EE 330
Lecture 21

- Small Signal Analysis
- Small Signal Modelling
Small-Signal Operation

- If slope is steep, output range can be much larger than input range
- This can be viewed as voltage gain in the circuit
Small signal operation of nonlinear circuits

\[ V_{IN} = V_M \sin \omega t \]

\( V_M \) is small

Review from Last Lecture

- Small signal concepts often apply when building amplifiers
- If small signal concepts do not apply, usually the amplifier will not perform well
- Small signal operation is usually synonymous with “locally linear”
- Small signal operation is relative to an “operating point”
Amplification with Transistors

From Wikipedia: (Oct. 2015)

An **amplifier**, **electronic amplifier** or (informally) **amp** is an electronic device that increases the **power** of a **signal**.

- It is difficult to increase the voltage or current very much with passive RC circuits.
- Voltage and current levels can be increased a lot with transformers but not practical in integrated circuits.
- Power levels cannot be increased with passive elements (R, L, C, and Transformers).
- Often an amplifier is defined to be a circuit that can increase power levels (be careful with Wikipedia and WWW even when some of the most basic concepts are discussed).
- Transistors can be used to increase not only signal levels but power levels to a load.
- In transistor circuits, power that is delivered in the signal path is supplied by a biasing network.
Consider the following MOSFET and BJT Circuits

**BJT**
- $V_{CC}$
- $R_1$
- $V_{OUT}$
- $Q_1$
- $V_{EE}$
- $V_{IN}(t)$

**MOSFET**
- $V_{DD}$
- $R_1$
- $M_1$
- $V_{OUT}$
- $V_{IN}(t)$
- $V_{SS}$

- MOS and BJT Architectures often Identical
- Circuit are Highly Nonlinear
- Nonlinear Analysis Methods Must be used to analyze these and almost any other nonlinear circuit
Small signal analysis using nonlinear models

By selecting appropriate value of $V_{SS}$, $M_1$ will operate in the saturation region

Assume $M_1$ operating in saturation region

$$V_{OUT} = V_{DD} - (I_D R)$$

$$I_D = \frac{\mu C_{ox} W}{2L} (V_{IN} - V_{SS} - V_T)^2$$

$$I_{DQ} = \frac{\mu C_{ox} W}{2L} (V_{SS} + V_T)^2$$

$$V_{OUT} = V_{DD} - \frac{\mu C_{ox} W}{2L} (V_{IN} - V_{SS} - V_T)^2 R$$

$$V_{OUT} = V_{DD} - \frac{\mu C_{ox} W}{2L} (V_M \sin \omega t - [V_{SS} + V_T])^2 R$$

$V_{IN}=V_M \sin \omega t$

$V_M$ is small

$V_{DD}$

$R$

$M_1$

$V_{OUT}$

$V_{SS}$

$V_{IN}$

$V_M$

$t$

Termed Load Line

Review from Last Lecture
Small signal analysis example

Assume $M_1$ operating in saturation region

Quiescent Output

$$V_{OUT} \approx \left[ V_{DD} - \frac{\mu C_{ox} W}{2L} \left( V_{SS} + V_T \right)^2 R \right] + \left[ \frac{\mu C_{ox} W}{L} \left( V_{SS} + V_T \right) R \right] V_M \sin \omega t$$

ss Voltage Gain

$$A_v = \frac{\mu C_{ox} W}{L} \left( V_{SS} + V_T \right) R$$

$$V_{OUTQ} = \left[ V_{DD} - \frac{\mu C_{ox} W}{2L} \left( V_{SS} + V_T \right)^2 R \right]$$

$$V_{OUT} \approx V_{OUTQ} + A_v V_M \sin \omega t$$

Note the ss voltage gain is negative since $V_{SS} + V_T < 0$!
Small signal analysis example

Assume $M_1$ operating in saturation region.

$$V_{\text{OUT}} \approx V_{\text{OUTQ}} + A_{V} V_{M} \sin \omega t$$

$$A_{V} = \frac{\mu C_{\text{ox}} W}{L} \left[ V_{SS} + V_{T} \right] R$$

$$V_{\text{OUTQ}} = \left\{ V_{DD} - \frac{\mu C_{\text{ox}} W}{2L} \left[ V_{SS} + V_{T} \right]^2 R \right\}$$

