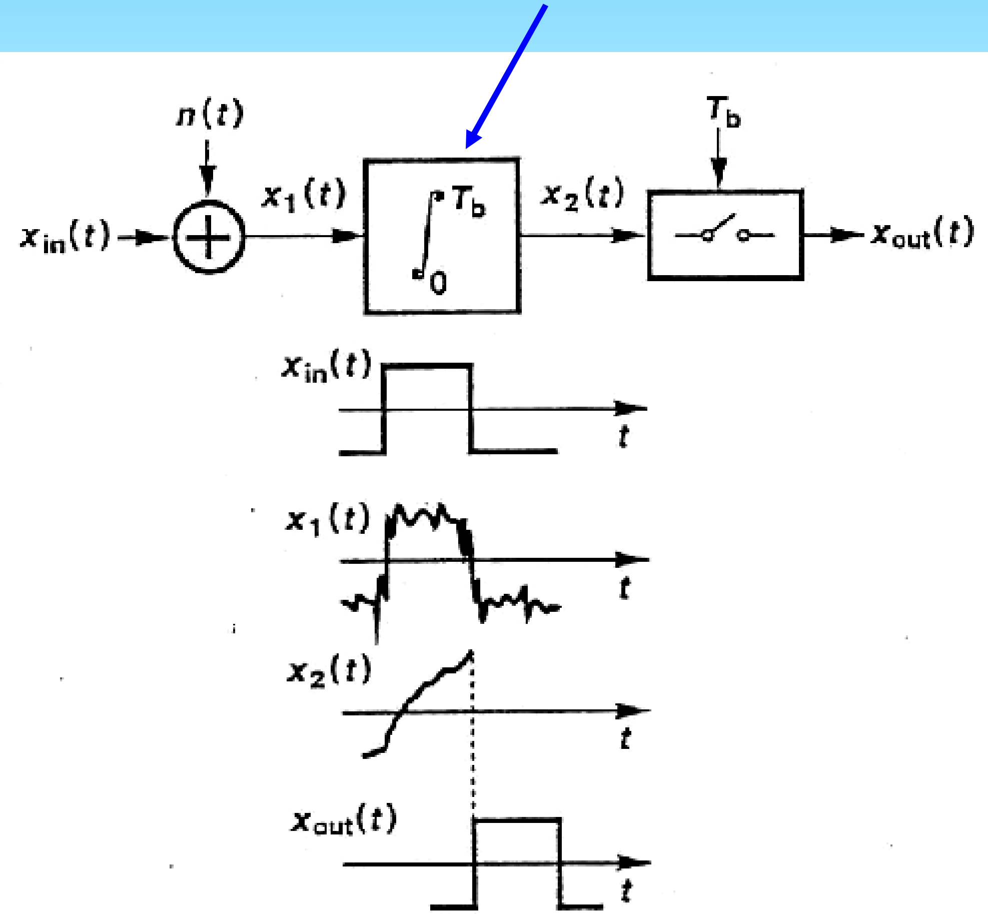
- Waveform of BPSK after adding noise



Detection with Matched Filter:

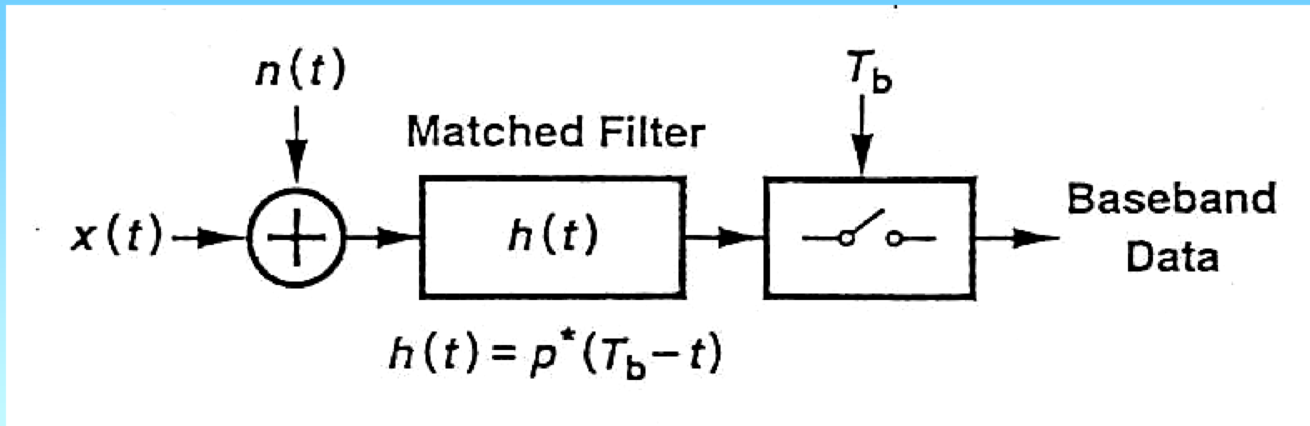


(a)



(b)

Filter impulse response is a square pulse, matching the pulse shapes in x_{in}



$$y(t) = p(t) * h(t)$$
$$= \int_{-\infty}^{+\infty} p(t-t)h(t)dt,$$

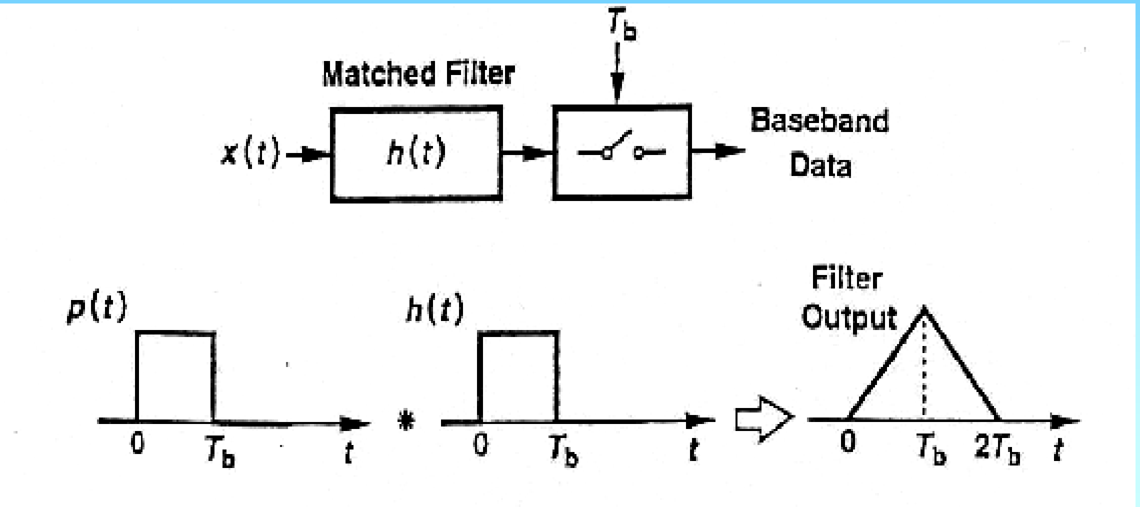
$$n_o(t) = p(t) * n(t)$$
$$\overline{n_o^2 * t)} = \int_{-\infty}^{+\infty} |H(f)|^2 N(f)df$$

$$E_P = \int_{-\infty}^{+\infty} |p(t)|^2 dt$$

$$\overline{n_o^2} = N_o / 2$$

$$SNR_{\max} = \frac{2E_p}{N_0}$$

Optimum Detection:



$$y(T_b) = \int_{-\infty}^{+\infty} x(t)h(T_b - t)dt$$

$$= \int_{-\infty}^{+\infty} x(t)p(t)dt,$$

$$y(T_b) = \int_{t=0}^{t=T_b} x(t)p(t)dt$$

Demodulation

- BPSK Coherent Receiver :

$$s_{BPSK}(t) = m(t)\sqrt{\frac{2E_b}{T_b}}\cos(2pf_c t + q_c + q_{ch})$$

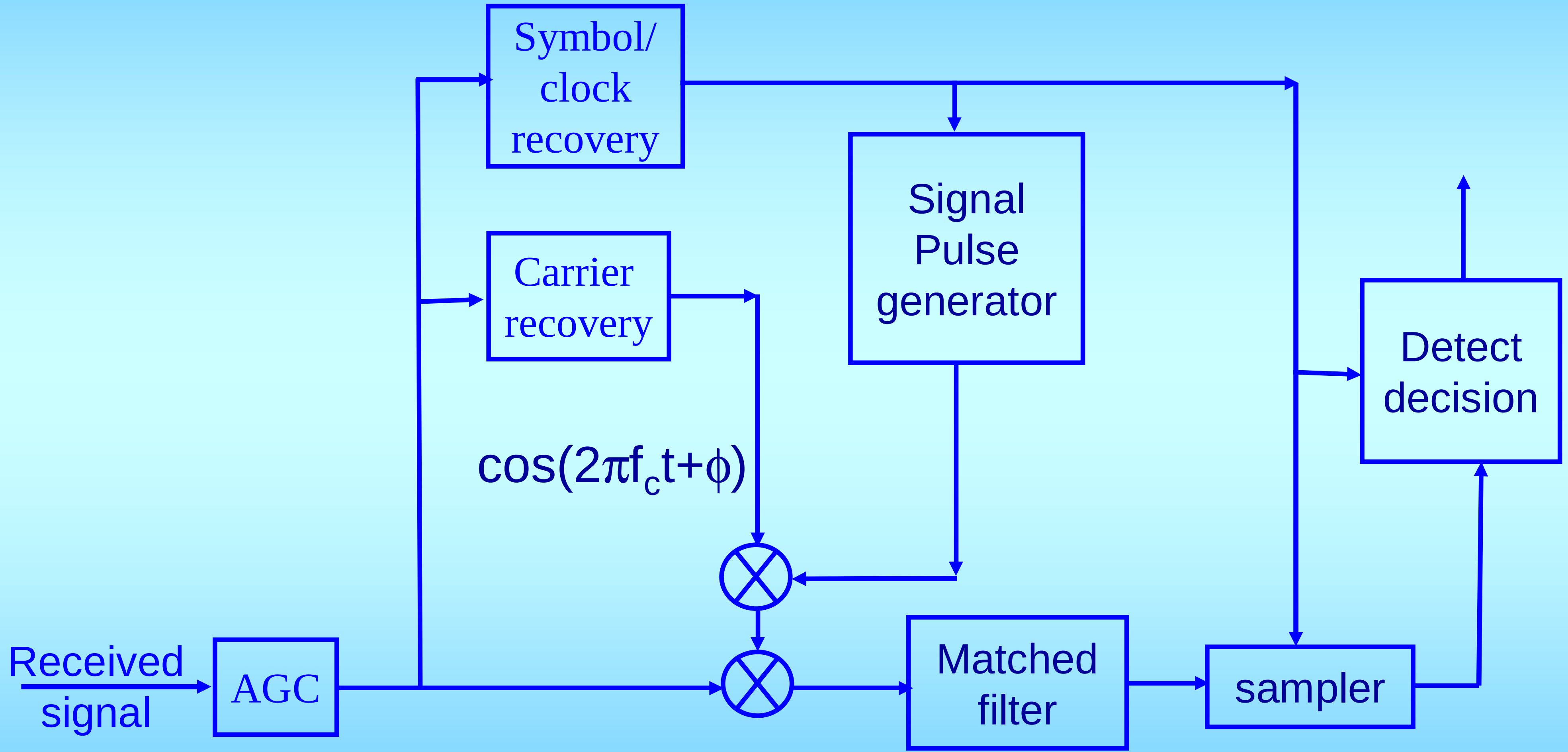
$$= m(t)\sqrt{\frac{2E_b}{T_b}}\cos(2pf_c t + q\quad)$$

Using Carrier Recovery to get carrier frequency:

$$2m(t)\sqrt{\frac{2E_b}{T_b}}\cos^2(2pf_c t + q\quad) = m(t)\sqrt{\frac{2E_b}{T_b}}[1 + 1\cos 2(2pf_c t + q_c)]$$

Bit Error Rate : $P_{e,BPSK} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

