Transfer Functions and Transfer Characteristics

This document was prepared as review material for students in EE 230

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Last Updates: Jan 16, 2010

Electronic circuits and electronic systems are designed to perform a wide variety of tasks. The performance requirements from task to task are often significantly different. Although the performance requirements vary widely, there are considerable benefits from both design and assessment viewpoints of having standard methods for characterizing the performance of these systems. The concepts of transfer characteristics and transfer functions are used extensively to characterize these circuits and systems.

An electronic system, an electronic circuit, or a more general system that may have some or no electrical relationships are often characterized by the transfer functions and the transfer characteristics as well and the concepts are universal. And within the context of electronic circuits and systems, terminology such as circuit, network, amplifier, filter, system, architecture, structure, along with several other terms are used interchangeably. The difference between an amplifier, a circuit, a filter, a network, an architecture, or even a system is often only in how the "entity" is intended to be used with possible no difference in how the "entity" is characterized or in how it operates. In what follows, these terms will be used interchangeably but the concepts of transfer functions and transfer characteristics are universal.

Transfer Characteristics

A system with an input X_{IN} and an output X_{OUT} is shown in Fig. 1. In electrical systems the input and output quantities, typically termed signals, are often dependent on a single additional input variable, time. In this case the input and output signals would time-dependent voltages or currents. In Fig. 2, three different methods of indicating a system that serves as a voltage amplifier with time-dependent input and output voltages is shown. For simplicity, the notation in part c) of this figure is most commonly used.

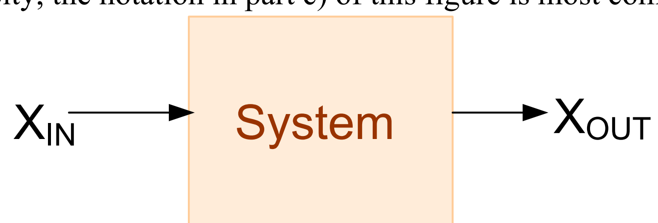

Fig. 1 Single-Input Single-Output System

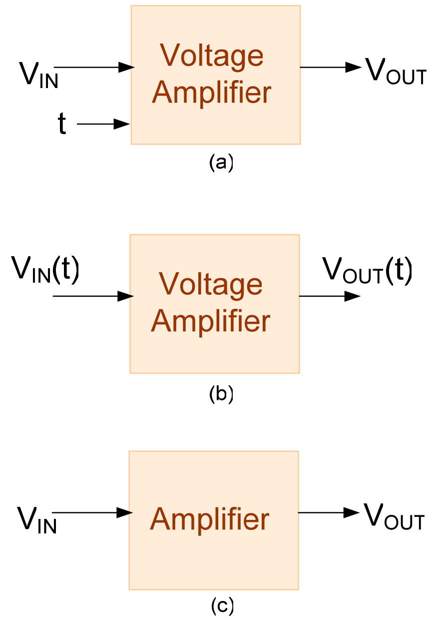

Fig. 2 Representations of a Voltage Amplifier

The transfer characteristics of a system is defined to be the pseudo-static relationship between the input and output variable. If the system is a voltage-in, voltage-out system we would term this pseudo-static relationship the dc transfer characteristics. Three different transfer characteristics are shown in Fig. 3. In top part of the figure, the relationship between the input and output is a straight line. In the middle part, it is weakly nonlinear. And, in the bottom part it is highly nonlinear.

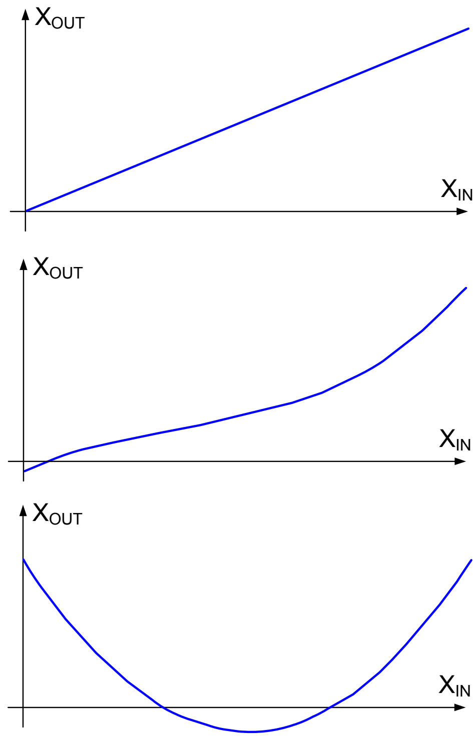

Fig. 3 Transfer characteristics of linear, weakly nonlinear, and highly nonlinear systems

