EE 230
Lecture 34

Small Signal Models
Small Signal Analysis
Determine the small-signal Model for a MOSFET with $W=10\mu$, $L=1\mu$ if operating with a quiescent gate-source voltage of $3V$ and a quiescent drain-source voltage of $8V$. Assume $u_{COX}=100E^{-4}A/V^2$, $V_T=1V$, and $\lambda=0$. 

Quiz 34
And the number is ?

1 3 8

5

2

9

4

6

7
Determine the small-signal Model for a MOSFET with $W=10\mu$, $L=1\mu$ if operating with a quiescent gate-source voltage of $3\text{V}$ and a quiescent drain-source voltage of $8\text{V}$. Assume $uC_{\text{OX}}=100\text{E-4A/V}^2$, $V_T=1\text{V}$, and $\lambda=0$.

Solution:

$$g_m = \mu C_{\text{OX}} \frac{W}{L} (V_{\text{GSQ}} - V_T)$$

$$g_o = \lambda I_D$$

$$g_m = 10^{-4} \frac{10}{1} (3 - 1) = 2\text{E-3}$$

$$g_o = 0$$
Small-signal Operation of Nonlinear Circuits

- Small-signal principles
- Example Circuit

Small-Signal Models

- Small-Signal Analysis of Nonlinear Circuits
Solution for the example was based upon solving the nonlinear circuit for $V_{OUT}$ and then linearizing the solution by doing a Taylor’s series expansion.

- Solution of nonlinear equations very involved with two or more nonlinear devices
- Taylor’s series linearization can get very tedious if multiple nonlinear devices are present

**Standard Approach to small-signal analysis of nonlinear networks**

1. Solve nonlinear network
2. Linearize solution

**Alternative Approach to small-signal analysis of nonlinear networks**

1. Linearize nonlinear devices
2. Replace all devices with small-signal equivalent
3. Solve linear small-signal network
Alternative Approach to small-signal analysis of nonlinear networks

1. Linearize nonlinear devices

2. Replace all devices with small-signal equivalent

3. Solve linear small-signal network

- Must only develop linearized model once for any nonlinear device
  
  e.g. once for a MOSFET, once for a JFET, and once for a BJT

  Linearized model for nonlinear device termed “small-signal model”

  derivation of small-signal model for most nonlinear devices is less complicated than solving even one simple nonlinear circuit

- Solution of linear network much easier than solution of nonlinear network
Review from Last Time:

*Linearized nonlinear devices*
### Dc and small-signal equivalent elements

<table>
<thead>
<tr>
<th>Element</th>
<th>ss equivalent</th>
<th>dc equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>dc Voltage Source</td>
<td>$V_{DC}$</td>
<td>$V_{DC}$</td>
</tr>
<tr>
<td>ac Voltage Source</td>
<td>$V_{AC}$</td>
<td>$V_{AC}$</td>
</tr>
<tr>
<td>dc Current Source</td>
<td>$I_{DC}$</td>
<td>$I_{DC}$</td>
</tr>
<tr>
<td>ac Current Source</td>
<td>$I_{AC}$</td>
<td>$I_{AC}$</td>
</tr>
<tr>
<td>Resistor</td>
<td>$R$</td>
<td>$R$</td>
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</table>
Review from Last Time:

Dc and small-signal equivalent elements

<table>
<thead>
<tr>
<th>Element</th>
<th>ss equivalent</th>
<th>dc equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capacitors</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Large</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Small</td>
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</tbody>
</table>

Inductors

| L           |               |               |
| Large       |               |               |
| Small       |               |               |

MOS Transistors

|               |               | Simplified    |
|               |               | Simplified    |
Review from Last Time:

Dc and small-signal equivalent elements

**Bipolar Transistors**

<table>
<thead>
<tr>
<th>Element</th>
<th>ss equivalent</th>
<th>dc equivalent</th>
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</thead>
<tbody>
<tr>
<td><img src="image1" alt="Bipolar Transistor Diagram" /></td>
<td><img src="image2" alt="Small-Signal Equivalent Diagram" /></td>
<td><img src="image3" alt="Dc Equivalent Diagram" /></td>
</tr>
</tbody>
</table>

**Dependent Sources**

<table>
<thead>
<tr>
<th>Element</th>
<th>ss equivalent</th>
<th>dc equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image4" alt="Dependent Source Diagram" /></td>
<td><img src="image5" alt="Small-Signal Equivalent Diagram" /></td>
<td><img src="image6" alt="Dc Equivalent Diagram" /></td>
</tr>
</tbody>
</table>
How is the small-signal equivalent circuit obtained from the nonlinear circuit?