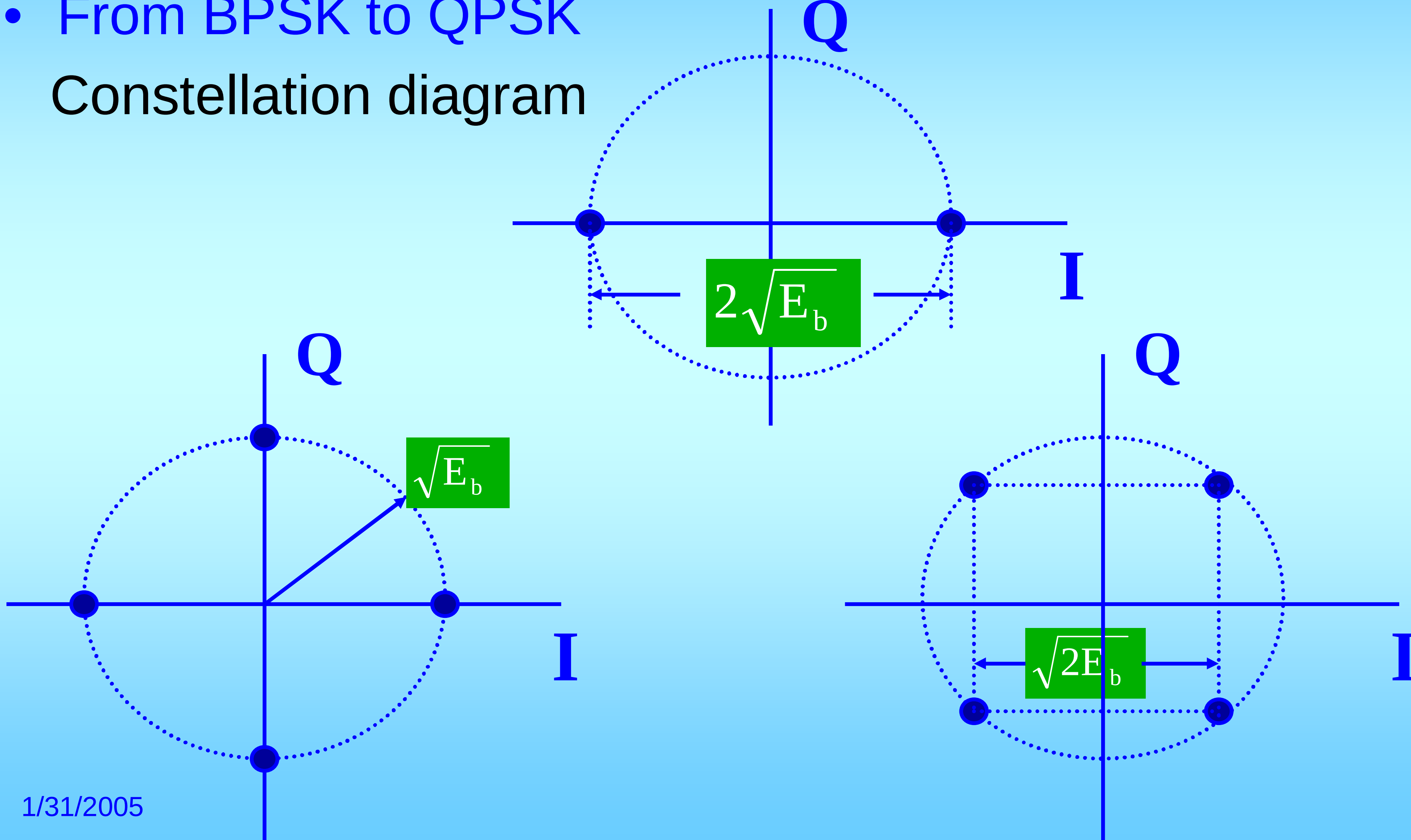
- BPSK Coherent Receiver Architecture



Linear Modulation Techniques

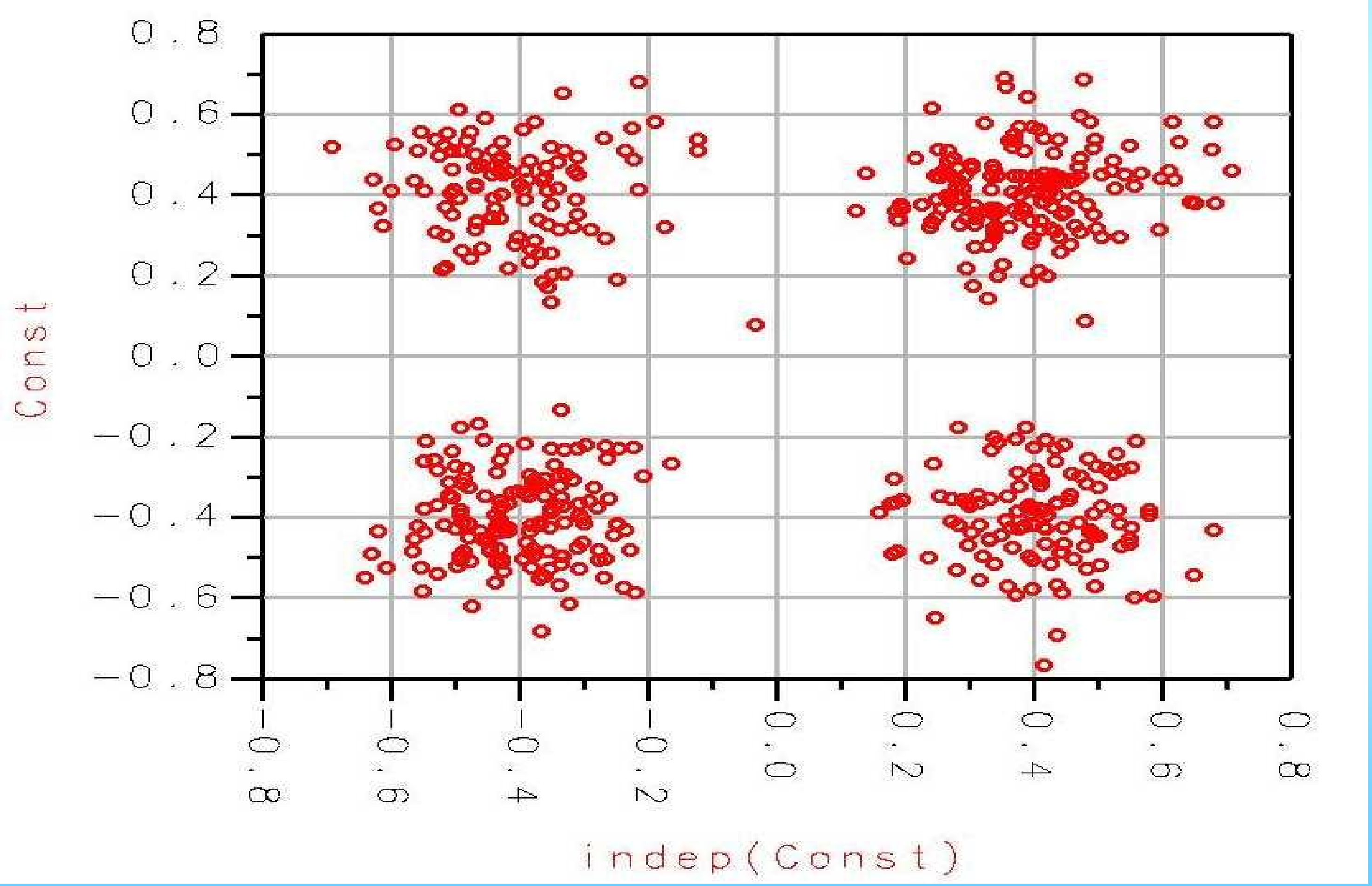
- From BPSK to QPSK

Constellation diagram



Linear Modulation Techniques

- Constellation diagram after adding noise



QPSK transmitter

$$s_{QPSK}(t) = \sqrt{\frac{2E_b}{T_s}} \cos[2pf_c t + (i-1)\frac{p}{2}]$$

$$0 \le \mathrm{t} \le \mathrm{T_s} \quad \mathrm{i} = 1,2,3,4$$

$$s_{QPSK}(t) = \sqrt{\frac{2E_b}{T_s}} \cos[(i-1)\frac{p}{2}] \cos(2pf_c t)$$

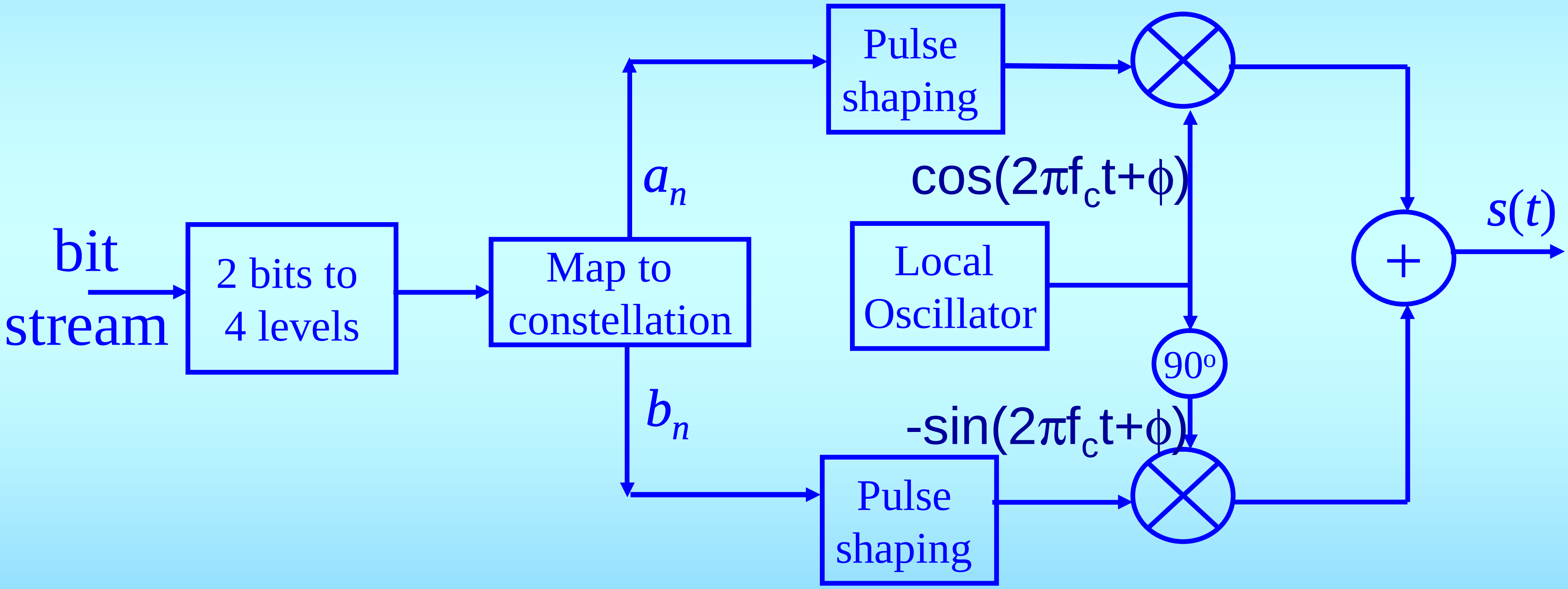
$$- \sqrt{\frac{2E_b}{T_s}} \sin[(i-1)\frac{p}{2}] \sin(2pf_c t)$$

$$s_{QPSK}(t) = \left\{ \sqrt{E_b} \cos[(i-1)\frac{p}{2}] f_1(t) - \sqrt{E_b} \sin[(i-1)\frac{p}{2}] f_2(t) \right\}$$

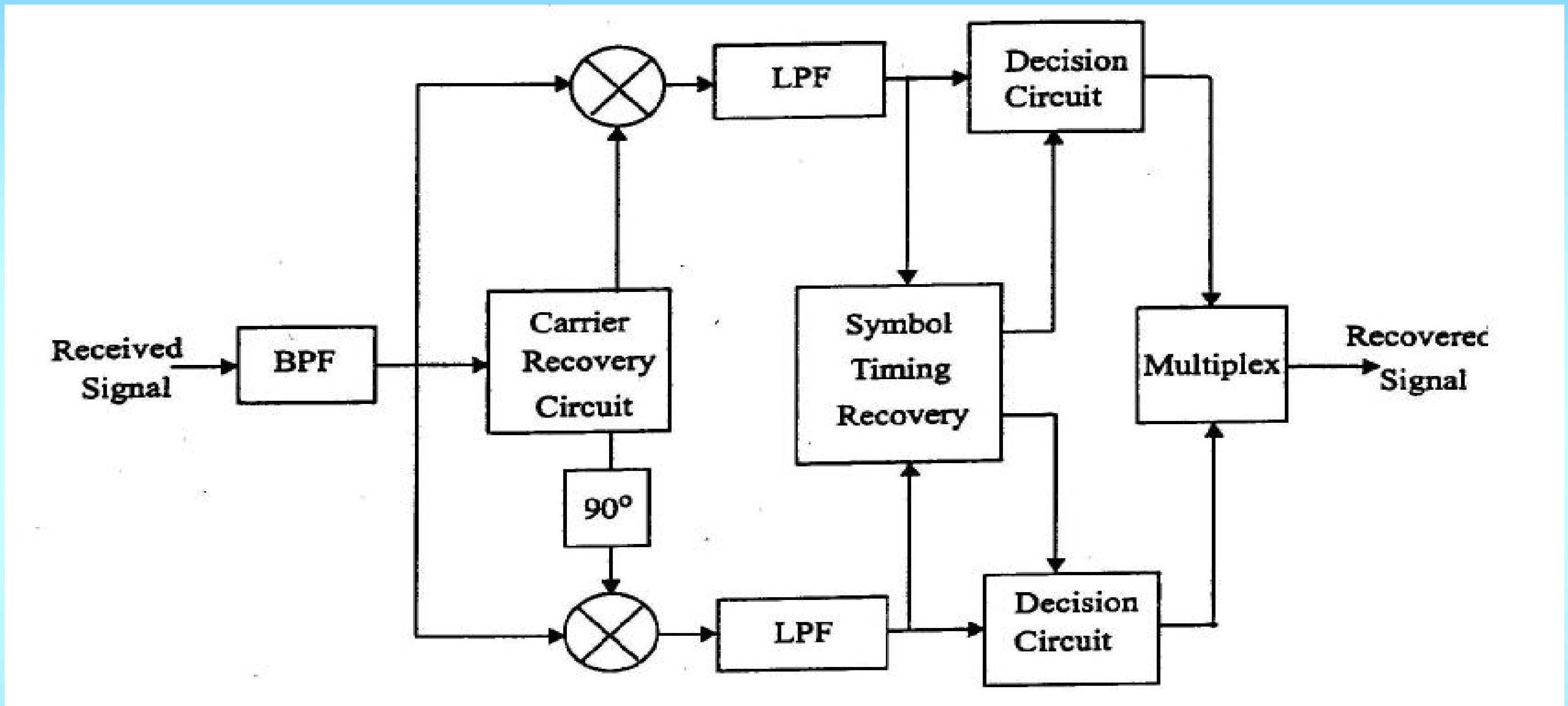
$$P_{\mathrm{BPSK}} = P_p(f + f_c) + P_p(f - f_c), \qquad P_p(f) = 2CT_b \left(\frac{\sin 2pfT_b}{2pfT_b} \right)^2$$

C is average magnitude squared

QPSK transmitter



QPSK Receiver



Bit Error Rate : $P_{e,QPSK} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

BER & BW efficiency comparison

- BPSK v.s QPSK
  - BER is the same !!!

Bit Error Rate : $$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

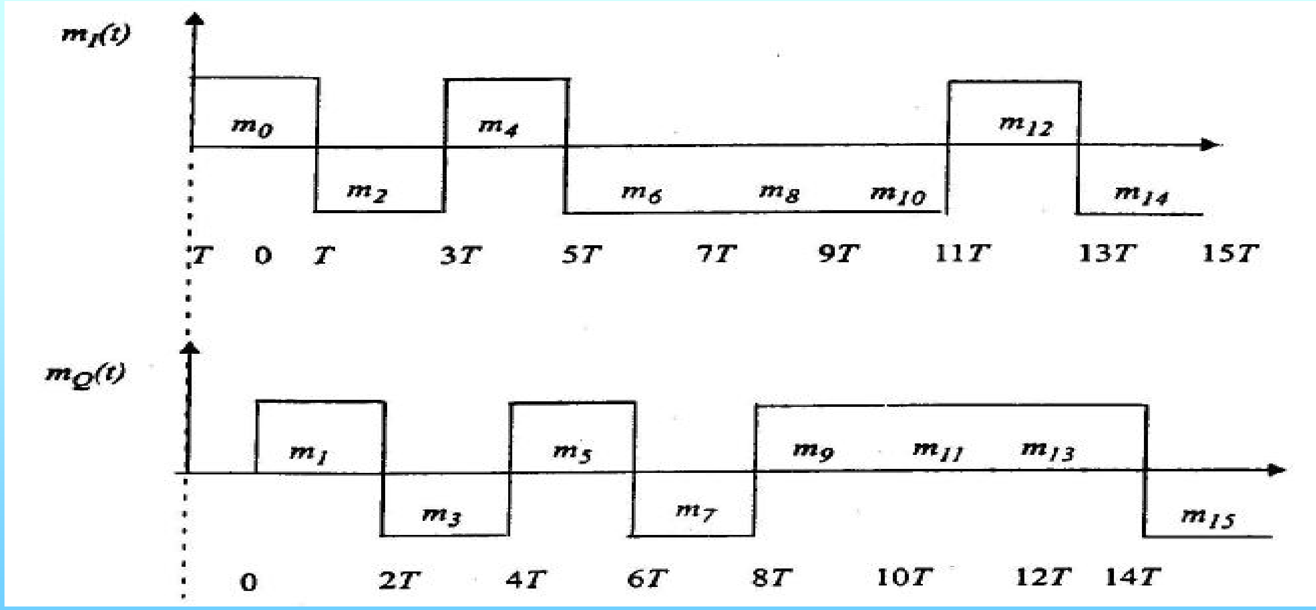
But QPSK only need ½ the bandwidth of BPSK !!!

