EE 230
Lecture 6

Linear Systems

• Poles/Zeros/Stability
• Stability
Quiz 5

A system has the transfer function \( T(s) \)
Determine the poles of the system.

\[
T(s) = \frac{4 + \frac{1}{s}}{7 + s + \frac{10}{s}}
\]
And the number is?
And the number is ?
Quiz 5
A system has the transfer function $T(s)$
Determine the poles of the system.

$$T(s) = \frac{4 + \frac{1}{s}}{7 + s + \frac{10}{s}}$$

Solution:

$$T(s) = \frac{4 + \frac{1}{s}}{7 + s + \frac{10}{s}} = \frac{4 \left( s + \frac{1}{4} \right)}{s^2 + 7s + 10}$$

$$T(s) = \frac{4 \left( s + \frac{1}{4} \right)}{s^2 + 7s + 10} = \frac{4 \left( s + \frac{1}{4} \right)}{(s+2)(s+5)}$$

Poles at $s = -2$ and $s = -5$
Many times interested in the step response of a linear system when the system is first-order.

For any first-order linear system, the unit step response is given by

\[ X_{OUT} = F + (I-F)e^{\frac{t-T_0}{t_c}} \]

I is the initial value, F is the final value and \( t_c \) is the time constant.
Review from Last Time

Step Response of First-Order Networks

\[ X_{\text{OUT}} = F + (1-F)e^{-\frac{t-T_0}{t_C}} \]

\[ \text{u}(t-T_0) \quad \text{u}(t-T_0) \]

Linear System

\[ X_{\text{OUT}} \]

\[ T_0 \quad T_0 + t_C \]

\[ F \quad I \]

\[ I \quad I \]

\[ T_0 \quad T_0 + t_C \]
Step Response of First-Order Networks

Example:

Obtain the step response of the circuit shown if the step is applied at time $T=1\text{msec}$ and prior to $V_{\text{OUT}}(t)=0$ for $t<1\text{msec}$. Assume $R=1\text{K}, C=0.1\mu\text{F}$

Solution:

$$T(s) = \frac{1}{1+RCs}$$

$$T(s) = \frac{1}{s+1/RC}$$

This is first order and of the form:

$T(s) = \frac{K}{s-p}$

$\therefore p = -\frac{1}{RC}$

$t_c = -p^{-1} = RC$

Thus, the output can be expressed as:

$$V_{\text{OUT}} = F + (I-F)e^{\frac{t-T_0}{t_c}}$$
Impedance and Conductance Notation

Circuit Analysis with Impedance Notation (Z) and Conductance Notation (G)

Ohms Law

\[ V = I \cdot Z \quad \text{and} \quad I = V \cdot G \]

KCL

\[ V_x (G_1 + G_2 + G_3 + \ldots + G_k) = V_1 G_1 + V_2 G_2 + V_3 G_3 + \ldots + V_k G_k \]

Formally:

\[ V_x \left( \sum_{i=1}^{k} G_i \right) = \sum_{i=1}^{k} V_i G_i \]

KCL is often the fastest way to analyze electronic circuits

Why?

Conductance notation is often much less cumbersome than impedance notation when analyzing electronic circuits

Why?
For any linear system, $T(s)$ can be expressed as

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i}$$

where $a_i$ and $b_i$ are all real, $b_n \neq 0$, $a_m \neq 0$, and $n \geq m$.

Can always make $b_n = 1$.

Numerator often termed $N(s)$
Denominator often termed $D(s)$

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i} = \frac{N(s)}{D(s)}$$

Definition: The roots of $D(s)$ are the poles of $T(s)$ and the roots of $N(s)$ are the zeros of $T(s)$.

The poles of $T(s)$ are often termed the poles of the system.
Claim: A system with a 1st order lowpass transfer function with a pole $p$ and a dc gain $K$ has a unit step response of

$$X_{OUT} = K + (I-K)e^{p(t-T_0)}$$

where $I$ is the initial value of the output

$$T(s) = \frac{-Kp}{s-p}$$
Poles and Zeros of Linear Networks

\[ T(s) = \sum_{i=0}^{m} a_i s^i \]
\[ \sum_{i=0}^{n} b_i s^i \]