Linear Systems

If the relationship between the input and output can be expressed as

$$X_{OUT} = KX_{IN} \qquad (1)$$

where K is a constant, we say the system is linear. Actually, the class of linear systems extends somewhat beyond the case where K is constant. The following definition defines a linear system:

Definition: A system is linear iff

$$X_{OUT}\left(a_1X_{IN1}+a_2X_{IN2}\right) = X_{OUT}\left(a_1X_{IN1}\right)+X_{OUT}\left(a_2X_{IN2}\right) \qquad \textbf{(2)}$$

where a_1 and a_2 are any real numbers and X_{IN1} and X_{IN2} are any two inputs.

With this definition of a linear system, the relationship of (1) is satisfied. But the I/O characteristics of many linear systems do not satisfy (1). For example, if sinusoidal inputs at two different frequencies are applied to a linear system, the outputs due to each will also be a sinusoid but the relative magnitude of the input and output sinusoidal signals may be different. If a system is linear, the relationship of (1) is satisfied for dc inputs. But, it is possible to have a system that is nonlinear where (1) is satisfied for dc inputs.

Transfer Functions

Any linear system is characterized by a transfer function. A linear system also has transfer characteristics. But, if a system is not linear, the system does not have a transfer function. The following definition will be used to define a transfer function.

Definition: The transfer function, T(s), of a linear system with input X_{IN} and output X_{OUT} is given by the expression

$$T(s)=\frac{\mathcal{X}_{OUT}(s)}{\mathcal{X}_{IN}(s)} \qquad (3)$$

where $\mathcal{X}_{OUT}(s)$ and $\mathcal{X}_{IN}(s)$ are the Laplace transforms of the output and input.

Although the input signal and correspondingly the output signal can take on arbitrary values, the ratio of the Laplace Transforms of these two quantities is not dependent upon the particular input that is applied provided the system is linear.

The input/output relationship defined by (3) is often termed a frequency domain characterization of the circuit The relationship between the actual input/output and the frequency domain input/output is depicted in Fig. 4.

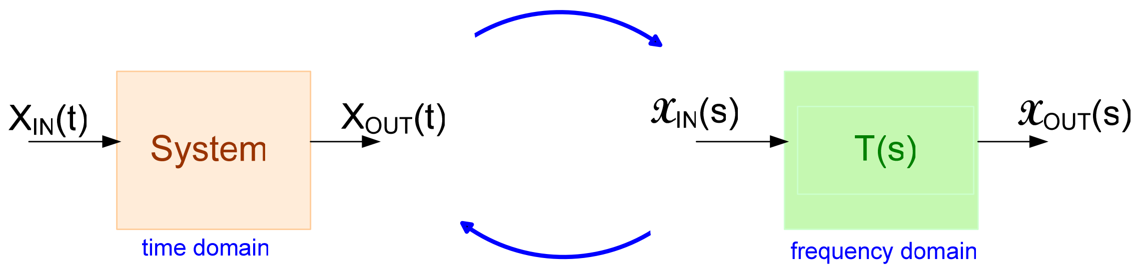

Fig. 4 Time-domain and frequency-domain transformations

The frequency domain characterization of a linear system and correspondingly the transfer function is of particular use in determining the sinusoidal steady state response of the network. A key theorem, and one of the major reasons that the frequency domain was studied in EE 201, follows.

Theorem 1: If a linear network has transfer function T(s) and input given by the expression $X_{IN}(t)=X_M\sin(\omega t+\theta)$, then the steady state output is given by

$$X_{OUT}(t)=X_M\left|T(j\omega)\right|\sin\left(\omega t+\theta+\angle T(j\omega)\right) \qquad (4)$$

This theorem states the steady state output is a sinusoid of the same frequency as the excitation but scaled in magnitude by the magnitude of the transfer function evaluated at s=jω and shifted in phase by the phase of the transfer function evaluated at s=jω.