What is the small-signal equivalent of the MOSFET and BJT?
Review from Last Time:

4-terminal small-signal network summary

Small signal model:

\[
\begin{align*}
i_1 &= y_{11} v_1 + y_{12} v_2 + y_{13} v_3 \\
i_2 &= y_{21} v_1 + y_{22} v_2 + y_{23} v_3 \\
i_3 &= y_{31} v_1 + y_{32} v_2 + y_{33} v_3 \\
y_{ij} &= \left. \frac{\partial f_i(V_1, V_2, V_3)}{\partial V_j} \right|_{V=V_Q}
\end{align*}
\]

\[
\begin{align*}
I_1 &= f_1(V_1, V_2, V_3) \\
I_2 &= f_2(V_1, V_2, V_3) \\
I_3 &= f_3(V_1, V_2, V_3)
\end{align*}
\]
Review from Last Time:

3-terminal small-signal network summary

\[ I_1 = f_1(V_1, V_2) \]
\[ I_2 = f_2(V_1, V_2) \]

Small signal model:

\[ i_1 = y_{11} V_1 + y_{12} V_2 \]
\[ i_2 = y_{21} V_1 + y_{22} V_2 \]

[Diagram of circuit with equations and symbols]
Small-Signal Model

\[
\begin{align*}
\mathbf{i}_1 &= \mathbf{y}_{11} \mathbf{V}_1 \\
y_{11} &= \frac{\partial f_1(V_1)}{\partial V_1} \mid_{V=V_0} \\
\tilde{V} &= V_{1Q}
\end{align*}
\]

A Small Signal Equivalent Circuit
Small Signal Model of MOSFET

Large Signal Model

\[ I_G = 0 \]

3-terminal device

\[ I_D = \begin{cases} 
0 & \text{if } V_{GS} \leq V_T \\
\mu C_{OX} \frac{W}{L} \left( V_{GS} - V_T - \frac{V_{DS}}{2} \right) V_{DS} & \text{if } V_{GS} > V_T \text{ and } V_{DS} < V_{GS} - V_T \\
\mu C_{OX} \frac{W}{2L} (V_{GS} - V_T)^2 \left( 1 + \lambda V_{DS} \right) & \text{if } V_{GS} > V_T \text{ and } V_{DS} \geq V_{GS} - V_T
\end{cases} \]

MOSFET is usually operated in saturation region in linear applications where a small-signal model is needed so will develop the small-signal model in the saturation region.
Small Signal Model of MOSFET

by convention, \( y_{21} = g_m, \ y_{22} = g_0 \)

\[
y_{21} \approx g_m = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T)
\]

\[
y_{22} = g_o \approx \lambda I_{DQ}
\]

\[
i_g = 0
\]

\[
i_d = g_m V_{GS} + g_o V_{DS}
\]
Small Signal Model of MOSFET

\[ g_m = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \]

\[ g_o \approx \lambda I_{DQ} \]

Alternate equivalent expressions:

\[ I_{DQ} = \mu C_{ox} \frac{W}{2L} (V_{GSQ} - V_T)^2 \left(1 + \lambda V_{DSQ}\right) \approx \mu C_{ox} \frac{W}{2L} (V_{GSQ} - V_T)^2 \]

\[ g_m = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \]

\[ g_m = \sqrt{2\mu C_{ox} \frac{W}{L} \cdot \sqrt{I_{DQ}}} \]

\[ g_m = \frac{2I_{DQ}}{V_{GSQ} - V_T} \]
Small signal analysis example

$$V_{IN} = V_M \sin \omega t$$

$$A_v = \frac{2I_{DQ} R}{V_{SS} + V_T}$$

Observe the small signal voltage gain is twice the Quiescent voltage across $R$ divided by $V_{SS} + V_T$

- This analysis which required linearization of a nonlinear output voltage is quite tedious.