But – this expression gives little insight into how large the gain is!
And the analysis for even this very simple circuit was messy!
Small signal analysis example

\[ V_{\text{OUT}} \cong V_{\text{OUTQ}} + A_v V_M \sin \omega t \]
Small signal analysis example

\[ V_{\text{OUT}} \simeq V_{\text{OUTQ}} + A_v V_M \sin \omega t \]

\[ A_v = \frac{\mu C_{ox} W}{L} \left[ V_{ss} + V_T \right] R \]
Small signal analysis example

\[ V_{\text{OUT}} \approx V_{\text{OUTQ}} + A_v V_M \sin \omega t \]

\[ A_v = \frac{\mu C_{\text{ox}} W}{L} \left[ V_{\text{ss}} + V_T \right] R \]
Small signal analysis example

\[ V_{\text{OUT}} \approx V_{\text{OUTQ}} + A_V V_M \sin \omega t \]

\[ V_{\text{IN}} = V_M \sin \omega t \]

Serious Distortion occurs if signal is too large or Q-point non-optimal.

Here “clipping” occurs for high \( V_{\text{OUT}} \).
Small signal analysis example

\[ V_{\text{OUT}} \approx V_{\text{OUTQ}} + A_V V_M \sin \omega t \]

\[ A_v = \frac{\mu C_{\text{ox}} W}{L} [V_{\text{ss}} + V_T] R \]

Serious Distortion occurs if signal is too large or Q-point non-optimal
Here “clipping” occurs for low \( V_{\text{OUT}} \)
Small signal analysis example

\[ V_{\text{IN}} = V_M \sin \omega t \]

\[ V_{\text{OUT}} \approx V_{\text{OUTQ}} + A_V V_M \sin \omega t \]

\[ A_V = \frac{\mu C_{\text{ox}} W}{L} \left[ V_{\text{SS}} + V_T \right] R \]

But recall:

Thus, substituting from the expression for \( I_{\text{DQ}} \) we obtain

\[ I_{\text{DQ}} = \frac{\mu C_{\text{ox}} W}{2L} \left( V_{\text{SS}} + V_T \right)^2 \]

\[ A_V = \frac{2I_{\text{DQ}} R}{\left[ V_{\text{SS}} + V_T \right]} \]

Note this is negative since \( V_{\text{SS}} + V_T < 0 \)
Observe the small signal voltage gain is twice the Quiescent voltage across $R$ divided by $V_{SS} + V_T$

Can make $|A_V|$ large by making $|V_{SS} + V_T|$ small

- This analysis which required linearization of a nonlinear output voltage is quite tedious.

- This approach becomes unwieldy for even slightly more complicated circuits

- A much easier approach based upon the development of small signal models will provide the same results, provide more insight into both analysis and design, and result in a dramatic reduction in computational requirements
Small signal analysis example

(Consider what was neglected in the previous analysis)

\[ V_{\text{OUT}} \approx V_{\text{OUTQ}} + A_V V_M \sin \omega t \]

However, there are invariably small errors in this analysis

\[ V_{\text{OUT}} = V_{\text{OUTQ}} + A_V V_M \sin \omega t + \epsilon(t) \]

To see the effects of the approximations consider again

\[ V_{\text{OUT}} = V_{DD} - \frac{\mu C_{Ox} W}{2L} \left( V_M \sin \omega t - [V_{SS} + V_T] \right)^2 R \]

\[ V_{\text{OUT}} = V_{DD} - \frac{\mu C_{Ox} RW}{2L} \left( V_M^2 \sin^2(\omega t) - 2[V_{SS} + V_T] V_M \sin \omega t + [V_{SS} + V_T]^2 \right) \]

\[ V_{\text{OUT}} = V_{DD} - \frac{\mu C_{Ox} RW}{2L} \left( V_M^2 \left[ \frac{1 - \cos 2 \omega t}{2} \right] - 2[V_{SS} + V_T] V_M \sin \omega t + [V_{SS} + V_T]^2 \right) \]

\[ V_{\text{OUT}} = \left\{ V_{DD} - \frac{\mu C_{Ox} RW}{2L} \left( V_M^2 + [V_{SS} + V_T]^2 \right) \right\} + \left\{ \frac{\mu C_{Ox} W}{L} [V_{SS} + V_T] R \right\} V_M \sin \omega t + \left\{ \frac{\mu C_{Ox} RW}{4L} V_M^2 \right\} \cos 2 \omega t \]