Although this theorem is useful, there are still some challenges that need to be overcome to simplify the sinusoidal steady state analysis of linear networks. The challenge is in obtaining the transfer function T(s). The straightforward way to obtain T(s) from (3) is to write a set of differential equations relating the input and output variables of a circuit and then take the Laplace Transform of this set of equations to obtain a set of transformed equations. These equations become algebraic and can be solved to obtain T(s). But this process is still very tedious. A more practical approach is to transform the time domain circuit itself to a frequency domain circuit and then analyze the frequency domain circuit. This circumvents the need for writing any differential equations.

Two methods of obtaining the transfer function will be described here. One is a transformation of the circuit into what is often termed an s-domain or "Laplace domain" circuit. The other uses phasors. The methods are almost identical but circuits and electronics textbooks typically treat these as different approaches so both will be described.

s-domain circuit analysis

The concept of transforming the time-domain circuit to the s-domain circuit is depicted in Fig. 5. The s-Domain Circuit is topologically identical to the

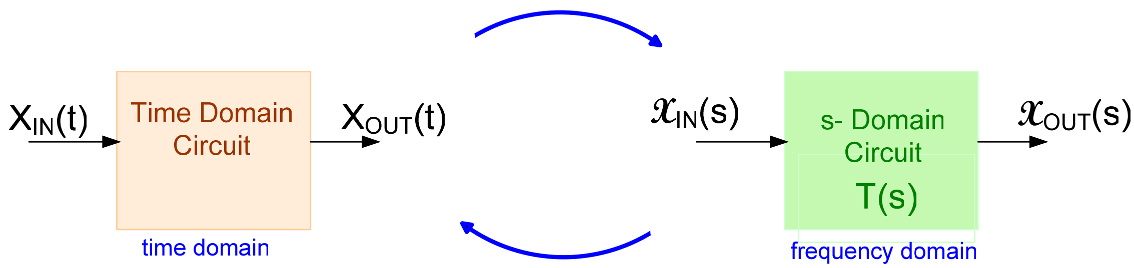

Fig. 5 Transformation from time-domain circuit to s-domain circuit

time-domain circuit, the only difference is how the circuit elements are modeled. All elements in the two circuits are identical except for the capacitors and inductors. The inductors and capacitors are simply replaced with impedances given in Fig 6.

Fig.6 Impedances of Inductors and Capacitors in s-domain network

Mathematically, these mappings of an element to an impedance can be expressed as

$$C \rightarrow \frac{1}{sC} \quad (5)$$

$$L \rightarrow sL \quad (6)$$

After the s-domain circuit is obtained, it can be analyzed using standard circuit analysis techniques to obtain the transfer function T(s). Once T(s) is obtained, Theorem 1 can be used to obtain the sinusoidal steady state response.

Phasor-domain circuit analysis

The concept of transforming a circuit to the phasor-domain is shown in Fig. 7. .

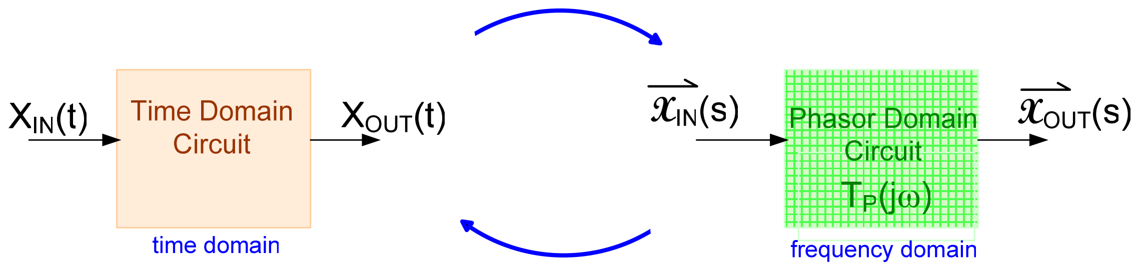

Fig. 7 Transformation from time-domain circuit to phasor-domain circuit

The phasor-domain circuit is also topologically identical to the time-domain circuit, the only difference being in how the circuit elements are modeled. All elements are the same except for the inductors and capacitors. The inductors and capacitors are simply replaced with impedances given in Fig.8.