- This approach becomes unwieldy for even slightly more complicated circuits

- A much easier approach based upon the development of small signal models will provide the same results, provide more insight into both analysis and design, and result in a dramatic reduction in computational requirements
Small Signal-Large Signal Model of MOSFET in Saturation Region

**Summary**

**Large Signal Model for Q-Point Calculations**

\[
\lambda \cong \frac{W}{2L} (V_{gsq} - V_t)^2
\]

**Small Signal Model**

**Basic Model**

\[
g_m = \mu C_{ox} \frac{W}{L} (V_{gsq} - V_t)
\]

\[
g_o \cong \lambda_{dq}
\]

**Simplified**

\[
g_m = \mu C_{ox} \frac{W}{L} (V_{gsq} - V_t)
\]
Consider again:

**Small signal analysis example**

\[ A_v = \frac{2I_{DQ} R}{[V_{SS} + V_T]} \]

Derived for \( \lambda = 0 \) (i.e. \( g_0 = 0 \))

\[ I_{DQ} = \mu C_{ox} \frac{W}{2L} (V_{GSQ} - V_T)^2 \]
Consider again:

Small signal analysis example

\[ A_v = \frac{2I_{DQ} R}{V_{SS} + V_T} \]

The gain expressions appear to be different!

\[ A_v = \frac{V_{OUT}}{V_{IN}} = -g_m R \]

\[ g_m = \frac{2I_{DQ}}{V_{GSQ} - V_T} \]

\[ V_{GSQ} = -V_{SS} \]

\[ A_v = \frac{2I_{DQ} R}{V_{SS} + V_T} \]
Consider again:

Small signal analysis example

\[ A_v = \frac{V_{OUT}}{V_{IN}} = -g_m R \]

More accurate gain can be obtained if \( \lambda \) effects are included and does not significantly increase complexity of small signal analysis.
Small Signal Model of BJT

3-terminal device

Forward Active Model:

\[ I_C = J_S A_E e^{\frac{V_{BE}}{V_t}} \left( 1 + \frac{V_{CE}}{V_{AF}} \right) \]

\[ I_B = \frac{J_S A_E}{\beta} e^{\frac{V_{BE}}{V_t}} \]

Usually operated in Forward Active Region when small-signal model is needed
Small Signal Model of BJT

\[ I_1 = f_1(V_1, V_2) \quad \leftrightarrow \quad I_B = \frac{J_S A_E}{\beta} e^{\frac{V_{BE}}{V_t}} \]

\[ I_2 = f_2(V_1, V_2) \quad \leftrightarrow \quad I_C = J_S A_E e^{\frac{V_{BE}}{V_t}} \left( 1 + \frac{V_{CE}}{V_{AF}} \right) \]

Small-signal model:

\[ y_{ij} = \left. \frac{\partial f_i(V_1, V_2)}{\partial V_j} \right|_{V=V_Q} \]

\[ y_{11} = g_\pi = \left. \frac{\partial I_B}{\partial V_{BE}} \right|_{V=V_Q} \]

\[ y_{12} = \left. \frac{\partial I_B}{\partial V_{CE}} \right|_{V=V_Q} \]

\[ y_{21} = g_m = \left. \frac{\partial I_C}{\partial V_{BE}} \right|_{V=V_Q} \]

\[ y_{22} = g_o = \left. \frac{\partial I_C}{\partial V_{CE}} \right|_{V=V_Q} \]
Small Signal Model of BJT

\[ I_B = \frac{J_S A_E}{\beta} e^{\frac{V_{BE}}{V_t}} \]

\[ I_C = J_S A_E e^{\frac{V_{BE}}{V_t}} \left( 1 + \frac{V_{CE}}{V_{AF}} \right) \]

Small-signal model:

\[ g_m = \frac{\partial I_C}{\partial V_{BE}} \bigg|_{V_{BE}=V_{Q}} = \frac{1}{V_t} \frac{J_S A_E}{\beta} e^{\frac{V_{BE}}{V_t}} \bigg|_{V_{BE}=V_{Q}} = \frac{I_{BO}}{V_t} \approx \frac{I_{CO}}{\beta V_t} \]

\[ y_{12} = \frac{\partial I_B}{\partial V_{CE}} \bigg|_{V_{CE}=V_{Q}} = 0 \]

\[ y_{21} = g_m = \frac{\partial I_C}{\partial V_{BE}} \bigg|_{V_{BE}=V_{Q}} = \frac{1}{V_t} J_S A_E e^{\frac{V_{BE}}{V_t}} \left( 1 + \frac{V_{CE}}{V_{AF}} \right) \bigg|_{V_{BE}=V_{Q}} = \frac{I_{CO}}{V_t} \]

\[ y_{22} = g_o = \frac{\partial I_C}{\partial V_{CE}} \bigg|_{V_{CE}=V_{Q}} = \frac{J_S A_E e^{\frac{V_t}{V_{AF}}}}{V_{AF}} \bigg|_{V_{CE}=V_{Q}} \approx \frac{I_{CO}}{V_{AF}} \]
Small Signal Model of BJT

\[ i_B = y_{11} V_{BE} + y_{12} V_{CE} \]
\[ i_C = y_{21} V_{BE} + y_{22} V_{CE} \]

\[ g_\pi = \frac{I_C}{\beta V_t} \quad g_m = \frac{I_C}{V_t} \quad g_o = \frac{I_C}{V_{AF}} \]

\[ i_B = g_\pi V_{BE} \quad i_C = g_m V_{BE} + g_o V_{CE} \]

\[ g_0 \text{ can often be neglected!} \]
Alternate Small Signal Model of BJT

Observe: \( g_m V_{BE} = g_m \frac{I_B}{g_\pi} = \beta I_B \)