Alternate representation of transfer function

\[ T(s) = K \prod_{i=1}^{m} (s-z_i) \]
\[ \prod_{i=1}^{n} (s-p_i) \]

where \( K = \frac{a_m}{b_n} \), \( \{z_1, \ldots, z_m\} \) are the poles and \( \{p_1, \ldots, p_n\} \) are the zeros.
Response of Linear Networks

If \( X_{IN}(s) \) is the input (in the s-domain) to a linear system, then the output (in the s-domain) is given by

\[
X_{out}(s) = T(s)X_{IN}(s)
\]

\[
X_{OUT}(s) = K \frac{\prod_{i=1}^{m} (s-z_i)}{\prod_{i=1}^{n} (s-p_i)} X_{IN}(s)
\]

The Laplace transform of the excitation can be written as

\[
X_{IN}(s) = H \frac{\prod_{i=1}^{r} (s-\alpha_i)}{\prod_{i=1}^{q} (s-\beta_i)}
\]

\[
X_{OUT}(s) = KH \frac{\prod_{i=1}^{m} (s-z_i)}{\prod_{i=1}^{n} (s-p_i)} \frac{\prod_{i=1}^{r} (s-\alpha_i)}{\prod_{i=1}^{q} (s-\beta_i)}
\]
Response of Linear Networks

If $X_{IN}(s)$ is the input (in the s-domain) to a linear system, then the output (in the s-domain) is given by

$$X_{OUT}(s) = KH \frac{\prod_{j=1}^{m}(s-z_j)}{\prod_{i=1}^{n}(s-p_i)} \frac{\prod_{i=1}^{r}(s-\alpha_i)}{\prod_{i=1}^{q}(s-\beta_i)}$$

If the denominator terms are all unique, this can be expanded in a partial fraction as

$$X_{OUT}(s) = \left( \frac{\theta_1}{s-p_1} + \frac{\theta_2}{s-p_2} + ... + \frac{\theta_n}{s-p_n} \right) + \left( \frac{\gamma_1}{s-\beta_1} + \frac{\gamma_2}{s-\beta_2} + ... + \frac{\gamma_m}{s-\beta_m} \right)$$

The time-domain output can be obtained from the inverse Laplace transform as:

$$X_{OUT}^{-1}(s) = \left( \theta_1 e^{p_1 t} + \theta_2 e^{p_2 t} + ... + \theta_n e^{p_n t} \right) + \left( \gamma_1 e^{\beta_1 t} + \gamma_2 e^{\beta_2 t} + ... + \gamma_m e^{\beta_m t} \right)$$

The term in the left parenthesis is the natural response due to the network.
The term in the right parenthesis is the forced response due to the excitation.
Response of Linear Networks

If $X_{\text{IN}}(s)$ is the input (in the s-domain) to a linear system, then the output (in the s-domain) is given by

$$X_{\text{OUT}}(s) = KH \frac{\prod_{j=1}^{m}(s-z_i)}{\prod_{i=1}^{r}(s-\alpha_i)} \frac{\prod_{i=1}^{n}(s-p_i)}{\prod_{i=1}^{q}(s-\beta_i)}$$

$$X^{-1}_{\text{OUT}}(s) = \left( \theta_1 e^{p_1 t} + \theta_2 e^{p_2 t} + \ldots + \theta_n e^{p_n t} \right) + \left( \gamma_1 e^{\beta_1 t} + \gamma_2 e^{\beta_2 t} + \ldots + \gamma_m e^{\beta_m t} \right)$$

If the real part of the poles are negative, the natural response will decay out but if the real part of the poles are positive, it will grow without bound!

The forced response will have properties that are similar to those of the excitation.
Poles and Zeros of Linear Networks

\[ T(s) = K \frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)} \]

A plot of the poles and zeros in the complex plane is often used to visually show their location.

Example: \[ T(s) = \frac{s-2}{(s+2)(s+4)} \]
Poles and Zeros of Linear Networks

\[ T(s) = K \frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)} \]

A plot of the poles and zeros in the complex plane is often used to visually show their location.

Example: \[ T(s) = \frac{10(s-4)(s+2)}{(s+6+j5)(s+6-j5)(s+8)} \]
Poles and Zeros of Linear Networks

\[ T(s) = K \prod_{i=1}^{m}(s-Z_i) \prod_{i=1}^{n}(s-p_i) \]

A plot of the poles and zeros in the complex plane is often used to visually show their location.

Example: \[ T(s) = \frac{4s+1}{s^5 + s^4 + 2s^3 + 3s^2 + 2s + 1} \]

Zero as \( s = -1/4 \)

Poles?