Fig.8 Impedances of Inductors and Capacitors in phasor-domain network

Mathematically, these mappings of an element to an impedance can be expressed as

$$C \rightarrow \frac{1}{j\omega C} \qquad (7)$$

$$L \rightarrow j\omega L \qquad (8)$$

After the phasor-domain circuit is obtained, it can be analyzed using standard circuit analysis techniques to obtain the phasor-domain transfer function $T_P(j\omega)$. Whereas the transfer function T(s) includes the variable s and thus is neither a real or complex quantity until further information about s is given, the phasor-domain transfer function is a complex quantity as are all impedances in the phasor-domain circuit. As such, manipulations of equations leading to $T_P(j\omega)$ can be made as part of the analysis process. Unfortunately this additional flexibility, when exercised, often causes unnecessary arithmetic calculations when calculating $T_P(j\omega)$.

Since the phasor-domain circuit and the s-domain circuits differ only in how the energy storage elements are characterized and since this characterization is similar, it follows that

$$T(j\omega) = T(s)\Big|_{s=j\omega} = T_P(j\omega) \qquad (9)$$

Thus, the sinusoidal steady state response can also be obtained from the phasor-domain transfer function $T_P(j\omega)$. From (), it is apparent that $T_P(j\omega)$ can be readily obtained from T(s). Obtaining T(s) from $T_P(j\omega)$ is not so easy to do but seldom would one want to do this anyway.

Relationship between time-domain, s-domain and phasor-domain analysis

The process of obtaining the sinusoidal steady-state response has been discussed using the concept of the transfer function, Theorem 1, and either the s-domain circuit or the phasor-domain circuit. In this section, the relationship between these two methods of analysis will be contrasted with using the time domain analysis. The block diagram in Fig. depicts three methods for obtaining the sinusoidal steady state response of a linear system. The central path shows the time-domain approach. It involves writing the differential equations that characterize the actual circuit and solving these equations to obtain the response. The left path uses the s-domain approach and the right path uses the phasor-domain approach. The s-domain approach and the time-domain approach can be used even if the input is not a sinusoid though the analysis becomes more complicated. The phasor-domain approach is only applicable to when the input is a sinusoidal function and the steady-state response is desired.

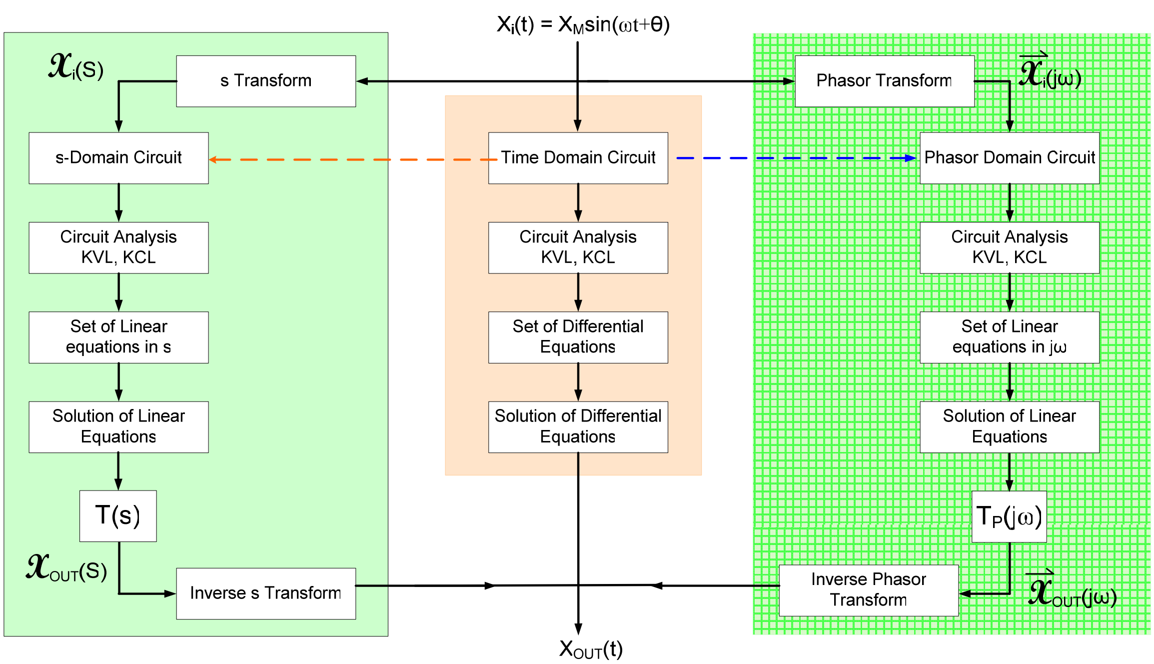

Fig.9 Time-domain, s-domain and phasor-domain analysis of linear circuits

It should be noted in the block diagram that the last blocks, the inverse transforms, were not discussed in the previous section. The inverse transforms needed to obtain the sinusoidal steady state response are provided by Theorem 1 and the inverse transforms are given by (4).