BPSK: symbol rate = bit rate

QPSK: symbol rate = ½ bit rate

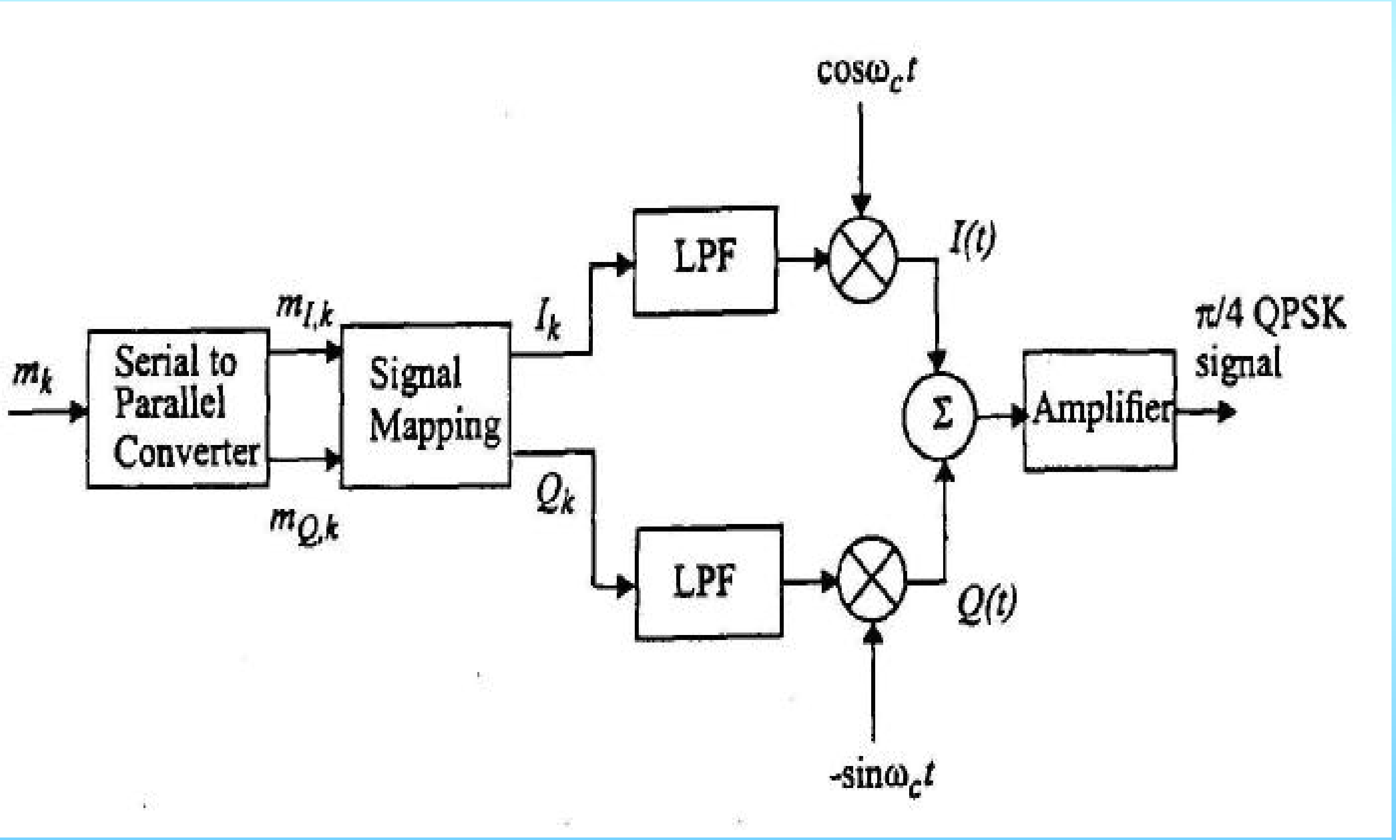
Reducing the maximum phase jump

- Offset QPSK
  - To overcome 180° phase shift in QPSK thus prevent the overhead to overcome signal envelop distortion caused in QPSK



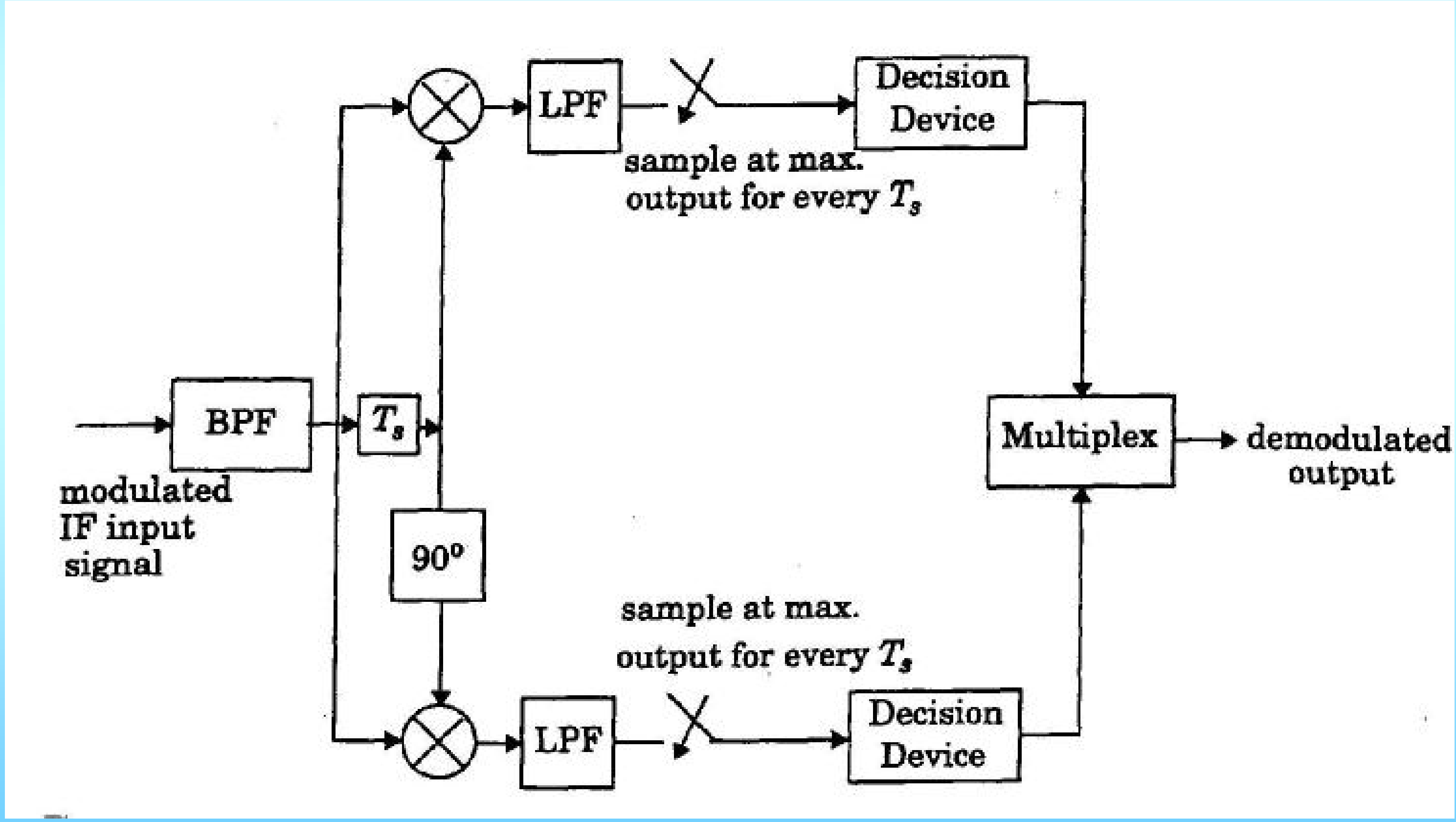
Reducing the maximum phase jump

- Offset QPSK Tx



Reducing the maximum phase jump

- Offset QPSK Rx



Further reduction of phase jump

- π/4-QPSK
  - – Alternate between the 2 QPSK constellations
  - – If the current two bits corresponds to a point in the left constellation
  - – Next two bits will be represented by a point on the right constellation
  - – Vice versa
  - – Also called π/4-DQPSK

Concerns over linear modulation

- On each of the I/Q axis, we are performing AM modulation
- Transmitted signal amplitude changes with time
  - Cause receiver challenges
  - Transmitter power utilization
  - Sensitive to additive noise

è use nonlinear modulation

Constant Envelope Modulation

- Class C Amp. can be used
  - saving power
- Limiter-discriminator detection can be used
  - easy and simple architecture
- Good performance against random noise and signal fluctuation due to Rayleigh fading
  - good performance
- But! BW is larger than linear modulation

Binary frequency shift keying

- BFSK
  - General Form

$$s_{\text{FSK}}(t) = u_H(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2pf_c + 2p\Delta f)t \quad 0 \le t \le T_b \text{ (bit 1)}$$

$$s_{\text{FSK}}(t) = u_L(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2pf_c - 2p\Delta f)t \quad 0 \le t \le T_b \text{ (bit 0)}$$

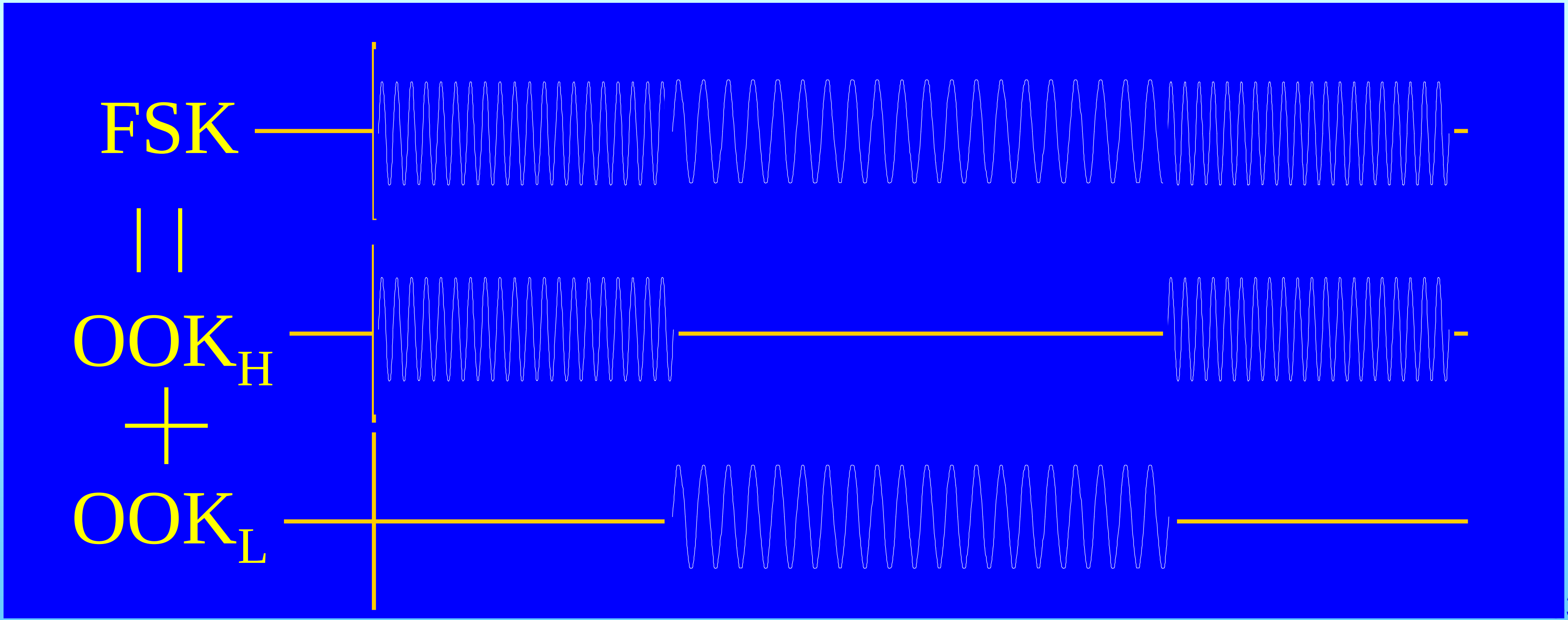
  - Discontinuous phase FSK

$$s_{\text{FSK}}(t) = u_H(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2pf_H t + q_1) \quad 0 \le t \le T_b \text{ (bit 1)}$$

$$s_{\text{FSK}}(t) = u_L(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2pf_L t - q_1) \quad 0 \le t \le T_b \text{ (bit 0)}$$

Discontinuous phase FSK

- Discontinuous phase FSK
  - can be combined by two OOK
  - cause spectral spreading and spurious



Continuous phase FSK

- Continuous phase FSK
  - similar to FM except that m(t) is binary

$$s_{\text{FSK}}(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2pf_c t + q(t))$$

$$= \sqrt{\frac{2E_b}{T_b}}\cos[2pf_c t + 2pf_d \int_{-\infty}^{t} m(h)\,dh]$$

$$= \sqrt{\frac{2E_b}{T_b}}\left[\cos 2pf_c t \cdot \cos q(t) - \sin 2pf_c t \cdot \sin q(t)\right]$$

Binary frequency shift keying

- Coherent detection of BFSK

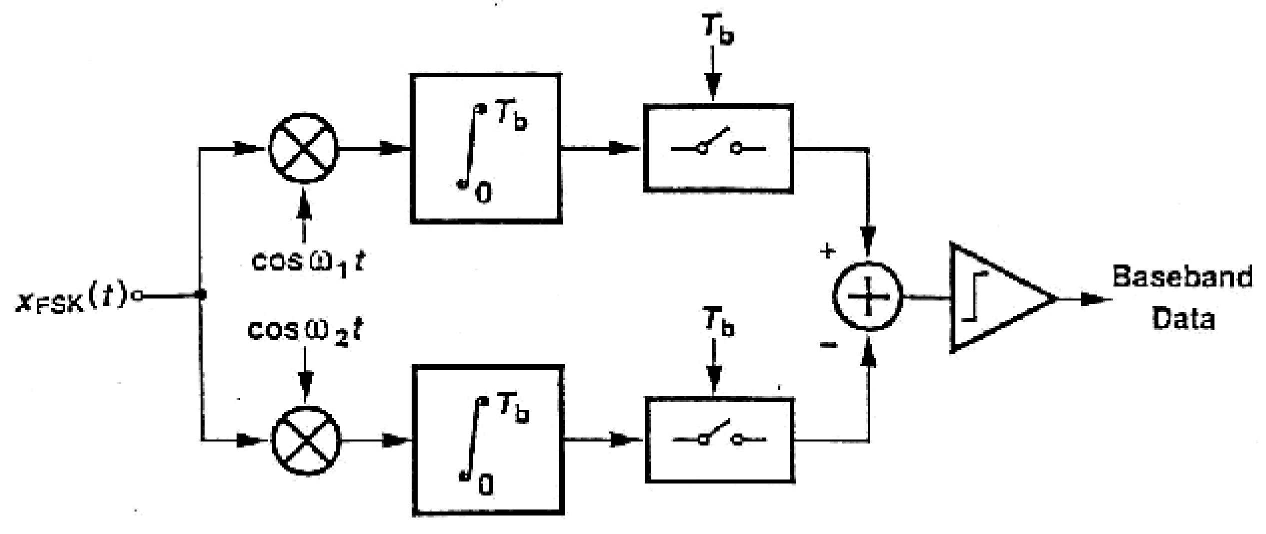


Figure 3.24 Coherent FSK detector.

$$P_{e,FSK} = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Binary frequency shift keying

- Non coherent detection of BFSK

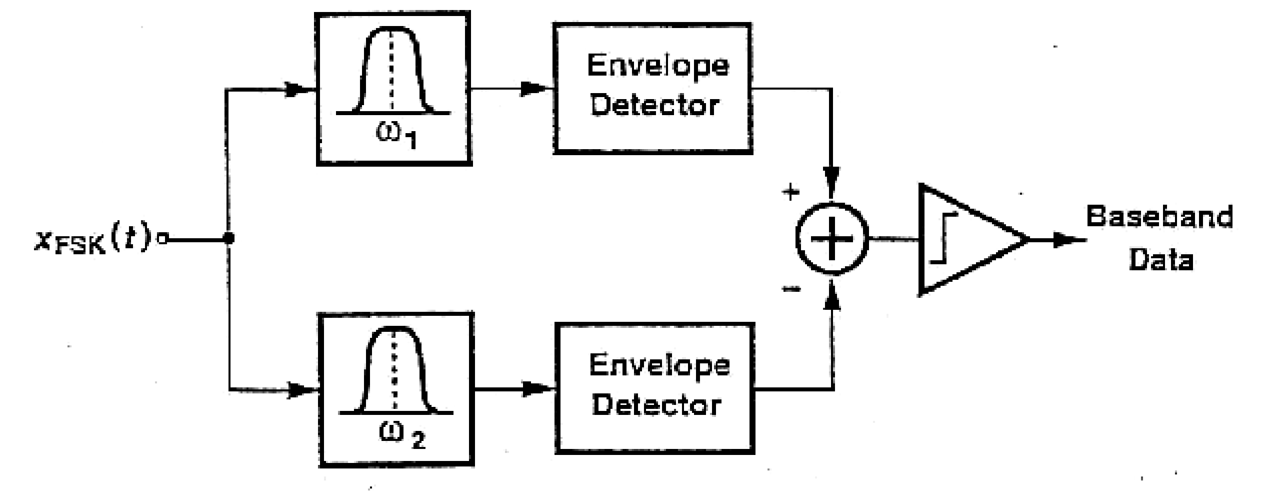


Figure 3.25 Noncoherent FSK detector.