Numerical techniques must be used, closed form expression does not exist for polynomials of order 5 or higher.
The Complex Plane

- Left Half Plane
- Right Half Plane

Re

Im
Theorem: Any network comprised of resistors, capacitors, and inductors will have all poles in the Left Half of the s-plane.

Note: This theorem is not true in the more general case where the circuit may contain amplifiers or dependent sources.
Poles and Zeros of Linear Networks

Theorem: A system is stable iff all poles lie in the left half-plane

If a system is not stable, it is said to be unstable

What does “stable” mean?

A linear system is stable iff any bounded input will result in a bounded output

A system is stable iff the output due to any appropriately small input does not cause the output to diverge to \( \pm \infty \) and does not create a time varying output that persists indefinitely
An unstable system will have one or more poles outside of the left half-plane.

If a linear system is unstable, practically one of three things will happen:

a) A periodic output of constant amplitude will persist for ever.

b) A time-varying output will grow in amplitude with time.

c) The output will grow without bound towards $+\infty$ or $-\infty$.

Practically, if a system is unstable, as amplitudes grow nonlinearities will be introduced and these nonlinearities will limit the growth of the output.
Stability of Linear Systems

Example: Determine if the following systems are stable

\[ T(s) = \frac{1}{s+1} \]  
\[ \text{single pole at } p = -1 \]  
\[ \therefore \text{system is stable} \]

Step response (at \( t=0 \)) if \( I=0 \)

Since \( I=0 \) and \( F=1 \)
\[ X_{OUT} = F + (I-F)e^{pt} \]
\[ X_{OUT} = 1-e^{-t} \]

\[ T(s) = \frac{1}{s-1} \]  
\[ \text{single pole at } p = 1 \]  
\[ \therefore \text{system is not stable} \]

Can be shown that step response (at \( t=0 \)) if \( I=0 \) is
\[ X_{OUT} = 1-e^t \]

This diverges to \(-\infty\) as \( t \) increases!
Stability of Linear Systems

Example: Determine if the following systems are stable

\[ T(s) = \frac{4}{s^2 + 9} \]  
poles at \( p = -3j \) and \( p = +3j \)  
\[ \therefore \text{ system is not stable} \]

It can be shown that the step response will include a term

\[ r(t) = H \sin 3t \]

This is a time-varying input that persists indefinitely!
Stability of Linear Systems

Example: Determine if the following systems are stable

\[ T(s) = \frac{4}{s^2 + 0.1s + 1} \]

poles at \( p = -0.05 + 0.9988j \) and \( p = -0.05 - 0.9988j \)

\[ \therefore \text{system is stable} \]

But since poles are so close to imaginary axis, natural response may ring for a while.
Stability of Linear Systems

Is stability good or bad?

Some unstable systems

Tacoma Narrows Bridge

Chernobyl
Stability of Linear Systems

Is stability good or bad?

Some more unstable systems

Flagship product that got Hewlett Packard started
Stability of Linear Systems

Is stability good or bad?
Some more unstable systems

B1 Bomber (Stealth)

Taipei 101 (hopefully stable)
Stability of Linear Systems

Is stability good or bad?

It depends upon what is desired

Many times instability is very undesirable

But often instability is very desirable as well

But regardless, it is almost always necessary to know whether a system is stable or unstable!
Amplifiers

An ideal amplifier is linear and has a frequency independent transfer function that does not change with source or load impedance.

Ideally, \( X_{OUT} = KX_{IN} \)

\( K \) is termed the amplifier gain.

\( K = T(s) \)

Often \( K > 1 \) (when \( X_{IN} \) and \( X_{OUT} \) of same dimensions).
## Types of Amplifiers

Assuming input and output variables from \{ I, V \}

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
<th>Type</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td>V</td>
<td>Voltage</td>
<td>Dimensionless</td>
</tr>
<tr>
<td>I</td>
<td>I</td>
<td>Current</td>
<td>Dimensionless</td>
</tr>
<tr>
<td>V</td>
<td>I</td>
<td>Transconductance</td>
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Why are there so many types of amplifiers?

- Because we can have them?
- Some transducers have output variables different than what is needed at output

Sometimes performance can be optimized by using a particular amplifier type

Will show later that if amplifiers are not ideal, they all have the same model and differ only in the parameters in the model.
End of Lecture 6