Example – The three approaches to sinusoidal steady state analysis

Consider the circuit if Fig. 10. We will obtain the sinusoidal steady state response using the three methods of analysis, the s-domain approach, the phasor-domain approach, and the time-domain approach. In all cases, it will be assumed that

$$V_{IN} = V_M \sin(\omega t+\theta) \tag{10}$$

and the goal is to obtain the steady state response $V_{OUT}(t)$.

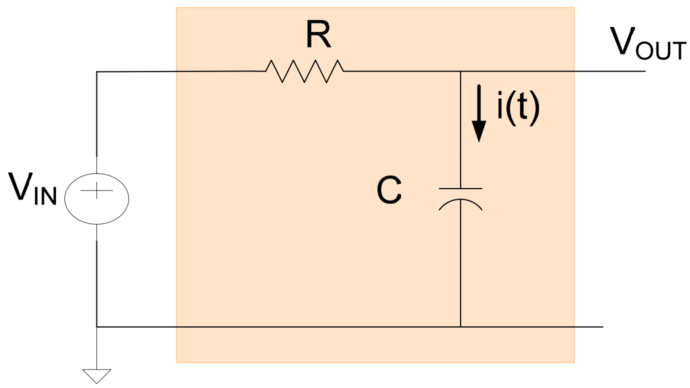

Fig. 10 Example time-domain circuit

s-domain analysis

From Fig.6 the s-domain equivalent of the circuit of Fig.10 can be obtained. This is shown in Fig. 11.

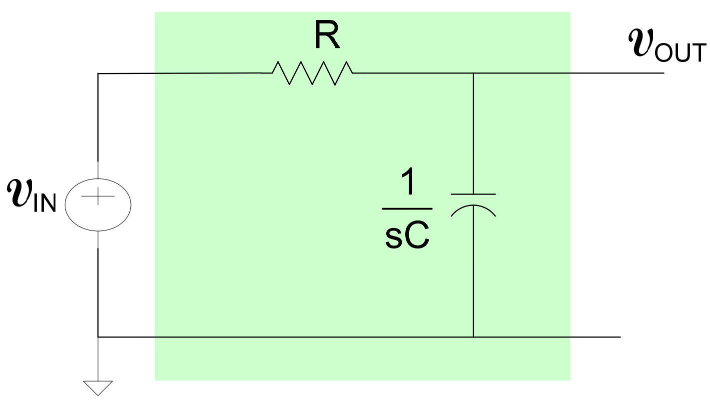

Fig. 11 s-domain circhit

It follows from KCL by summing currents at the output node that

$$\mathcal{v}_{OUT}\left(sC+\frac{1}{R}\right) = \mathcal{v}_{IN}\left(\frac{1}{R}\right) \tag{11}$$

Solving for $\mathcal{v}_{OUT}$ in terms of $\mathcal{v}_{IN}$, we obtain the transfer function

$$T(s)=\frac{1}{1+RCs} \tag{12}$$

Evaluating at s= jω we obtain

$$T(j\omega)=\frac{1}{1+j\omega RC} \tag{13}$$

The magnitude and phase of T(jω) are given by

$$\left|T(j\omega)\right|=\frac{1}{\sqrt{1+(\omega RC)^2}} \qquad \angle T(j\omega)=-\tan^{-1}(\omega RC) \tag{14}$$

It thus follows from (10) and Theorem 1 that the steady state output is given by

$$V_{OUT}(t)=V_M\left(\frac{1}{\sqrt{1+(\omega RC)^2}}\right)\sin\left(\omega t+\theta-\tan^{-1}(\omega RC)\right) \tag{15}$$

phasor-domain analysis

From Fig.8, the phasor-domain equivalent of the circuit of Fig.10 can be obtained. This is shown in Fig. 12