$$P_{e,\,\mathrm{FSK,\,NC}} = \frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$$

Modulation Index

- Modulation Index of FSK:

$$h = \frac{(2\Delta F)}{R_b} \quad ,$$

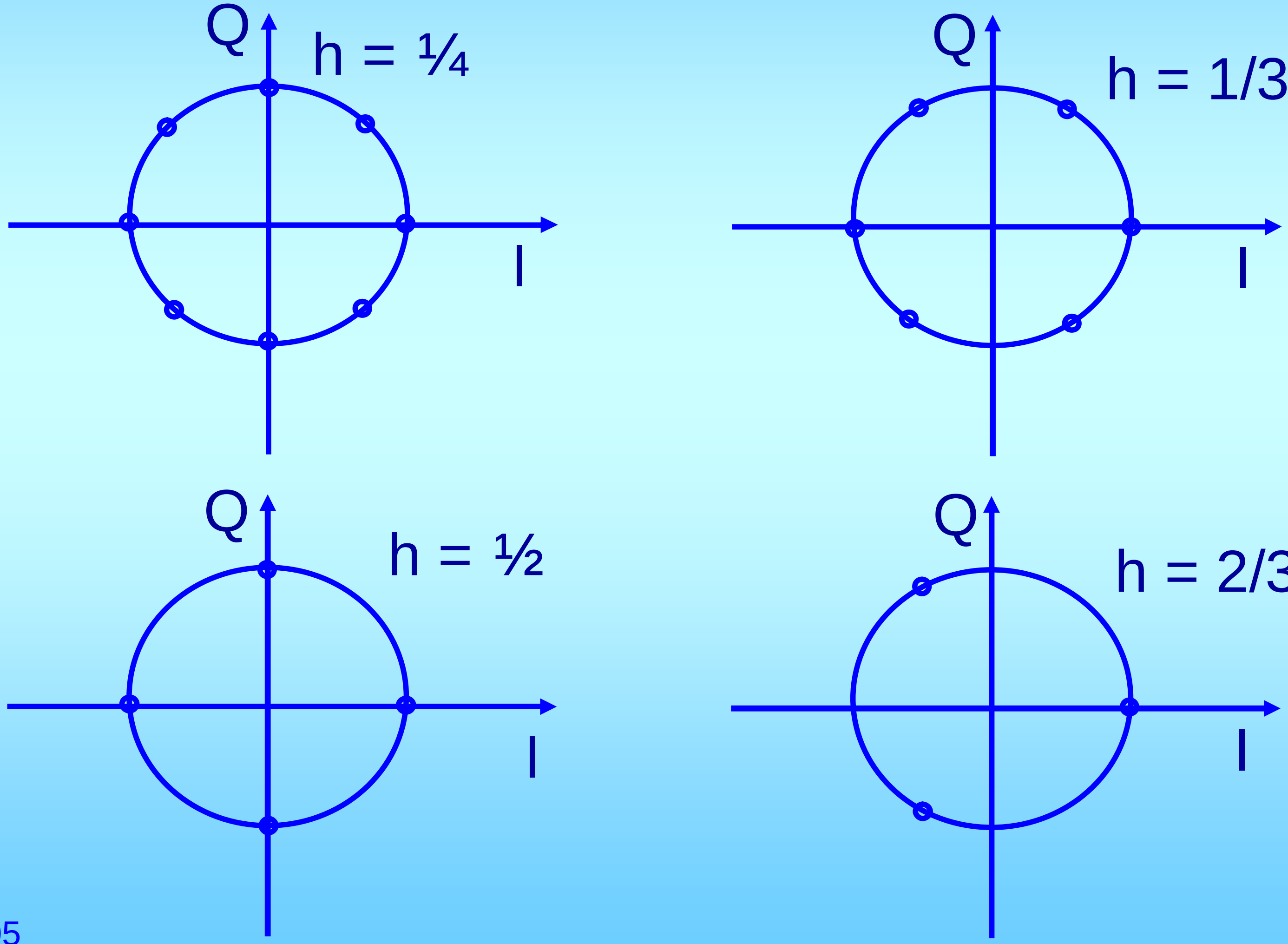
where ΔF is the peak RF frequency deviation and R_b is the bits rate

Example:

$$\Delta F = f_H - f_c = \frac{1}{4} R_b$$

$$\Rightarrow h = \frac{2 \times \frac{1}{4} R_b}{R_b} = 0.5$$

CPFSK and modulation index



Minimum shift keying

- a special type of Continuous phase FSK
- modulation index $h = 0.5$
- peak RF frequency deviation=Rb/4
- coherently orthogonal. i.e. $\int_0^{T_b} u_H(t) u_L(t)\,dt = 0$
- MSK=fast FSK
- MSK=OQPSK with baseband rectangular being replaced with half-sinusoidal
- MSK = FSK with binary signaling freq. of $f_c \pm 1/4T_b$

Minimum shift keying

- Advantage of MSK: particularly attractive for use in mobile radio communication systems:
  - constant envelope
  - spectral efficiency ?
  - good BER
  - self-synchronizing capability

Minimum shift keying

- MSK as OQPSK:

$$s_{MSK}(t) = m_I(t)\cos(\frac{pt}{2T_b})\cos(2pf_c t) + m_Q(t)\sin(\frac{pt}{2T_b})\sin(2pf_c t)$$

Pulse shape: half period cos

- MSK as CPFSK :

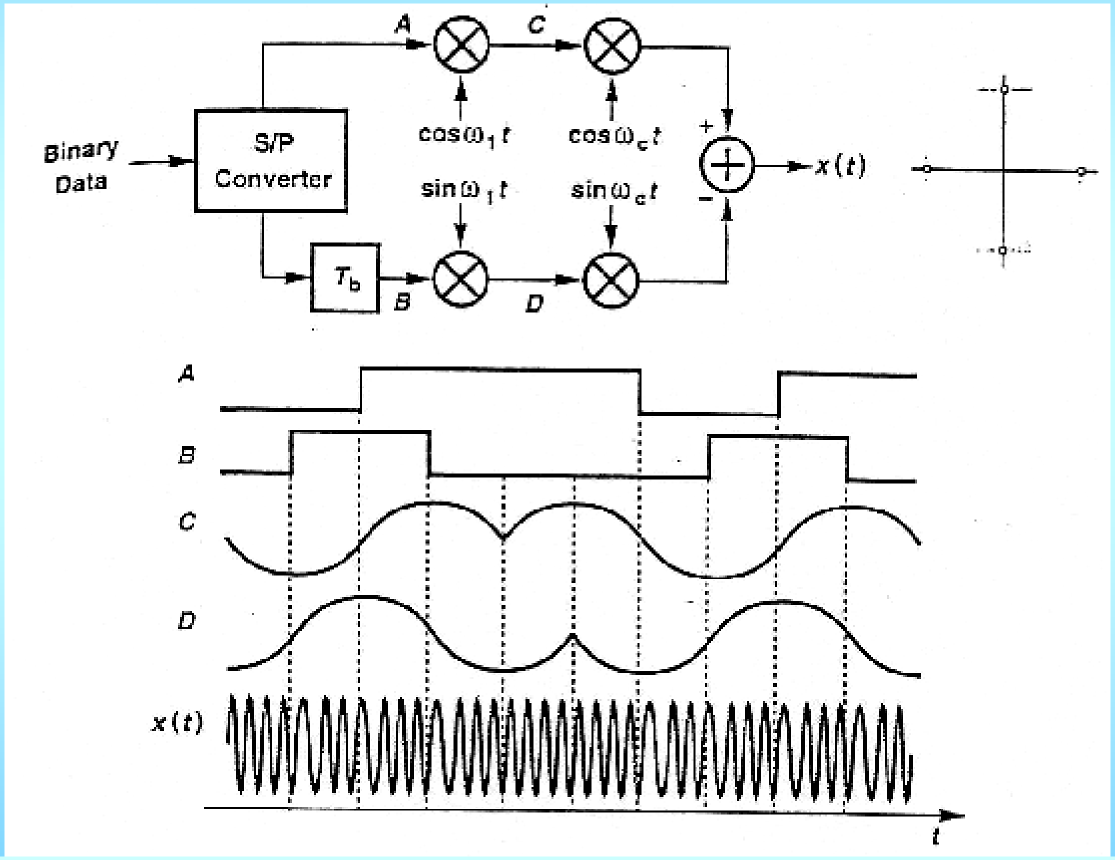
$$s_{\text{MSK}}(t) = \cos\left[2pf_c t - m_I(t)m_Q(t)\frac{pt}{2T_b} + f_k\right]$$

- MSK power spectrum

$$P_{\text{MSK}} = P_p(f + f_c) + P_p(f - f_c), \qquad P_p(f) = \frac{16A_c^2}{p^2}\left(\frac{\cos 2pfT_b}{1 - 16f^2T_b^2}\right)$$

Minimum shift keying

- MSK Transceiver



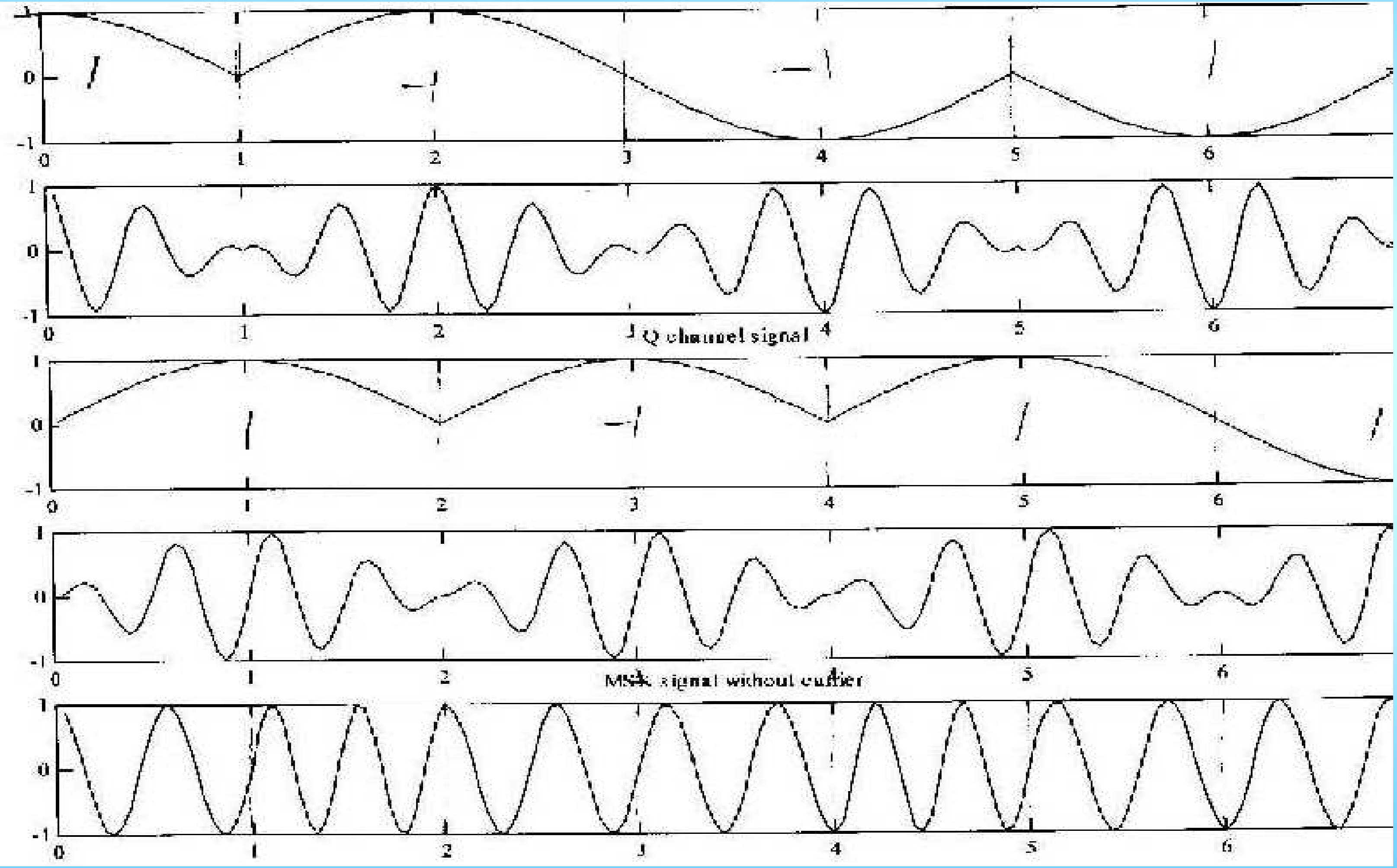
$$x_{MSK}(t) = \sqrt{2}A_c \cos\left[w_c t + \int_{-\infty}^{t} \sum_m b_m p(t - mT_b)dt \right]$$