Alternate Equivalent Small-signal Model

\[
g_\pi = \frac{I_{CQ}}{\beta V_t} \quad g_m = \frac{I_{CQ}}{V_t} \quad g_o = \frac{I_{CQ}}{V_{AF}}
\]
Large Signal Model of BJT in Forward Active Region for Q-point Calculations

\[ I_B = \frac{J_S A_E}{\beta} e^{\frac{V_{BE}}{V_t}} \]

\[ I_C = J_S A_E e^{\frac{V_{BE}}{V_t}} \left( 1 + \frac{V_{CE}}{V_{AF}} \right) \]

But when operating in Forward Active Region, \( V_{BE} \) will be around 0.6V

In most applications, the following model is adequate for Q-point calculations

\[ I_B = \frac{J_S A_E}{\beta} e^{\frac{V_{BE}}{V_t}} \]

\[ I_C = \beta I_B \]
Small Signal-Large Signal Model of BJT in Forward Active Region

**Summary**

**Large Signal Model for Q-Point Calculations**

**Small Signal Model**

**Basic Model**

\[ g_{\pi} = \frac{I_{CQ}}{\beta V_t} \]
\[ g_m = \frac{I_{CQ}}{V_t} \]
\[ g_o = \frac{I_{CQ}}{V_{AF}} \]

**Simplified**

\[ g_{\pi} = \frac{I_{CQ}}{\beta V_t} \]
\[ g_m = \frac{I_{CQ}}{V_t} \]
Small-signal Operation of Nonlinear Circuits

- Small-signal principles
- Example Circuit
- Small-Signal Models

Small-Signal Analysis of Nonlinear Circuits
Standard Approach to small-signal analysis of nonlinear networks

1. Linearize nonlinear devices
   *(have small-signal model for key devices!)*

2. Replace all devices with small-signal equivalent

3. Solve linear small-signal network
Example:

Determine the small signal voltage gain $A_v = \frac{v_{OUT}}{v_{IN}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$. 
Example: Determine the small signal voltage gain $A_V = \frac{v_{out}}{v_{in}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$. 

Small-signal circuit
Example: Determine the small signal voltage gain $A_V = \frac{v_{OUT}}{v_{IN}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$. 

Small-signal circuit

Small-signal MOSFET model for $\lambda = 0$
Example: Determine the small signal voltage gain $A_V = \frac{V_{OUT}}{V_{IN}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$. 

Small-signal circuit
Example:

Small-signal circuit

Analysis:

By KCL

\[ g_{m1} v_{GS1} = g_{m2} v_{GS2} \]

but

\[ v_{GS1} = v_{IN} \]

\[ -v_{GS2} = v_{OUT} \]

thus:

\[ A_v = \frac{v_{OUT}}{v_{IN}} = -\frac{g_{m1}}{g_{m2}} \]
Example:

Small-signal circuit

Analysis:

\[ A_v = \frac{\mathbf{v}_{\text{OUT}}}{\mathbf{v}_{\text{IN}}} = -\frac{g_{m1}}{g_{m2}} \]

Recall:

\[ g_m = -\sqrt{2I_D \mu C_{ox}} \sqrt{\frac{W_1}{L_1}} \]

\[ A_v = \frac{\sqrt{2I_D \mu C_{ox}} \frac{W_1}{L_1}}{\sqrt{2I_D \mu C_{ox}} \frac{W_2}{L_2}} = -\frac{W_1}{W_2} \sqrt{\frac{L_2}{L_1}} \]
Example:

**Small-signal circuit**

\[ A_v = \frac{V_{OUT}}{V_{IN}} = -\frac{g_{m1}}{g_{m2}} \]

If \( L_1 = L_2 \), obtain

\[ A_v = -\sqrt{\frac{W_1}{W_2}} \sqrt{\frac{L_2}{L_1}} = -\sqrt{\frac{W_1}{W_2}} \]

**The width and length ratios can be accurately set when designed in a standard CMOS process**
End of Lecture 34