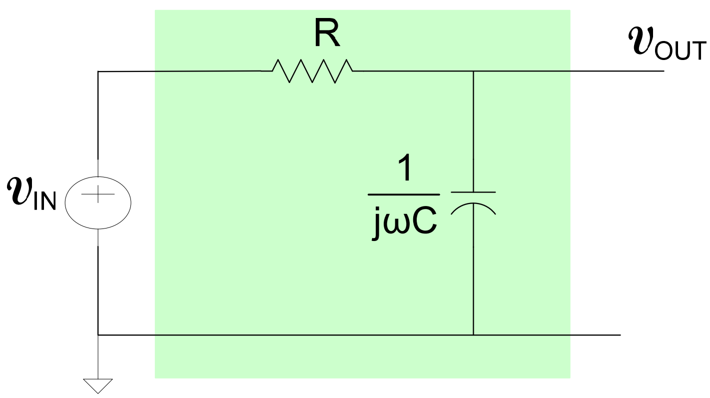

Fig.12 Phasor-domain circuit

It follows from KCL by summing currents at the output node that

$$\vec{v}_{OUT}\left(j\omega C+\frac{1}{R}\right)=\vec{v}_{IN}\left(\frac{1}{R}\right) \tag{16}$$

Solving for $\vec{v}_{OUT}$ in terms of $\vec{v}_{IN}$, we obtain the phasor-domain transfer function

$$T_P(j\omega)=\frac{1}{1+j\omega RC} \tag{17}$$

The magnitude and phase of $T_P(j\omega)$ are given by

$$\left|T_P(j\omega)\right|=\frac{1}{\sqrt{1+(\omega RC)^2}} \qquad \angle T_P(j\omega)=-\tan^{-1}(\omega RC) \tag{18}$$

It thus follows from (10), (9) and Theorem 1 that the steady state output is given by

$$V_{OUT}(t) = V_M\left(\frac{1}{\sqrt{1+(\omega RC)^2}}\right)\sin\left(\omega t + \theta - \tan^{-1}(\omega RC)\right) \tag{19}$$

As expected, it can be seen by comparing the results in (15) and (19) that the phasor-domain and s-domain analysis results are identical.

time-domain analysis

The time-domain analysis involves writing and solving the differential equations that describe the circuit. From Fig.8, we obtain the following three equations.

$$\left.\begin{aligned} i(t) &= C\frac{dV_{OUT}}{dt} \\ V_{IN}\text{-}V_{OUT} &= i(t)R \\ V_{IN} &= V_M\sin(\omega t+\theta) \end{aligned}\right\} \tag{20}$$

It remains to solve this set of three equations. One way to solve this simultaneous set of differential equations is to use Laplace transform methods. Taking the Laplace Transform of these three equations, we obtain the new set of three equations

$$\left.\begin{aligned} \mathcal{I} &= sC\mathcal{V}_{OUT} \\ \mathcal{V}_{IN}\text{-}\mathcal{V}_{OUT} &= \mathcal{I}R \\ \mathcal{V}_{IN} &= V_M\frac{(\sin\theta)s+\omega\cos\theta}{s^2+\omega^2} \end{aligned}\right\} \tag{21}$$

Solving this set of equations, we obtain the $\mathcal{V}_{OUT}$ as given by

$$\mathcal{V}_{OUT} = \left[V_M\frac{(\sin\theta)s+\omega\cos\theta}{s^2+\omega^2}\right]\left(\frac{1}{1+sRC}\right) \tag{22}$$

It remains to take the inverse Laplace transform of $\mathcal{V}_{OUT}$ to obtain $\tilde{V}_{OUT}(t)$ where the "~" operator is shown to indicate that the inverse Laplace transform will contain both a forced and natural response. The steady state response, $V_{OUT}(t)$, is the forced response. From a straightforward but tedious calculation, it follows that

$$\tilde{V}_{OUT}(t) = \left[V_M\frac{\sin\theta}{(RC)^2}\left(1\text{-}\frac{\omega RC}{\tan\theta}\right)e^{-t/RC}\right] + \left[V_M\left(\frac{1}{\sqrt{1+(\omega RC)^2}}\right)\sin\left(\omega t + \theta - \tan^{-1}(\omega RC)\right)\right] \tag{23}$$

The first term in [] on in this expression is the natural response and vanishes at $t \rightarrow \infty$. The forced response is the second term in []. It thus follows that the steady state response is given by

$$V_{OUT}(t) = V_M \left(\frac{1}{\sqrt{1+(\omega RC)^2}} \right) \sin\left(\omega t + \theta \ - \tan^{-1}(\omega RC)\right) \quad (24)$$

As expected, this is the same solution as was obtained from the s-domain analysis and the phasor-domain analysis.