Minimum shift keying

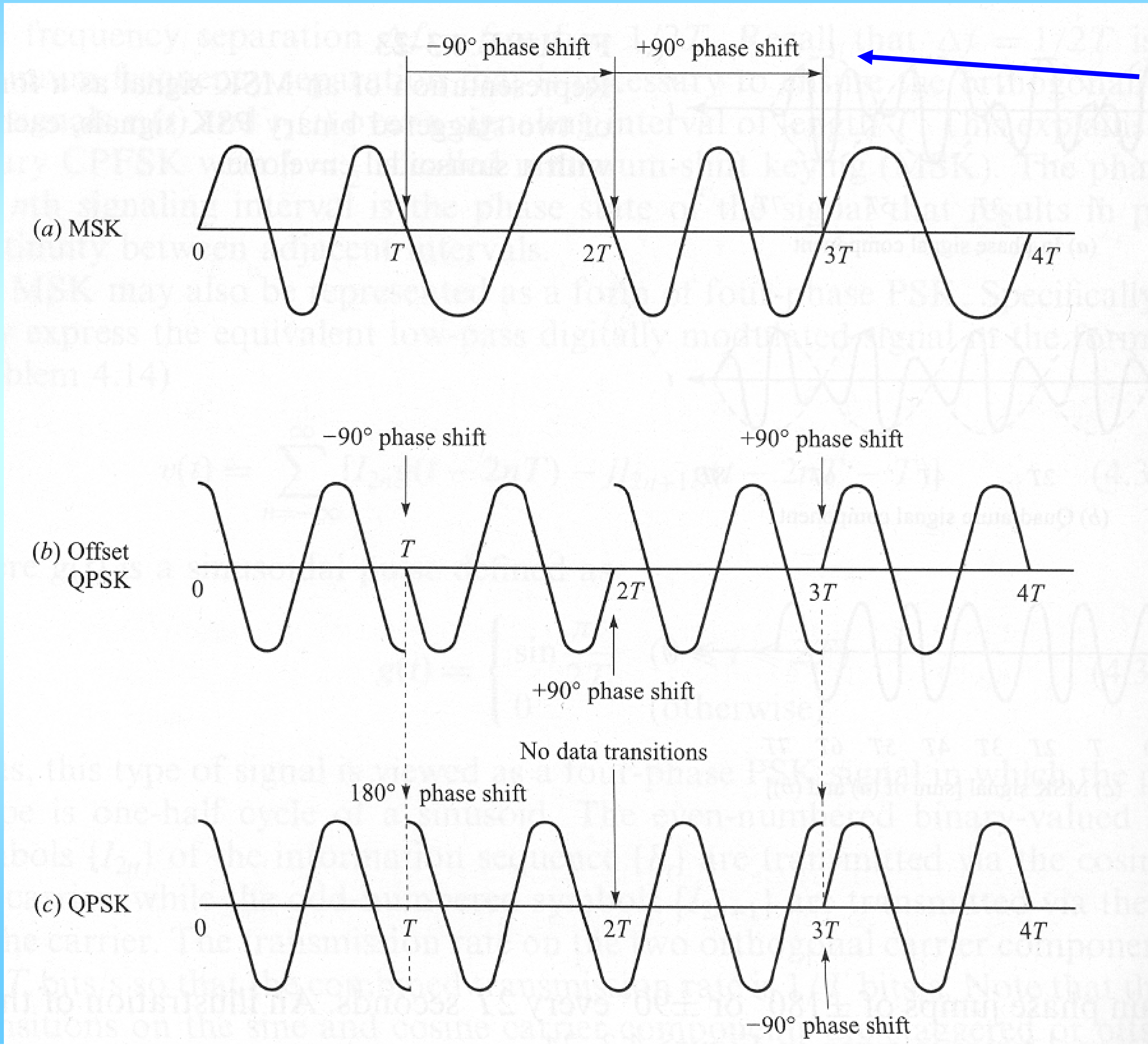
I

Q

MSK



Comparison of MSK, OQPSK, QPSK



90º phase shift in each symbol

MSK

- Receiver of MSK

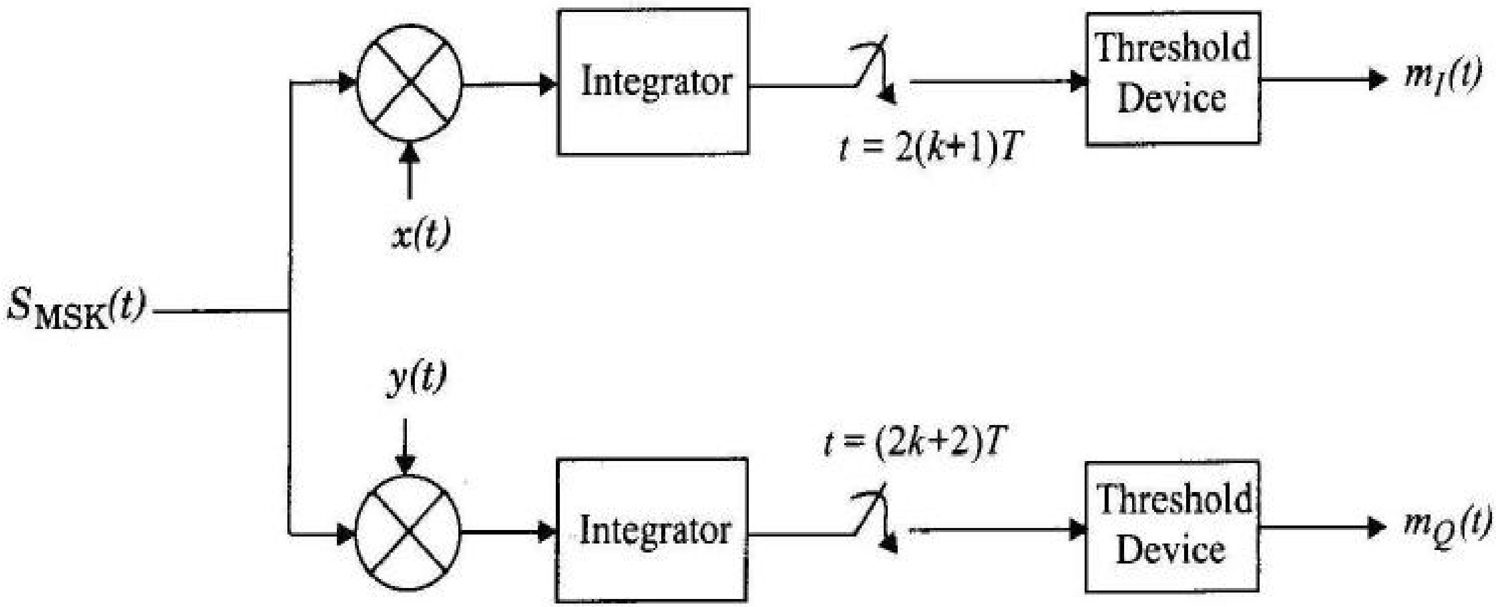


Figure 5.40
Block diagram of an MSK receiver.

Why is MSK more spectrally efficient?

- Power spectral density of MSK
  - 99% BW of MSK = $1.2/T_b$
  - 99% BW of QPSK or OQPSK = $8/T_b$

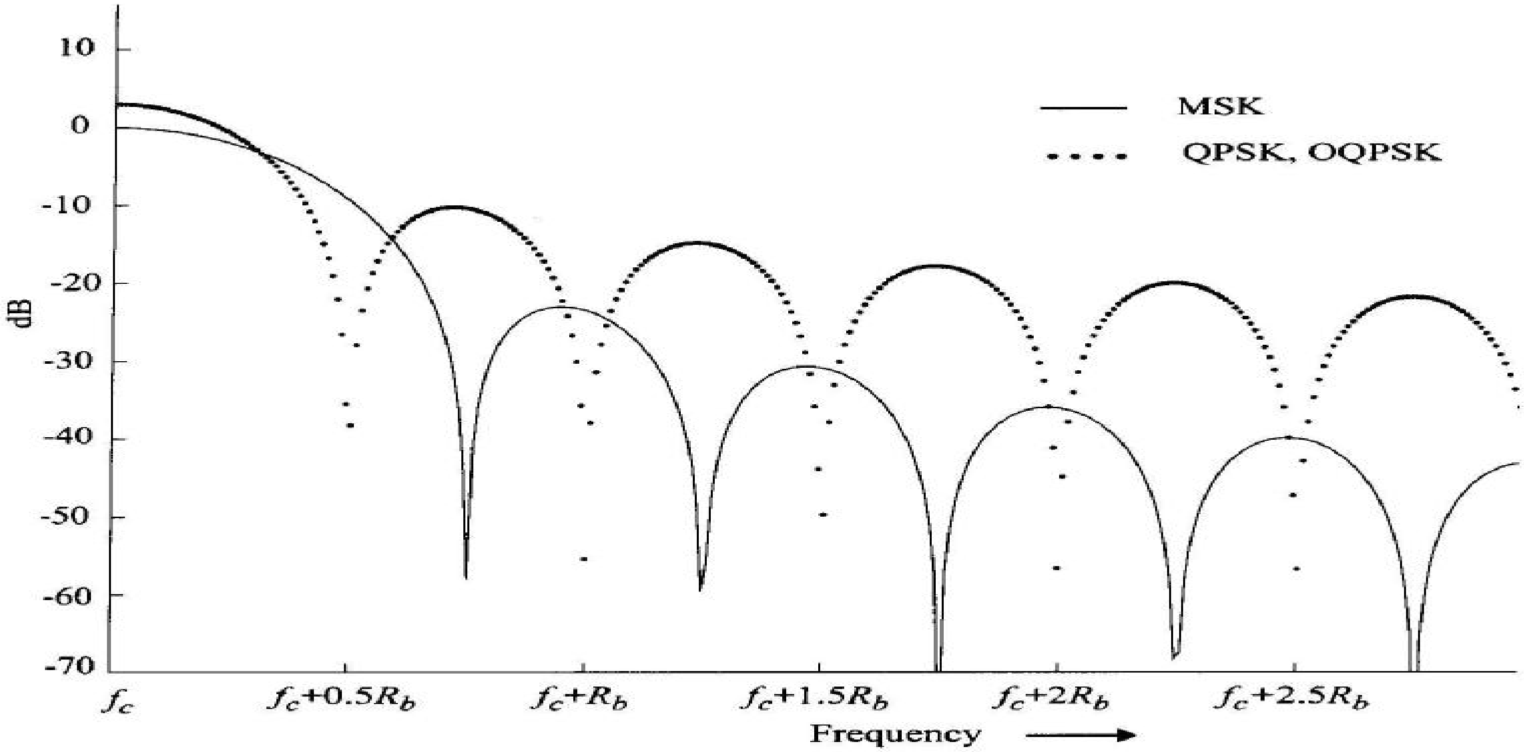
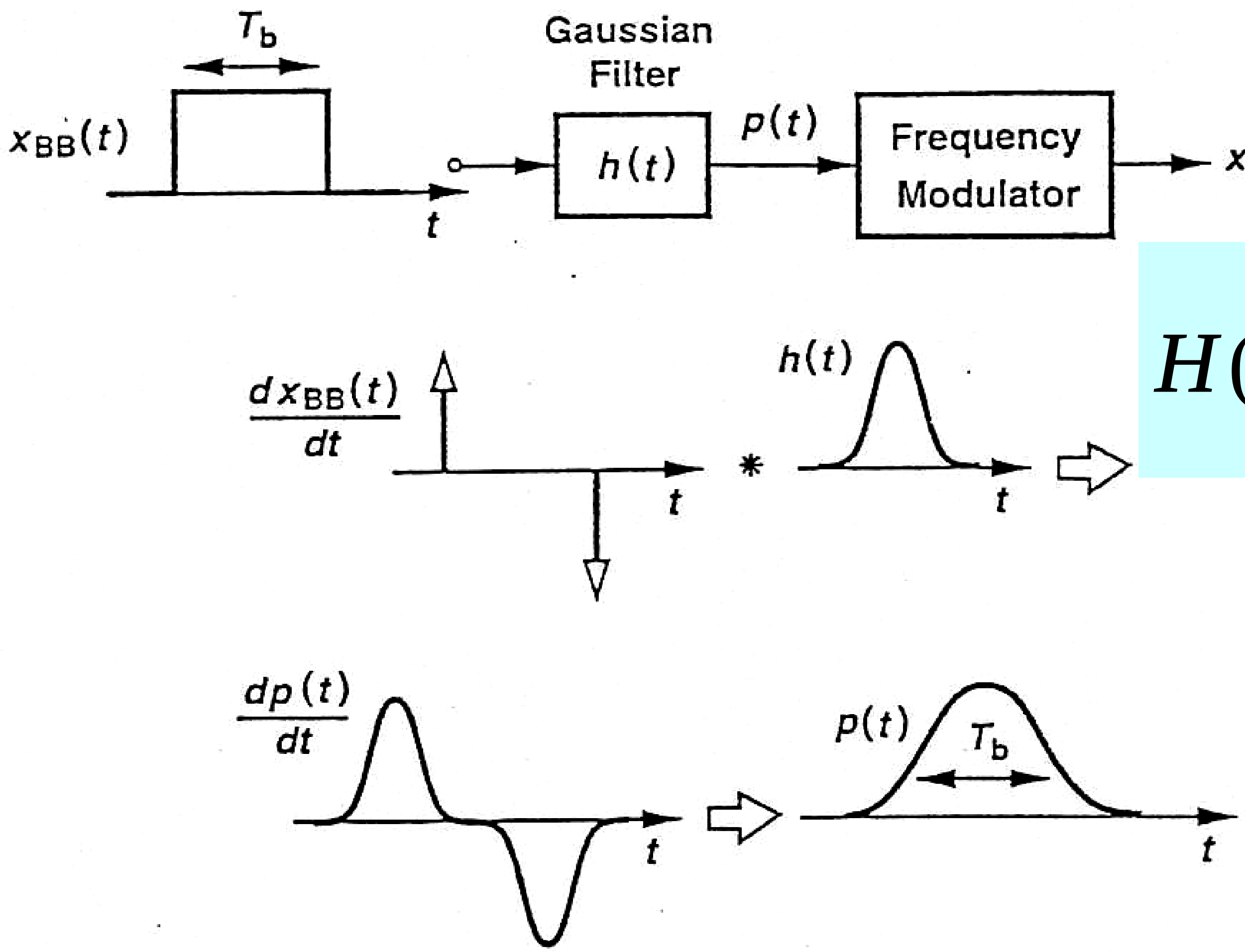


Figure 5.38
Power spectral density of MSK signals as compared to QPSK and OQPSK signals.

Even better spectral efficiency? Use Gaussian MSK



$$H(f) = \exp(-\frac{f^2}{B^2}\frac{\ln 2}{2}$$

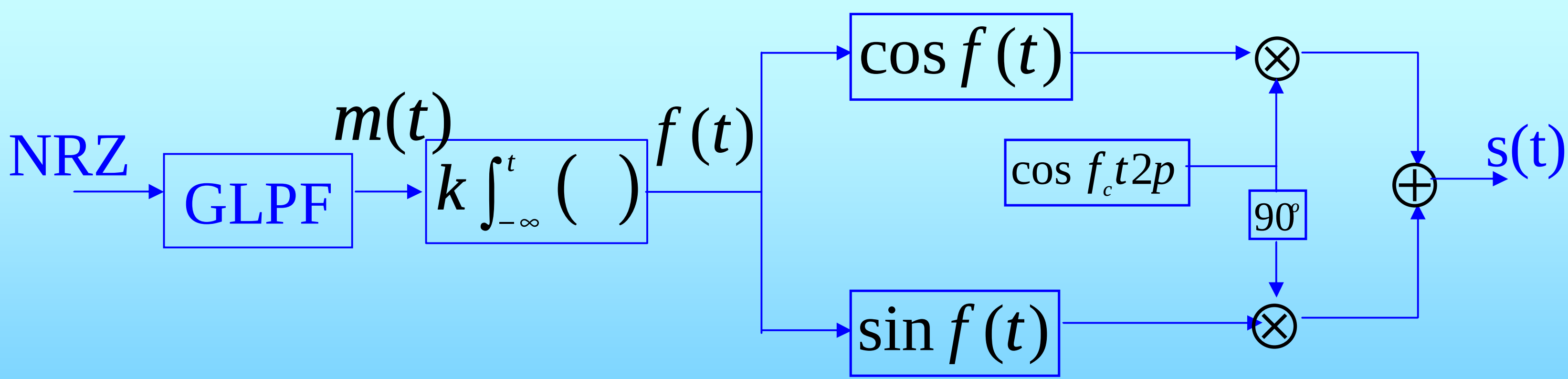
Figure 3.44 GMSK modulation.