Note presence of second harmonic distortion term!
Small signal analysis example

Nonlinear distortion term

\[ V_{\text{OUT}} \approx V_{\text{OUTQ}} + A_V V_M \sin \omega t \]

\[ V_{\text{OUT}} = V_{\text{OUTQ}} + A_V V_M \sin \omega t + \varepsilon(t) \]

\[ V_{\text{OUT}} = \left[ V_{\text{DD}} - \frac{\mu C_{\text{ox}} R W}{2L} \left( \frac{V_M^2}{2} + [V_{SS} + V_T]^2 \right) \right] + \left[ \frac{\mu C_{\text{ox}} W}{L} [V_{SS} + V_T] \right] V_M \sin \omega t + \left[ \frac{\mu C_{\text{ox}} R W}{4L} V_M^2 \right] \cos 2\omega t \]

\[ \tilde{V}_{\text{OUTQ}} = \left[ V_{\text{DD}} - \frac{\mu C_{\text{ox}} R W}{2L} \left( \frac{V_M^2}{2} + [V_{SS} + V_T]^2 \right) \right] \]

\[ A_V = \frac{\mu C_{\text{ox}} W}{L} [V_{SS} + V_T] R \]

\[ A_2 = \frac{\mu C_{\text{ox}} R W}{4L} V_M \]

\[ V_{\text{OUT}} = \tilde{V}_{\text{OUTQ}} + \{ A_V V_M \sin \omega t \} + \{ A_2 V_M \cos 2\omega t \} \]
Small signal analysis example

Nonlinear distortion term

\[ V_{\text{OUT}} = \tilde{V}_{\text{OUTQ}} + \left\{ A_v V_m \sin \omega t \right\} + \left\{ A_2 V_m \cos 2\omega t \right\} \]

\[ \tilde{V}_{\text{OUTQ}} = \left\{ V_{DD} - \frac{\mu C_{OX} R W}{2L} \left( \frac{V_m^2}{2} + [V_{SS} + V_T]^2 \right) \right\} \]

\[ A_v = \frac{\mu C_{OX} W}{L} [V_{SS} + V_T] R \]

\[ A_2 = \frac{\mu C_{OX} R W}{4L} V_m \]

Total Harmonic Distortion:

Recall, if \( x(t) = \sum_{k=0}^{\infty} b_k \sin(k\omega t + \phi_k) \) then

\[ \text{THD} = \sqrt{\sum_{k=2}^{\infty} b_k^2} \]

Thus, for this amplifier, as long as \( M_1 \) stays in the saturation region

\[ \text{THD} = \frac{\sqrt{(A_2 V_m)^2}}{A_v V_m} = \frac{A_2}{A_v} = \frac{\mu C_{OX} W}{4L} R V_m \frac{R V_m}{|V_{SS} + V_T|} = \frac{V_m}{4|V_{SS} + V_T|} \]

Distortion will be small for \( V_m << |V_{SS} + V_T| \)

Distortion will be much worse \( \text{larger and more harmonic terms} \) if \( M_1 \) leaves saturation region.
Consider the following MOSFET and BJT Circuits

**BJT**

- $V_{IN}(t)$
- $V_{CC}$
- $R_1$
- $V_{OUT}$
- $Q_1$
- $V_{EE}$

**MOSFET**

- $V_{DD}$
- $R_1$
- $M_1$
- $V_{OUT}$
- $V_{IN}(t)$
- $V_{SS}$

- Analysis was very time consuming
- Issue of operation of circuit was obscured in the details of the analysis

One of the most widely used amplifier architectures
Consider the following MOSFET and BJT Circuits

One of the most widely used amplifier architectures
Small signal analysis using nonlinear models

Assume $Q_1$ operating in forward active region

By selecting appropriate value of $V_{SS}$, $M_1$ will operate in the forward active region

$V_{IN} = V_M \sin \omega t$

$V_M$ is small

$I_C = J_S A_E e^{\frac{V_{IN}-V_{EE}}{V_t}}$

$I_{CQ} = J_S A_E e^{\frac{-V_{EE}}{V_t}}$

$V_{OUT} = V_{CC} - J_S A_E R e^{\frac{V_{IN}-V_{EE}}{V_t}}$

$V_{OUT} = V_{CC} - J_S A_E R e^{\frac{V_M \sin(\omega t)-V_{EE}}{V_t}}$
Small signal analysis using nonlinear models

\[ V_{\text{OUT}} = V_{\text{CC}} - J S A R e \]

\[ V_{\text{OUT}} = V_{\text{CC}} - J S A R e^e \]