Determining which analysis method is most useful

It should be apparent from this simple example that the total effort in obtaining the steady state response using either of the frequency domain approaches is much less than following the time domain approach and for more complicated circuits, the difference in effort will be even much larger. The issue of whether to use the s-domain or the phasor-domain analysis remains. The electronics community almost exclusively uses the s-domain analysis. There are probably several reasons for this but one of the most important, though not apparent from this simple example, is the amount of manipulative detail required with the two approaches. Since magnitude and phase operators are not defined in the s-domain analysis, one can not make intermediate "simplifications" whereas all equations present in a phasor-domain analysis are complex quantities and thus intermediate manipulations are possible and often implemented. Although this may appear to simplify the analysis, in actuality, unnecessary manipulations are often made and these intermediate manipulations thus often increase the effort required to solve the system of equations. A second is simply the number of characters that must be written when analyzing a circuit. It requires two characters to represent either an inductor or a capacitor in the s-domain and it requires three characters to represent these elements in the phasor domain. This actually results in considerably longer expressions when doing hand calculations in the phasor domain. A third reason also favors the s-domain analysis. By multiplying the transfer function T(s) by the Laplace transform of the excitation, the Laplace transform of the output can be obtained for arbitrary inputs and thus the s-domain approach can be used to obtain more than just the sinusoidal steady state response. In the latter case, there are some additional considerations that will not be discussed in detail here but they are associated with requiring the appropriate provisions for the initial conditions on all energy storage elements, specifically the inductors and capacitors. The s-domain analysis method, which is dominant in the electronics community, has been extracted from Fig. 9 and appears in Fig. 13.

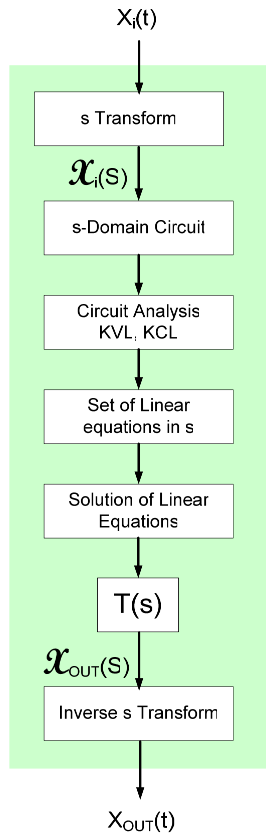

Fig. 13 s-domain approach to analysis of linear networks

Magnitude and Phase Plots and Amplifier/Filter Nomenclature

If a linear circuit has a sinusoidal excitation, it follows form Theorem that the magnitude and phase of the transfer function evaluated at the frequency of the excitation give the magnitude of the gain and the phase shift of the output relative to the input. Plots of the magnitude and phase are often used to visually show how gain and phase shift vary with frequency. These magnitude and phase plots are often termed Bode plots. From the magnitude and phase plots, the sinusoidal steady state response can be readily obtained for any frequency of the input.

The transfer function for the circuit of Fig. 10 was given in equation (12). The corresponding magnitude and phase for this transfer function are given by equation (14). The magnitude and phase are plotted in Fig.14. From this magnitude plot, it can be