- GMSK & GFSK
  - GMSK reduces the sidelobe levels of MSK
  - GMSK = Gaussian filter + MSK
  - MSK = GMSK with B = ∞
  - Important parameter : 3dB-BW – bit duration product ($B_{3dB}T_b$)

- Transmitter of GMSK :

  QUAD architecture

$$s(t) = \cos(2pf_c t + f(t))$$
$$= \cos 2pf_c t \cdot \cos f(t) - \sin 2pf_c t \cdot \sin f(t)$$



- Deciding *frequency modulation index* K_f

$$s_{\mathrm{FSK}}(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2pf_c t + f(t))$$

$$= \sqrt{\frac{2E_b}{T_b}}\cos[2pf_c t + K_f \int_{-\infty}^{t} m(h)\,dh]$$

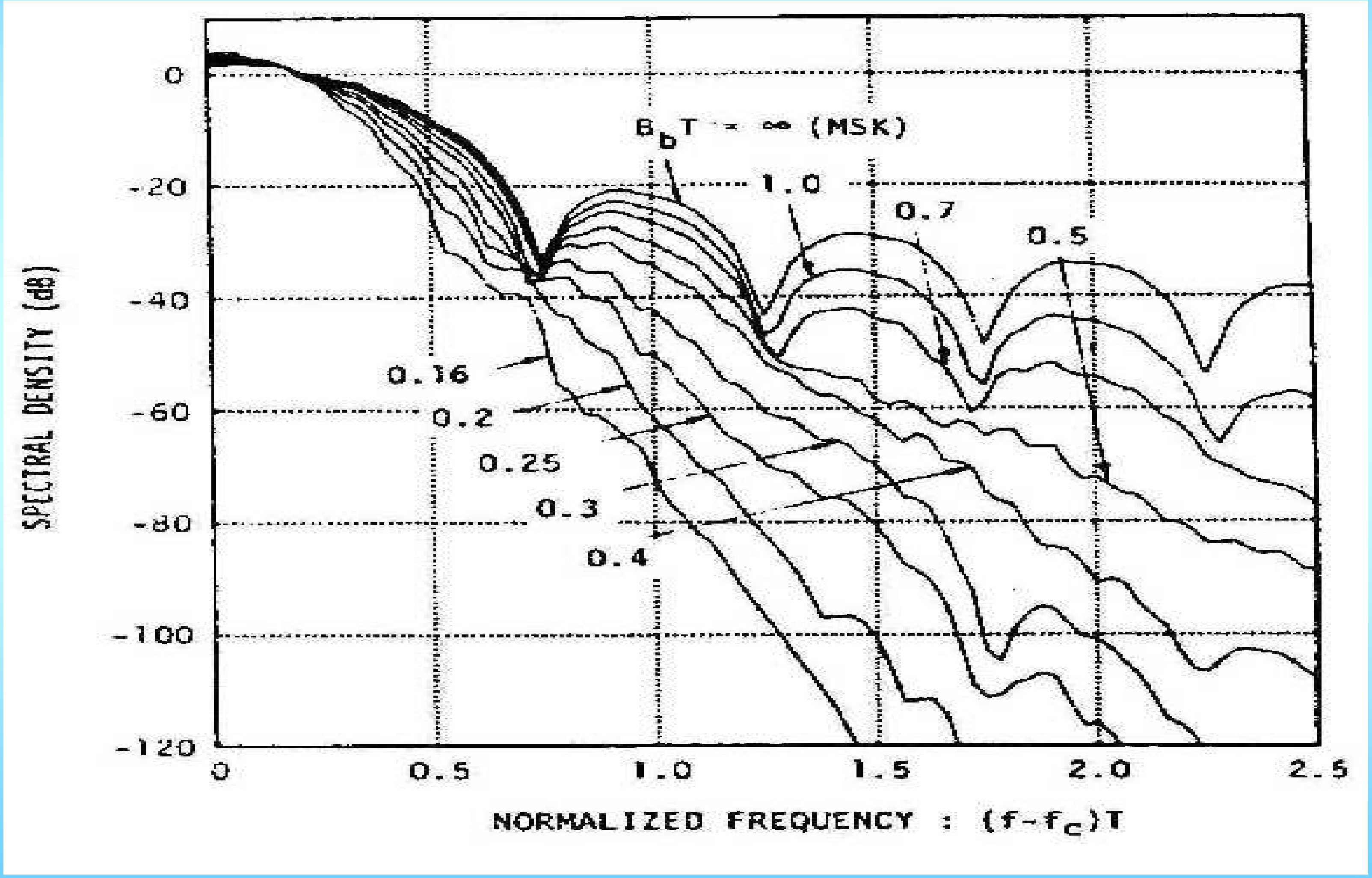
$$f(t) = K_f \int_{-\infty}^{t} m(h)\,dh = K_f \int_{-\infty}^{t} \sum_{n=-\infty}^{\infty} a_n r(h - nT)\,dh$$

$$\Rightarrow K_f \int_{-\infty}^{\infty} r(t)\,dt = p/2$$

$$\Rightarrow K_f \int_{-\infty}^{\infty} \Pi(t) * h_G(t)\,dt = p/2$$

$$\therefore K_f = \frac{p/2}{\int_{-\infty}^{\infty} \Pi(t) * h_G(t)\,dt}$$

- Power spectral density of a GMSK signal



- Receiver of GMSK

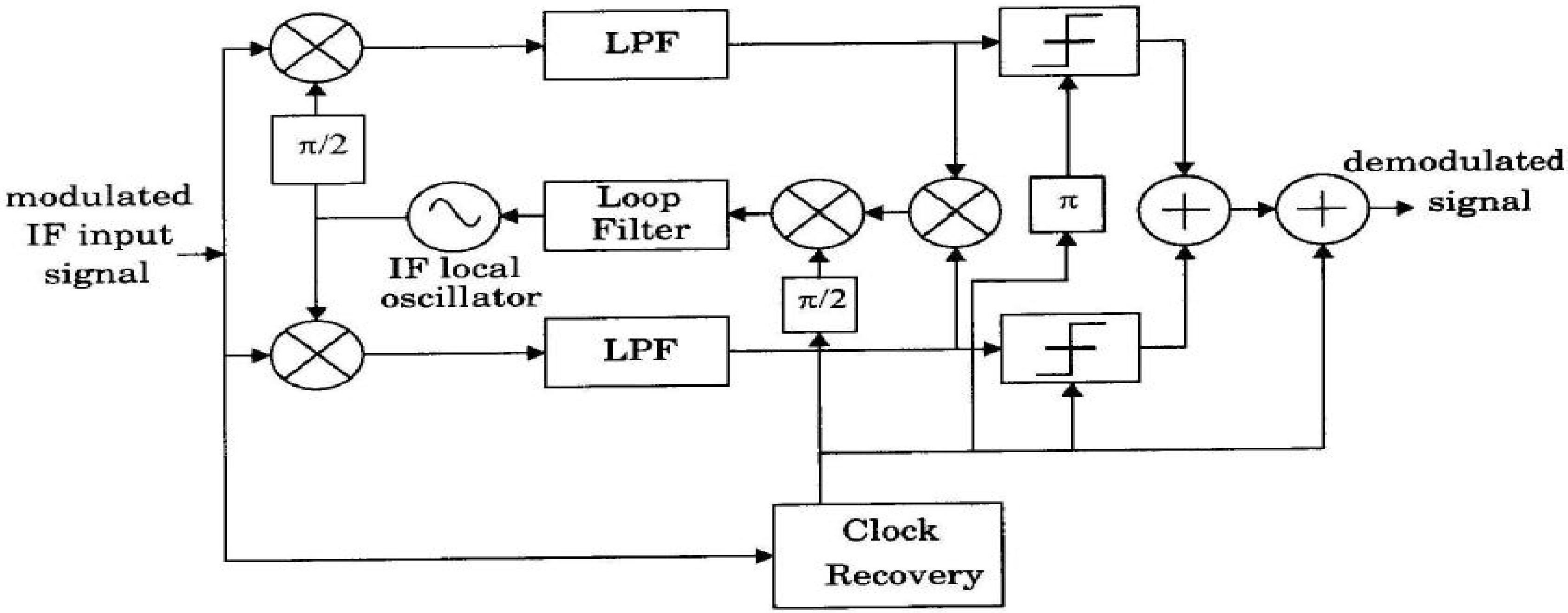


Figure 5.43
Block diagram of a GMSK receiver.

$$P_e = Q\left(\sqrt{\frac{2aE_b}{N_0}}\right)$$

$$a \cong \begin{cases} 0.68 & \text{for GMSK with } BT = 0.25 \\ 0.85 & \text{for simple MSK } (BT = \infty) \end{cases}$$

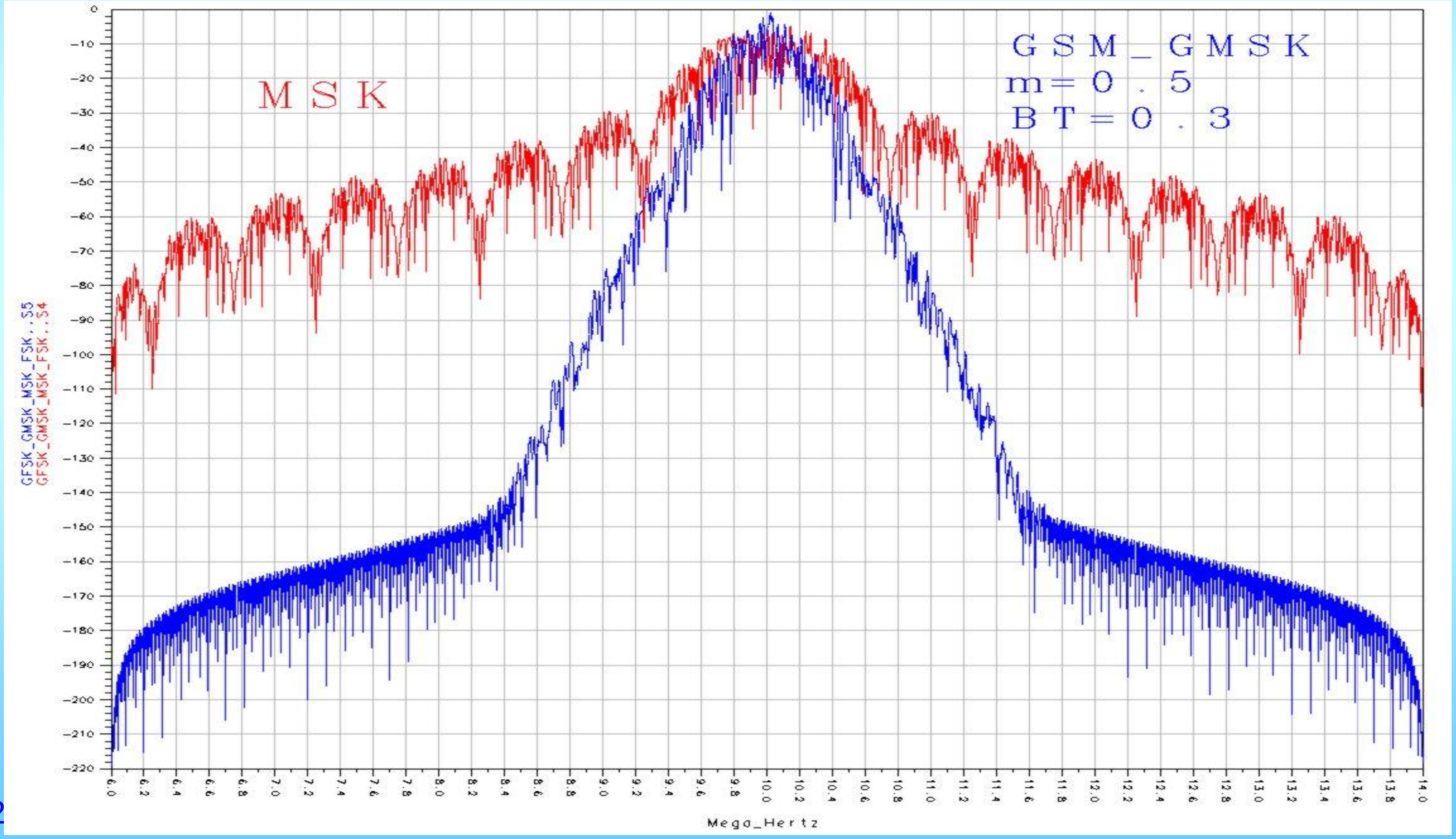
- Occupied RF Bandwidth

Table 5.3 Occupied RF Bandwidth (for GMSK and MSK as a fraction of R_b) Containing a Given Percentage of Power [Mur81]. Notice that GMSK is spectrally tighter than MSK.

BT	90%	99%	99.9%	99.99%
0.2 GMSK	0.52	0.79	0.99	1.22
0.25 GMSK	0.57	0.86	1.09	1.37
0.5 GMSK	0.69	1.04	1.33	2.08
MSK	0.78	1.20	2.76	6.00

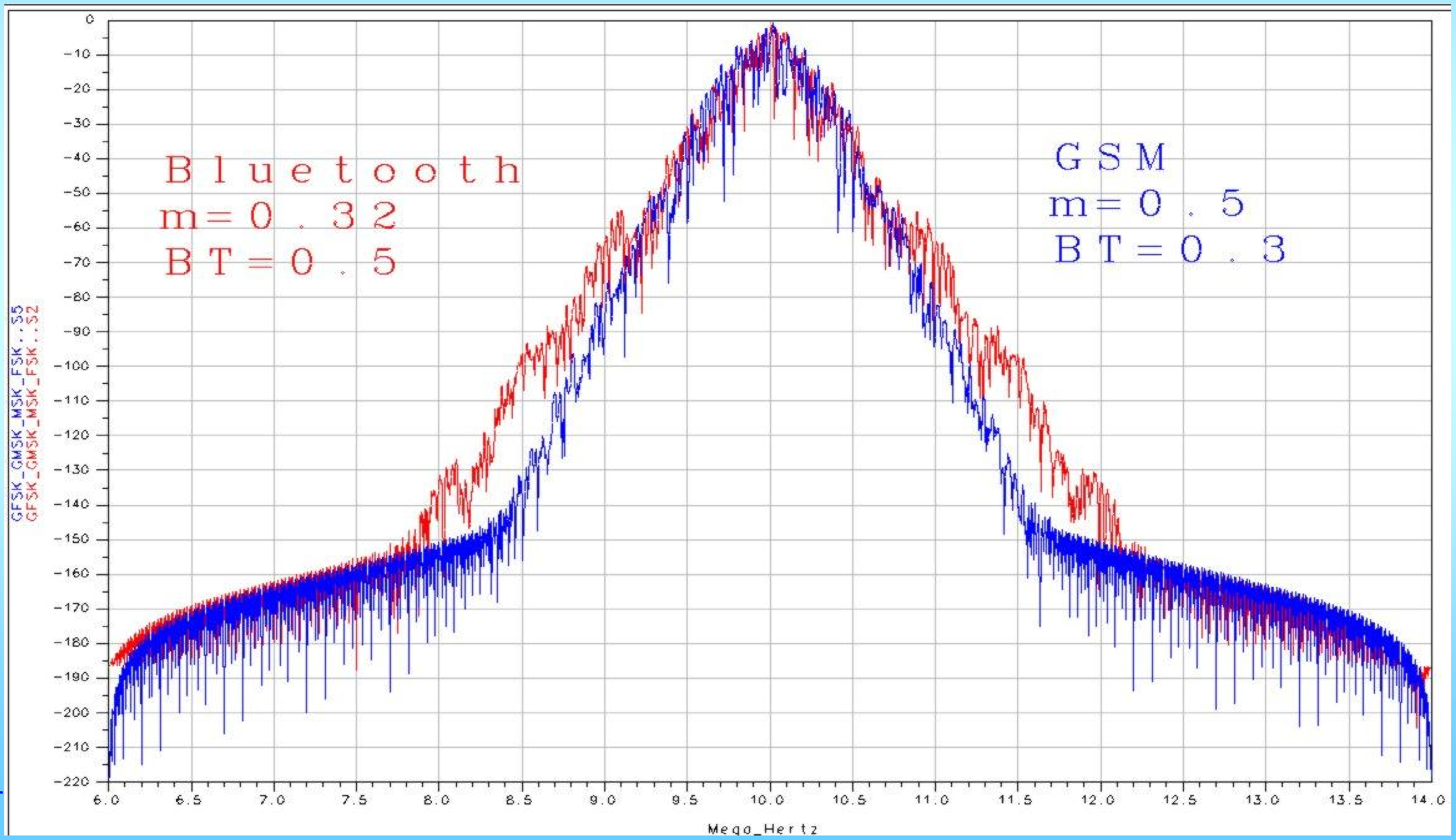
Simulated power spectral density

- GSM GMSK v.s MSK



Simulated power spectral density

- Bluetooth v.s GSM



M-ary PSK

- Two or more bits are grouped to form symbols and one of the possible M symbols may be sent

$$S_i(t) = \sqrt{\frac{2E_s}{T_s}} \cos\left(2pf_c t + \frac{2p}{M}(i-1) \right), 0 \le t \le T_s \quad i = 1, 2, \mathbf{K}, \mathrm{M}$$

$$= \sqrt{\frac{2E_s}{T_s}} \cos\left[(i-1)\frac{2p}{M} \right] \cos(2pf_c \mathrm{t}) - \sqrt{\frac{2E_s}{T_s}} \sin\left[(i-1)\frac{2p}{M} \right] \sin(2pf_c \mathrm{t})$$