Recall that if \( x \) is small

\[ e^x \approx 1 + x \] (truncated Taylor’s series)

\[ V_{\text{IN}} = V_M \sin \omega t \]

\[ V_M \text{ is small} \]

\[ V_{\text{OUT}} \approx \left[ V_{\text{CC}} - J S A R e \right] - \left[ J S A R e \right] \]

\[ I_{CQ} = J_S A_E e^{\frac{-V_{\text{EE}}}{V_t}} \]

\[ \frac{V_M \sin(\omega t) - V_{\text{EE}}}{V_t} \]
Small signal analysis using nonlinear models

\[ V_{\text{OUT}} \approx \left[ V_{CC} - J_S A_e \frac{-V_{EE}}{V_t} R_1 \right] - \left[ J_S A_e \frac{-V_{EE}}{V_t} \right] \frac{V_M \sin(\omega t)}{V_t} R_1 \]

\[ I_{CQ} = J_S A_e \frac{-V_{EE}}{V_t} \]

\[ V_{\text{IN}} = V_M \sin \omega t \]

\[ V_M \text{ is small} \]

**Quiescent Output**

\[ \text{ss Voltage Gain} \]
Comparison of Gains for MOSFET and BJT Circuits

**BJT**

\[ A_{VB} = -\frac{I_{CQ} R_1}{V_t} \]

If \( I_{DQ} R = I_{CQ} R_1 = 2V \), \( V_{SS} + V_T = -1V \), \( V_t = 25mV \)

\[ A_{VB} = -\frac{I_{CQ} R_1}{V_t} = -\frac{2V}{25mV} = -80 \]

\[ A_{VM} = \frac{2I_{DQ} R}{V_{SS} + V_T} \]

\[ A_{VM} = \frac{2I_{DQ} R}{-1V} = -4 \]

Observe \( A_{VB} >> A_{VM} \)

Due to exponential-law rather than square-law model

**MOSFET**

\[ A_{VM} = \frac{2I_{DQ} R}{V_{SS} + V_T} \]

\[ A_{VM} = \frac{2I_{DQ} R}{-1V} = -4 \]
Operation with Small-Signal Inputs

• Analysis procedure for these simple circuits was very tedious

• This approach will be unmanageable for even modestly more complicated circuits

• **Faster analysis method is needed!**
Small-Signal Analysis

- Will commit next several lectures to developing this approach
- Analysis will be MUCH simpler, faster, and provide significantly more insight
- Applicable to many fields of engineering
Small-Signal Analysis

- Simple dc Model
- Square-Law Model (with extensions for $\lambda, \gamma$ effects)
- Short-Channel $\alpha$-law Model
- Better Analytical dc Model
- Simpler dc Model
- BSIM Model
- Switch-Level Models
  - Ideal switches
  - $R_{SW}$ and $C_{GS}$
Operation with Small-Signal Inputs

Why was this analysis so tedious?

Because of the nonlinearity in the device models

What was the key technique in the analysis that was used to obtain a simple expression for the output (and that related linearly to the input)?

\[
V_{\text{OUT}} = V_{\text{CC}} - J_S A_m R \exp \left( \frac{-V_{\text{EE}}}{V_t} \right) e + \frac{V_M \sin(\omega t)}{V_t}
\]

\[
V_{\text{OUT}} \approx \left[ V_{\text{CC}} - I_{CQ} R_1 \right] - \left( \frac{I_{CQ} R_1}{V_t} \right) V_M \sin(\omega t)
\]

Linearization of the nonlinear output expression at the operating point
Operation with Small-Signal Inputs

\[ I_{CQ} = J_s A_E e^{\frac{-V_{EE}}{V_t}} \]

\[ V_{OUT} \approx \left[ V_{CC} - I_{CQ} R_1 \right] - \left( \frac{I_{CQ} R_1}{V_t} \right) V_m \sin(\omega t) \]

Quiescent Output

ss Voltage Gain

Small-signal analysis strategy

1. Obtain Quiescent Output (Q-point)
2. Linearize circuit at Q-point instead of linearize the nonlinear solution
3. Analyze linear “small-signal” circuit
4. Add quiescent and small-signal outputs to obtain good approximation to actual output
Small-Signal Principle

Nonlinear function
\[ y = f(x) \]