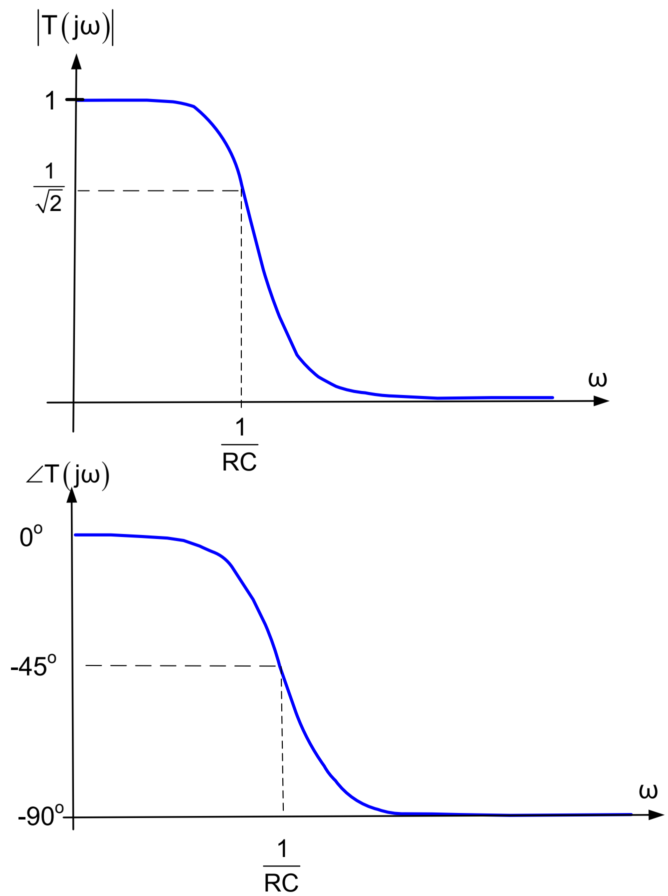

Fig. 14 Transfer function magnitude and phase plots for circuit of Fig. 10

seen that the magnitude of the gain is constant and nearly equal to 1 at low frequencies and drops at the frequency increases. The region where the gain is large and nearly constant is termed the pass band of the amplifier and the region where the gain is very low is termed the stop band of the filter. The transition between the passband and the stop band is not real abrupt in this circuit but it occurs around $\omega = (RC)^{-1}$. A circuit that passes sinusoids at low frequencies but blocks sinusoids at high frequency is termed a lowpass circuit. Some may term this a lowpass amplifier or a lowpass filter.

Several transfer function characteristics that vary intentionally with frequency that describe specific magnitude shapes are the lowpass, bandpass, highpass, band-reject, and notch functions. When circuits are designed to intentionally have these types of magnitude characteristics, the circuits are typically called filters. The transfer function magnitude for representative filters in each of these classes are shown in Fig. 15 . It should be noted that each of these classes of filters is large and that the magnitude responses shown in Fig.15 are simply representative characteristics. The phase response for these classical filter shapes is also of interest but is not shown in the figure.

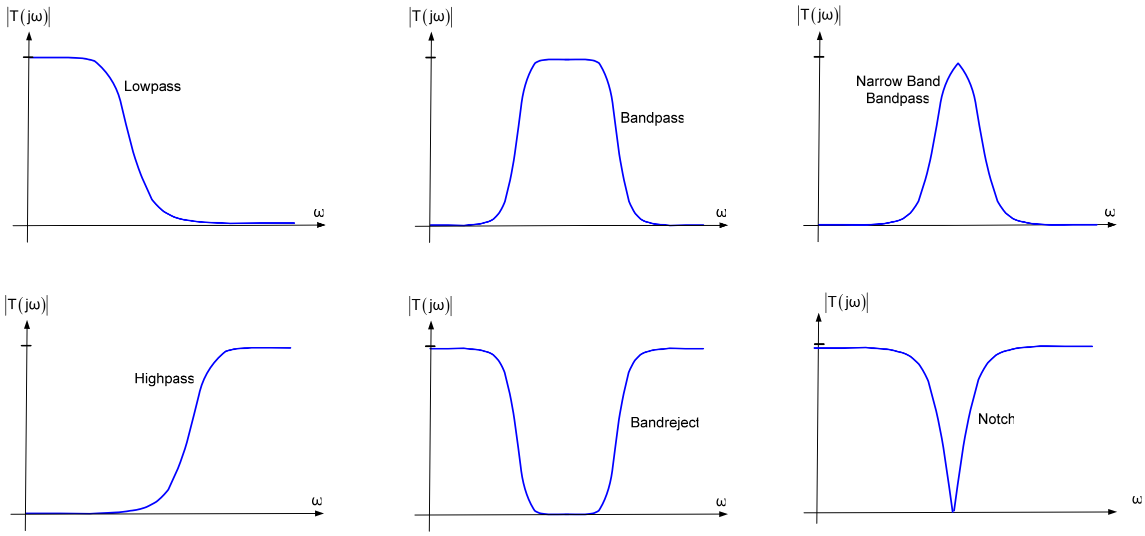

Fig. 15 Representative filter characteristics for lowpass, bandpass, highpass, bandreject and notch filters