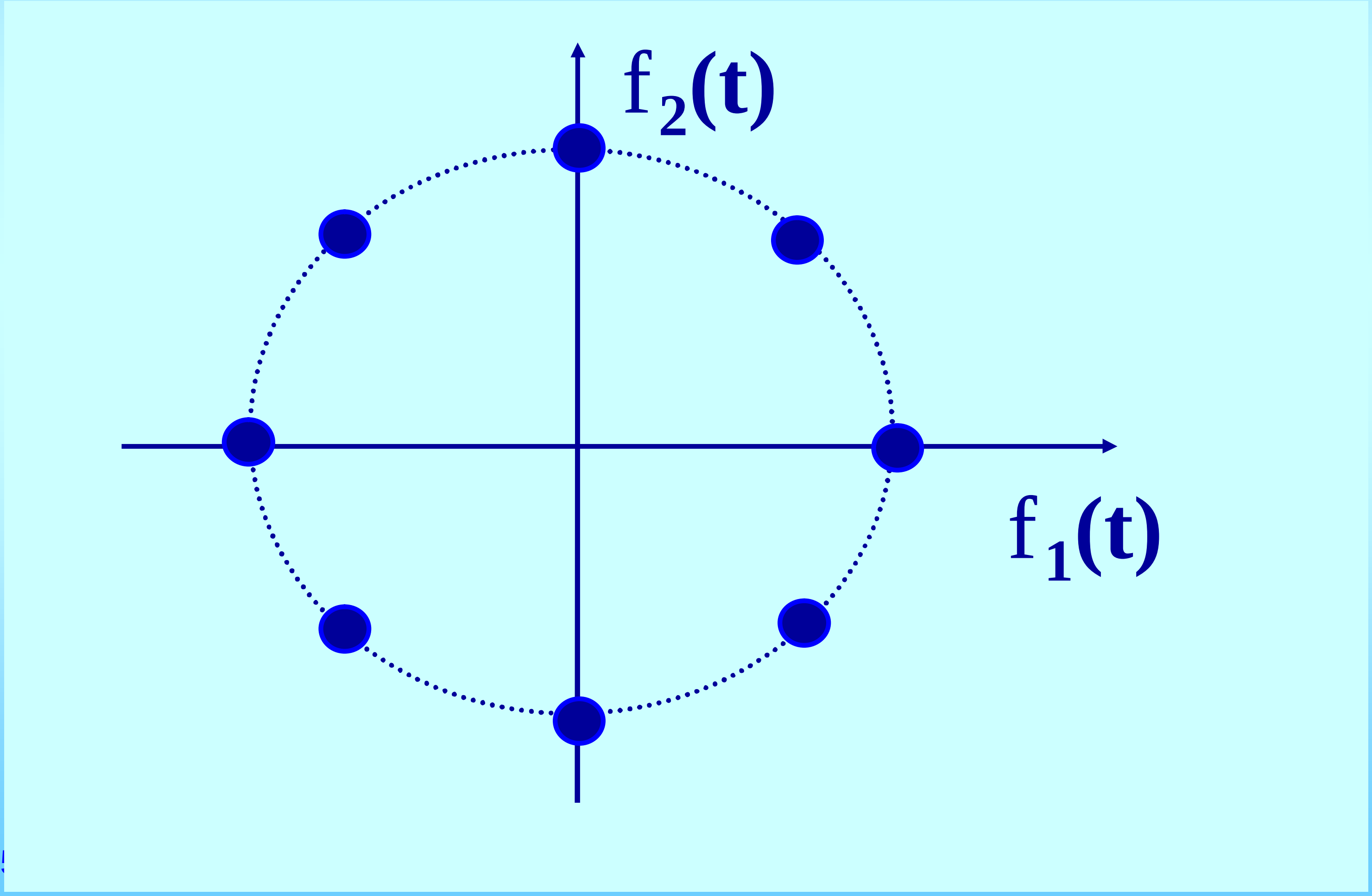
$$i = 1, 2, \mathbf{K}, \mathrm{M}$$

$$S_{\mathrm{M\text{-}PSK}}(t) = \left\{ \sqrt{E_s} \cos\left[(i-1)\frac{2\pi}{\mathrm{M}} \right] \varphi_1(t) - \sqrt{E_s} \sin\left[(i-1)\frac{2\pi}{\mathrm{M}} \right] \varphi_2(t) \right\}$$

$$i = 1, 2, \mathbf{K}, \mathrm{M}$$

M-ary PSK

- Constellation diagram of 8PSK



Quadrature amplitude modulation

- E_{min} : energy of the signal with the lowest amplitude
- a_i, b_i, index of the signal point

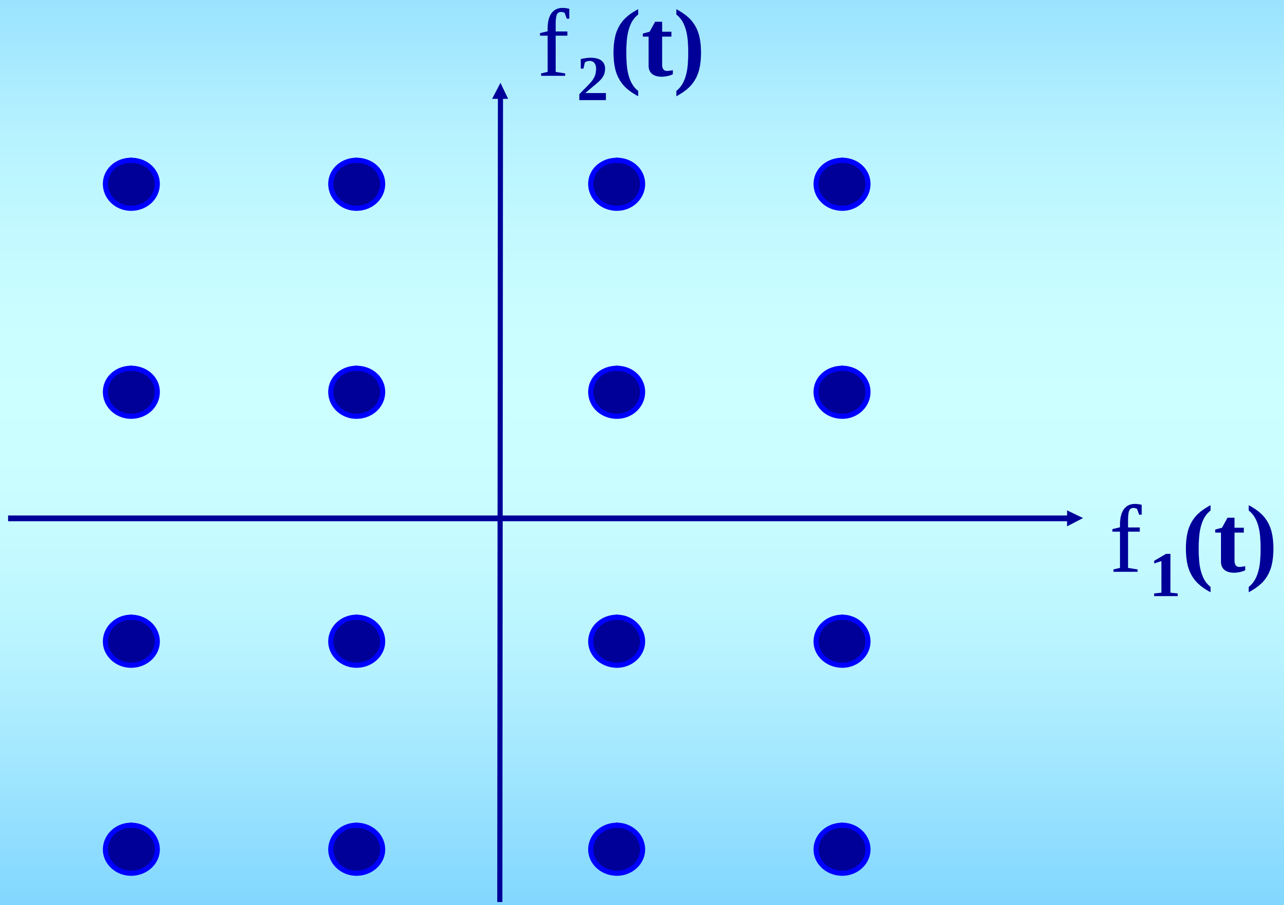
$$S_i(t) = \sqrt{\frac{2E_{min}}{T_s}}a_i\cos(2pf_c t) - \sqrt{\frac{2E_{min}}{T_s}}b_i\sin(2pf_c t)$$

$$0 \le t \le T \qquad i = 1, 2, \mathbf{K}, M$$

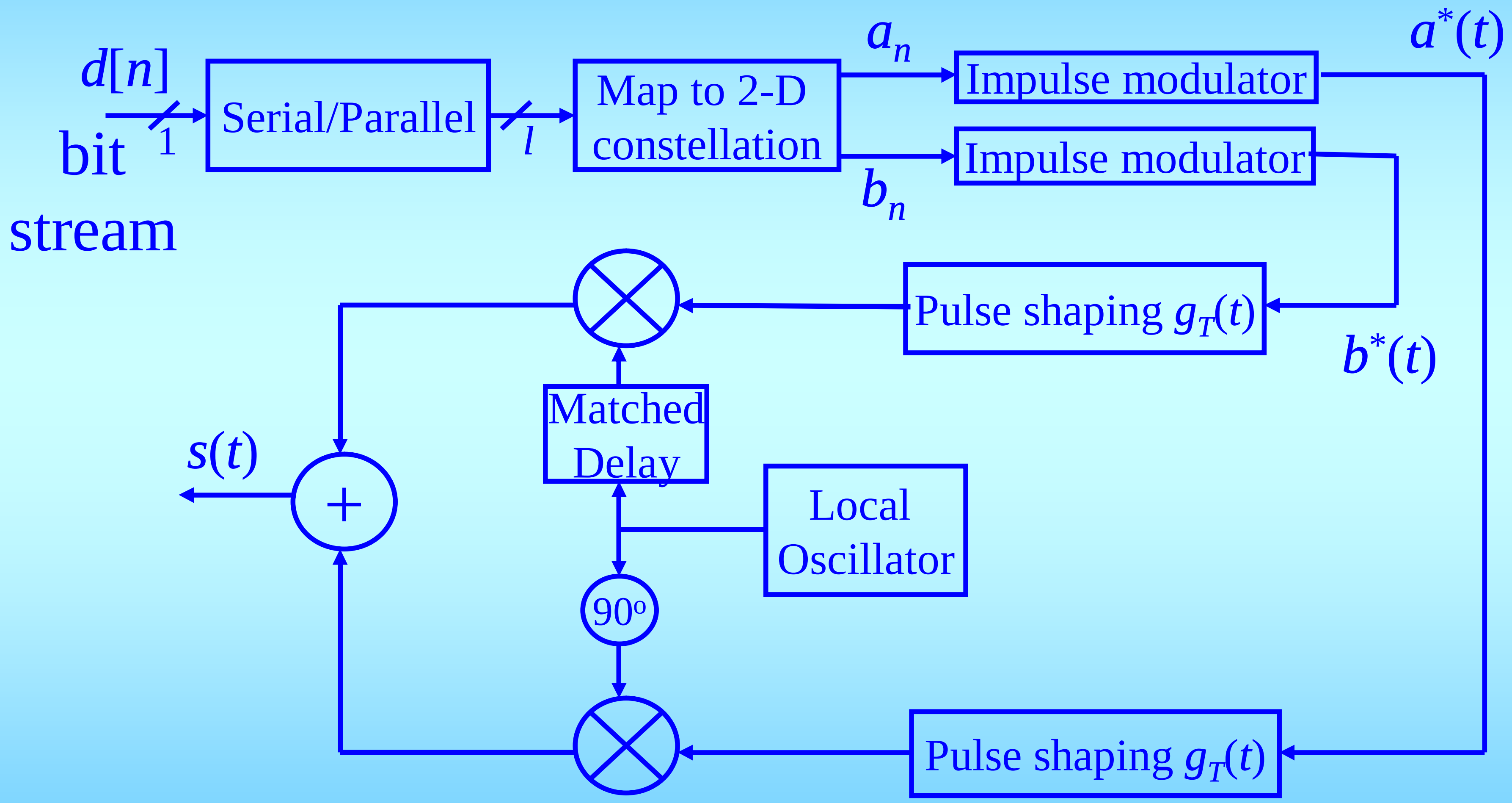
$$f_1(t) = \sqrt{\frac{2}{T_s}}\cos(2pf_c t) \quad 0 \le t \le T_s$$

$$f_2(t) = \sqrt{\frac{2}{T_s}}\sin(2pf_c t) \quad 0 \le t \le T_s$$

Constellation diagram of 16QAM



Digital QAM Modulator



Matched delay matches delay through 90^o phase shifter

Phase Shift by 90 Degrees

- 90º phase shift performed by Hilbert transformer

  cosine => sine

  $$\cos(2p\ f_0\ t) \Rightarrow \frac{1}{2} d(f + f_0) + \frac{1}{2} d(f - f_0)$$

  $$\sin(2p\ f_0\ t) \Rightarrow \frac{j}{2} d(f + f_0) - \frac{j}{2} d(f - f_0)$$

  sine => – cosine

- Frequency response of ideal Hilbert transformer:

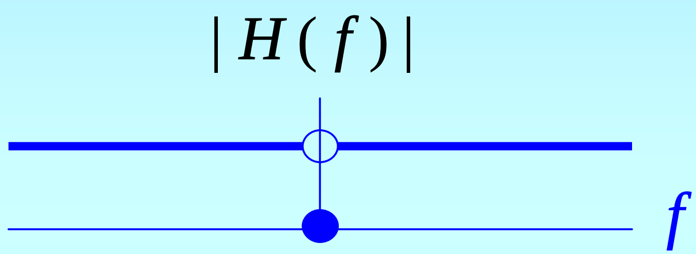
$$H(f) = -j\,\mathrm{sgn}(f)$$

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Hilbert Transformer

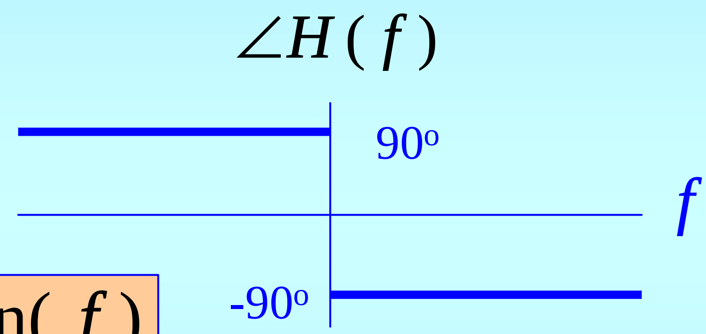
- Magnitude response

  All pass except at origin



- Phase response

  Piecewise constant



$$H(f) = -j\,\mathrm{sgn}(f)$$

- For $f_c > 0$

$$\cos(2pf_c t + \frac{p}{2}) = \sin(2pf_c t)$$

- For $f_c < 0$

$$\cos(2pf_c t - \frac{p}{2}) = \cos(-(2pf_c t + \frac{p}{2}))$$

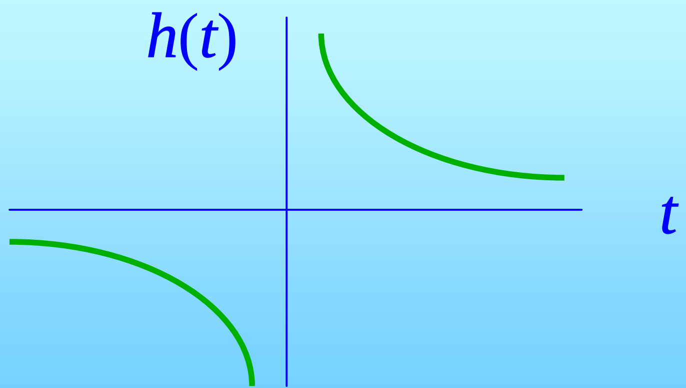
$$= \cos(2p(-f_c)t + \frac{p}{2}) = \sin(2p(-f_c)t)$$