Q-point

\[ Y_Q \]

\[ X_Q \]
Small-Signal Principle

Region around Q-Point

$y = f(x)$

$Y_Q$

$X_Q$

Q-point
Relationship is nearly linear in a small enough region around Q-point
Region of linearity is often quite large
Linear relationship may be different for different Q-points
Small-Signal Principle

Relationship is nearly linear in a small enough region around Q-point
Region of linearity is often quite large
Linear relationship may be different for different Q-points
Small-Signal Principle

Device Behaves Linearly in Neighborhood of Q-Point
Can be characterized in terms of a small-signal coordinate system
Small-Signal Principle

\[
y = \frac{y - y_Q}{x - x_Q} \frac{\partial f}{\partial x} \bigg|_{x=x_Q} = \frac{y - y_Q}{x - x_Q} \frac{\partial f}{\partial x} \bigg|_{x=x_Q} \]

\[
y = y_Q + \left( y_Q - x_Q \right) \frac{\partial f}{\partial x} \bigg|_{x=x_Q} \]

\[
m = \frac{\partial f}{\partial x} \bigg|_{x=x_Q} \]

\[
y - y_Q = \frac{\partial f}{\partial x} \bigg|_{x=x_Q} (x - x_Q) \]

Q-point \((x_Q, y_Q)\) or \((x_Q, y_Q)\)
Small-Signal Principle

Changing coordinate systems:

\[ y_{SS} = y - y_Q \]
\[ x_{SS} = x - x_Q \]

\[ y - y_Q = \left. \frac{\partial f}{\partial x} \right|_{x=x_Q} (x - x_Q) \]

\[ y_{SS} = \left. \frac{\partial f}{\partial x} \right|_{x=x_Q} x_{SS} \]
Small-Signal Model:

\[ y_{ss} = \frac{\partial f}{\partial x} \bigg|_{x=x_Q} x_{ss} \]

- Linearized model for the nonlinear function \( y = f(x) \)
- Valid in the region of the Q-point
- Will show the small signal model is simply Taylor’s series expansion of \( f(x) \) at the Q-point truncated after first-order terms
Small-Signal Principle

Observe:

\[ y - y_Q = \left. \frac{\partial f}{\partial x} \right|_{x=x_Q} (x - x_Q) \]

\[ y_Q = f(x_Q) \]

\[ y = f(x_Q) + \left. \frac{\partial f}{\partial x} \right|_{x=x_Q} (x - x_Q) \]

Recall Taylor's Series Expansion of nonlinear function \( f \) at expansion point \( x_0 \)

\[ y = f(x_0) + \sum_{k=1}^{\infty} \left( \frac{1}{k!} \left. \frac{\partial f}{\partial x} \right|_{x=x_0} (x-x_0)^k \right) \]

Truncating after first-order terms (and defining “o” as “Q”):

\[ y \approx f(x_Q) + \left. \frac{\partial f}{\partial x} \right|_{x=x_Q} (x - x_Q) \]

Mathematically, linearized model is simply Taylor’s series expansion of the nonlinear function \( f \) at the Q-point truncated after first-order terms with notation \( x_Q = x_0 \)
Small-Signal Principle

\[ y = \begin{cases} f(x_Q) \\ \frac{\partial f}{\partial x} \bigg|_{x=x_Q} x_{SS} \end{cases} \]

Quiescent Output

ss Gain

How can a circuit be linearized at an operating point as an alternative to linearizing a nonlinear function at an operating point?

Consider arbitrary nonlinear one-port network

Nonlinear One-Port
Arbitrary Nonlinear One-Port

\[ I = f(V) \]

**Q-point**

\[ I_{SS} = \frac{\partial I}{\partial V} \bigg|_{V=V_Q} \]

\[ V_{SS} = \mathbf{U} \]

\[ y = \frac{\partial I}{\partial V} \bigg|_{V=V_Q} \]

Linear model of the nonlinear device at the Q-point
Arbitrary Nonlinear One-Port

Linear small-signal model:

\[ i = y \cdot V \]

**A Small Signal Equivalent Circuit:**

- The small-signal model of this 2-terminal electrical network is a resistor of value \(1/y\) or a conductor of value \(y\)
- **One small-signal parameter** characterizes this one-port but it is dependent on Q-point
- This applies to **ANY** nonlinear one-port that is differentiable at a Q-point (e.g. a diode)