Hilbert Transformer

- Continuous-time ideal Hilbert transformer

$$H(f) = -j\,\mathrm{sgn}(f)$$

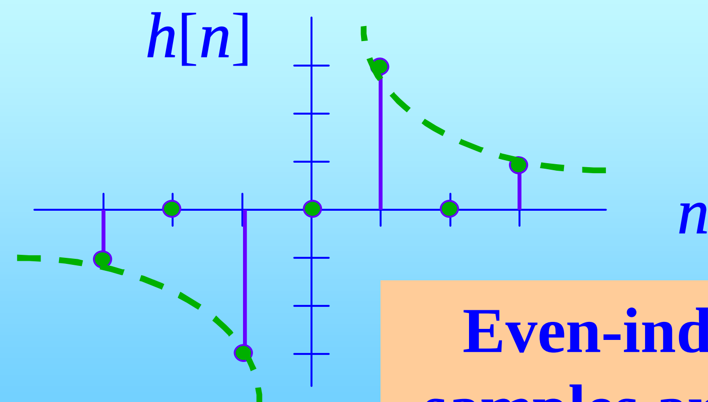
$$h(t) = \begin{cases} 1/(\pi t) & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$



- Discrete-time ideal Hilbert transformer

$$H(w) = -j\,\mathrm{sgn}(w)$$

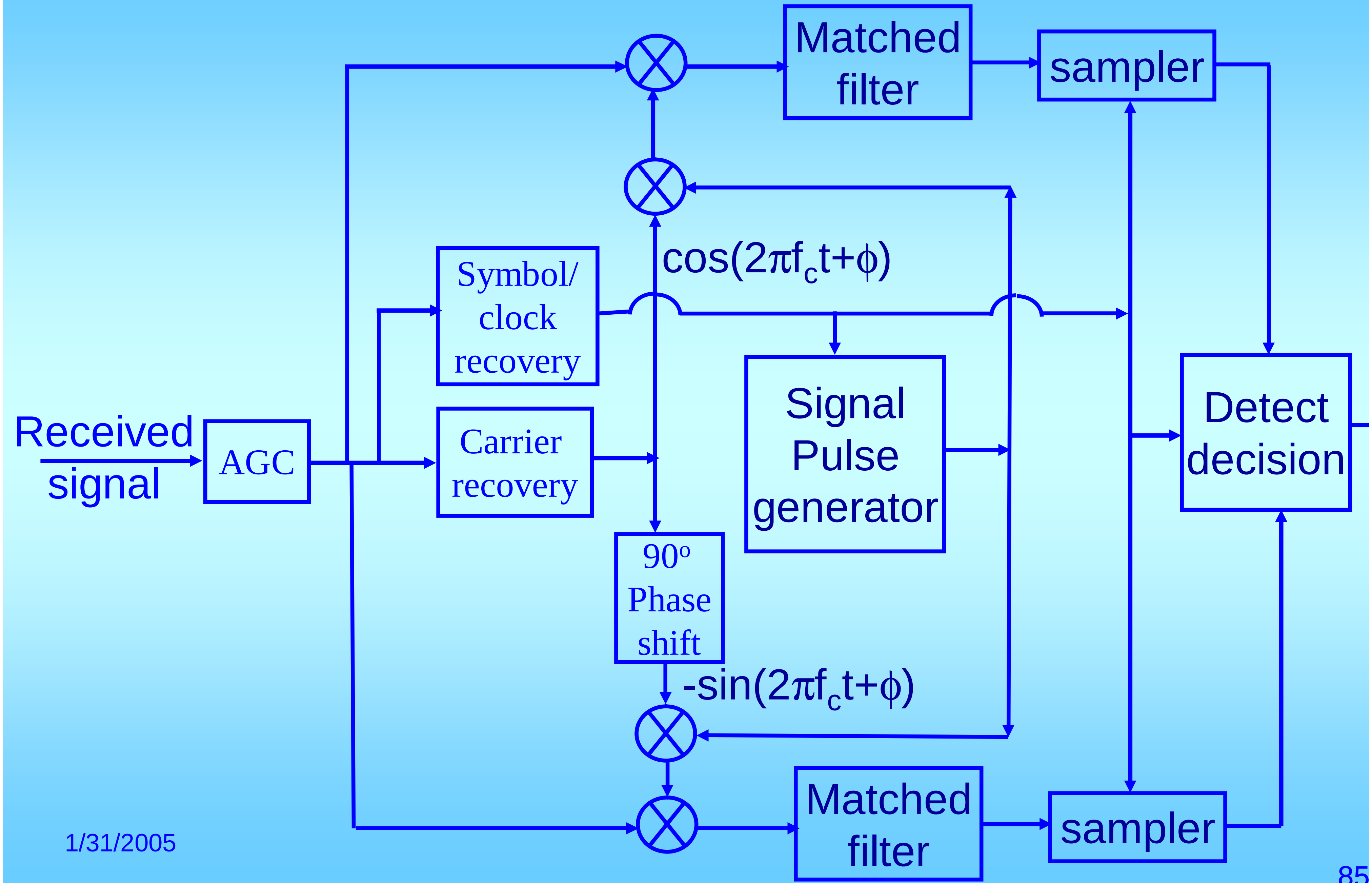
$$h[n] = \begin{cases} \frac{2}{p}\frac{\sin^2(pn/2)}{n} & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases}$$



Even-indexed samples are zero

QAM Receiver

- Channel has linear distortion, additive noise, and nonlinear distortion
- Adaptive digital FIR filter used to equalize linear distortion (magnitude/phase distortion in channel)
  - Channel equalizer coefficients adapted during startup
  - At startup, transmitter sends known PN training sequence


Matched filter
sampler
cos(2πf_c t+φ)
Symbol/ clock recovery
Received signal
AGC
Carrier recovery
Signal Pulse generator
Detect decision
90º Phase shift
-sin(2πf_c t+φ)
Matched filter
sampler

In-Phase/Quadrature Demodulation

- QAM transmit signal $x(t) = a(t)\cos(w_c\, t) + b(t)\sin(w_c\, t)$
- QAM demodulation by modulation then filtering
  - Construct in-phase *i*(*t*) and quadrature *q*(*t*) signals
  - Lowpass filter them to obtain baseband signals *a*(*t*) and *b*(*t*)

$$i(t) = 2x(t)\cos(w_c t) = 2a(t)\cos^2(w_c t) + 2b(t)\sin(w_c t)\cos(w_c t)$$
$$= \underbrace{a(t)}_{\text{baseband}} + \underbrace{a(t)\cos(2w_c t) + b(t)\sin(2w_c t)}_{\text{high frequency component centered at } 2\,w_c}$$

$$q(t) = 2x(t)\sin(w_c t) = 2a(t)\cos(w_c t)\sin(w_c t) + 2b(t)\sin^2(w_c t)$$
$$= \underbrace{b(t)}_{\text{baseband}} + \underbrace{a(t)\sin(2w_c t) - b(t)\cos(2w_c t)}_{\text{high frequency component centered at } 2\,w_c}$$

Performance Analysis of QAM

- Received QAM signal

$$x(nT) = s(nT) + v(nT)$$

- Information signal $s(nT)$

$$s(nT) = a_n + j\,b_n = (2i-1)d + j\,(2k-1)d$$

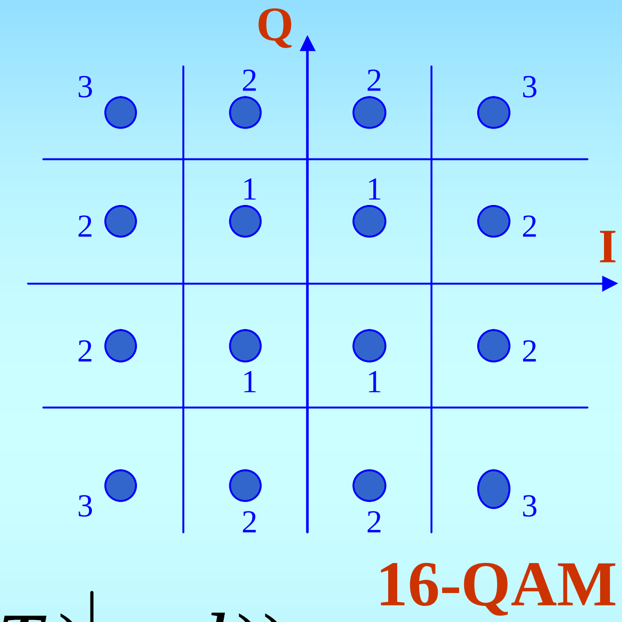
where $i,k \in \{ -1, 0, 1, 2 \}$ for 16-QAM

- Noise, $v_I(nT)$ and $v_Q(nT)$ are independent Gaussian random variables ~ $N(0; \sigma^2/T)$

$$v(nT) = v_I(nT) + j\,v_Q(nT)$$

Performance Analysis of QAM

- Type 1 correct detection



$$P_1(c) = P(|v_I(nT)| < d \ \& \ |v_Q(nT)| < d)$$

$$= P(|v_I(nT)| < d)P(|v_Q(nT)| < d)$$

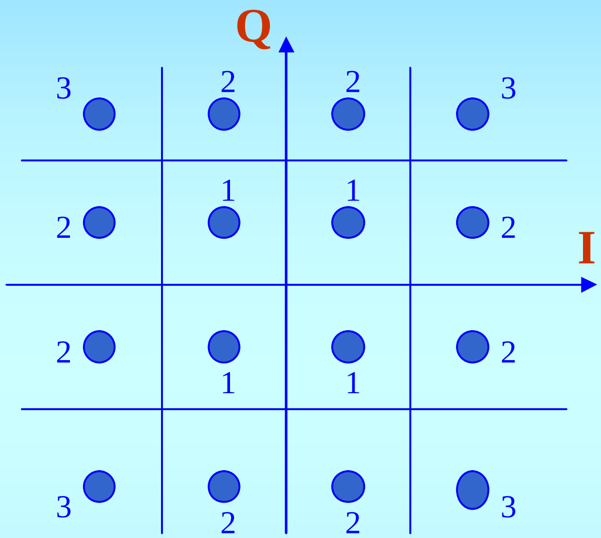
$$= (1 - \underbrace{P(|v_I(nT)| > d)}_{2Q(\frac{d}{s}\sqrt{T})})(1 - \underbrace{P(|v_Q(nT)| > d)}_{2Q(\frac{d}{s}\sqrt{T})})$$

$$= (1 - 2Q(\frac{d}{s}\sqrt{T}))^2$$

Performance Analysis of QAM

- Type 2 correct detection

$$P_2(c) = P(v_I(nT) < d \;\&\; |v_Q(nT)| < d)$$
$$= P(v_I(nT) < d)P(|v_Q(nT)| < d)$$
$$= (1 - 2Q(\frac{d}{s}\sqrt{T}))(1 - Q(\frac{d}{s}\sqrt{T}))$$



16-QAM

- Type 3 correct detection

$$P_3(c) = P(v_I(nT) < d \;\&\; v_Q(nT) > -d)$$
$$= P(v_I(nT) < d)P(v_Q(nT) > -d)$$
$$= (1 - Q(\frac{d}{s}\sqrt{T}))^2$$

Performance Analysis of QAM

- Probability of correct detection

$$P(c) = \frac{4}{16}(1 - 2Q(\frac{d}{s}\sqrt{T}))^2 + \frac{4}{16}(1 - Q(\frac{d}{s}\sqrt{T}))^2$$

$$+ \frac{8}{16}(1 - 2Q(\frac{d}{s}\sqrt{T}))(1 - Q(\frac{d}{s}\sqrt{T}))$$

$$= 1 - 3Q(\frac{d}{s}\sqrt{T}) + \frac{9}{4}Q^2(\frac{d}{s}\sqrt{T})$$

- Symbol error probability

$$P(e) = 1 - P(c) = 3Q(\frac{d}{s}\sqrt{T}) - \frac{9}{4}Q^2(\frac{d}{s}\sqrt{T})$$

Average Power Analysis

- PAM and QAM signals are deterministic
- For a deterministic signal $p(t)$, instantaneous power is $|p(t)|^2$
- 4-PAM constellation points: $\{ -3\,d,\ -d,\ d,\ 3\,d \}$
  - Total power $9\,d^2 + d^2 + d^2 + 9\,d^2 = 20\,d^2$
  - Average power per symbol $5\,d^2$
- 4-QAM constellation points: $\{ d + j\,d,\ -d + j\,d,\ d - j\,d,\ -d - j\,d \}$
  - Total power $2\,d^2 + 2\,d^2 + 2\,d^2 + 2\,d^2 = 8\,d^2$
  - Average power per symbol $2\,d^2$

Summary of QAM.

$$\{a_i, b_i\} = \begin{bmatrix} (-L+1, L-1) & (-L+3, L-1) & \cdots & (L-1, L-1) \\ (-L+1, L-3) & (-L+3, L-3) & \cdots & (L-1, L-3) \\ . & . & . & . \\ . & . & . & . \\ (-L+1, -L+1) & (-L+3, -L+1) & \cdots & (L-1, -L+1) \end{bmatrix}$$

$$\{a_i, b_i\} = \begin{bmatrix} (-3, 3) & (-1, 3) & (1, 3) & (3, 3) \\ (-3, 1) & (-1, 1) & (1, 1) & (3, 1) \\ (-3, -1) & (-1, -1) & (1, -1) & (3, -1) \\ (-3, -3) & (-1, -3) & (1, -3) & (3, -3) \end{bmatrix}$$

$$P_e \cong 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{2E_{min}}{N_0}}\right)$$

$$P_e \cong 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3E_{av}}{(M-1)N_0}}\right)$$

M-ary FSK

$$S_i(t) = \sqrt{\frac{2E_s}{T_s}} \cos\left[\frac{p}{T_s}(n_c + i)t\right] \quad 0 \le t \le T_s \quad i = 1, 2, \mathbf{K}, \mathrm{M}$$

– error probability under coherent detection

$$P_e \le (M-1)Q\left(\sqrt{\frac{2E_b \log_2 M}{N_0}}\right)$$

– error probability under non-coherent detection

$$P_e = \sum_{k=1}^{M-1}\left(\frac{(-1)^{k+1}}{k+1}\right)\binom{M-1}{k}\exp\left(\frac{-kE_s}{(k+1)N_0}\right)$$

M-ary FSK

- M_ary FSK
  - BW of coherent MFSK :

$$B = \frac{R_b(M+3)}{2\log_2 M}$$

  - BW of noncoherent MFSK :

$$B = \frac{R_b M}{2\log_2 M}$$