EE 330
Lecture 23

Small-Signal Operation, Analysis and Models
Consider the following MOSFET and BJT Circuits

BJT

MOSFET

Assume BJT operating in FA region, MOSFET operating in Saturation

Assume same quiescent output voltage and same resistor $R_1$

One of the most widely used amplifier architectures
Methods of Analysis of Nonlinear Circuits

Will consider three different analysis requirements and techniques for some particularly common classes of nonlinear circuits

1. Circuits with continuously differential devices

   Interested in obtaining transfer characteristics of these circuits or outputs for given input signals

2. Circuits with piecewise continuous devices

   Interested in obtaining transfer characteristics of these circuits or outputs for a given input signals

3. Circuits with small-signal inputs that vary around some operating point

   Interested in obtaining relationship between small-signal inputs and the corresponding small-signal outputs. Will assume these circuits operate linearly in some suitably small region around the operating point

Other types of nonlinearities may exist and other types of analysis may be required but we will not attempt to categorize these scenarios in this course.
3. Circuits with small-signal inputs that vary around some operating point

Interested in obtaining relationship between small-signal inputs and the corresponding small-signal outputs. Will assume these circuits operate linearly in some suitably small region around the operating point.

Analysis Strategy:

Use methods from previous class of nonlinear circuits.

More Practical Analysis Strategy:

Determine the operating point (using method 1 or 2 discussed above after all small signal independent inputs are set to 0).

Develop small signal (linear) model for all devices in the region of interest (around the operating point or “Q-point”).

Create small signal equivalent circuit by replacing all devices with small-signal equivalent.

Solve the resultant small-signal (linear) circuit.

Can use KCL, DVL, and other linear analysis tools such as superposition, voltage and current divider equations, Thevenin and Norton equivalence.

Determine boundary of region where small signal analysis is valid.
Comparison of Gains for MOSFET and BJT Circuits

**BJT**

\[ A_{VB} = - \frac{I_{CQ} R_1}{V_t} \]

If \( I_{DQ} R = I_{CQ} R_1 = 2V, V_{SS} + V_T = -1V, V_t = 25mV \)

\[ A_{VB} = - \frac{I_{CQ} R_1}{V_t} = - \frac{2V}{25mV} = -80 \]

**MOSFET**

\[ A_{VM} = \frac{2I_{DQ} R}{[V_{SS} + V_T]} \]

\[ A_{VM} = \frac{2I_{DQ} R}{[V_{SS} + V_T]} = \frac{4V}{-1V} = -4 \]

Observe \( A_{VB} \gg A_{VM} \)

Due to exponential-law rather than square-law model
Operation with Small-Signal Inputs

- Analysis procedure for these simple circuits was very tedious
- This approach will be unmanageable for even modestly more complicated circuits
- Faster analysis method is needed!
Operation with Small-Signal Inputs

Why was this analysis so tedious?

Because of the nonlinearity in the device models

What was the key technique in the analysis that was used to obtain a simple expression for the output?

\[
V_{\text{OUT}} = V_{\text{CC}} - J_S A_E R_1 \left( \frac{-V_{\text{EE}}}{V_i} + \frac{V_{\text{M}} \sin(\omega t)}{V_i} \right)
\]

\[
V_{\text{OUT}} \approx \left[ V_{\text{CC}} - I_{\text{CQ}} R_1 \right] - \left( \frac{I_{\text{CQ}} R_1}{V_i} \right) V_{\text{M}} \sin(\omega t)
\]

Linearization of the nonlinear output expression at the operating point
Operation with Small-Signal Inputs

\[ I_{CQ} = J_S A_E e^{\frac{-V_{EE}}{V_t}} \]

\[ V_{OUT} \cong \left[ V_{CC} - I_{CQ} R_1 \right] - \left( \frac{I_{CQ} R_1}{V_t} \right) V_m \sin(\omega t) \]

Quiescent Output

ss Voltage Gain

Small-signal analysis strategy

1. Obtain Quiescent Output (Q-point)
2. Linearize circuit at Q-point instead of linearize the nonlinear solution
   (trivially obtain small-signal circuit from linearized circuit elements)
3. Analyze linear “small-signal” circuit
4. Add quiescent and small-signal outputs to obtain good approximation to actual output
Small-Signal Principle

Nonlinear function
\[ y = f(x) \]

Q-point
\[ X_Q, Y_Q \]
Small-Signal Principle

Region around Q-Point

Q-point

y=f(x)

y

x
Small-Signal Principle

Relationship is nearly linear in a small enough region around Q-point
Region of linearity is often quite large
Linear relationship may be different for different Q-points
Small-Signal Principle

Relationship is nearly linear in a small enough region around Q-point.
Region of linearity is often quite large.
Linear relationship may be different for different Q-points.
Small-Signal Principle

Device Behaves Linearly in Neighborhood of Q-Point
Can be characterized in terms of a small-signal coordinate system
Small-Signal Principle

In region of Q-point, linear model and nonlinear model are the same! 
y=mx+b and y=f(x) give the same information!
y=mx+b is termed a linearized model of f(x) at the Q-point
Small-Signal Principle

How can $y = mx + b$ be obtained from $y = f(x)$?
How can $m$ and $b$ be obtained from $f(x)$?
Small-Signal Principle

\[ \begin{align*}
\frac{y - y_Q}{x - x_Q} &= \left. \frac{\partial f}{\partial x} \right|_{x=x_Q} \\
y - y_Q &= \left. \frac{\partial f}{\partial x} \right|_{x=x_Q} (x - x_Q) \\
y &= \left[ \left. \frac{\partial f}{\partial x} \right|_{x=x_Q} \right] x + \left[ y_Q - x_Q \left. \frac{\partial f}{\partial x} \right|_{x=x_Q} \right] \\
m &= \left[ \left. \frac{\partial f}{\partial x} \right|_{x=x_Q} \right] \\
b &= \left[ y_Q - x_Q \left. \frac{\partial f}{\partial x} \right|_{x=x_Q} \right]
\end{align*} \]
Changing coordinate systems:

\[
\begin{align*}
    y_{ss} &= y - y_Q \\
    x_{ss} &= x - x_Q
\end{align*}
\]

\[
y - y_Q = \left. \frac{\partial f}{\partial x} \right|_{x=x_Q} (x - x_Q) \quad \rightarrow \quad y_{ss} = \left. \frac{\partial f}{\partial x} \right|_{x=x_Q} x_{ss}
\]
Small-Signal Principle

Small-Signal Model:

\[ y_{ss} = \frac{\partial f}{\partial x} \bigg|_{x=x_Q} x_{ss} \]

- Linearized model for the nonlinear function \( y=f(x) \)
- Valid in the region of the Q-point
- Will show the small signal model is simply Taylor’s series expansion at the Q-point truncated after first-order terms
Small-Signal Principle

Observe:

\[ y - y_Q = \frac{\partial f}{\partial x} \bigg|_{x=x_Q} (x - x_Q) \]

\[ y = y_Q + \frac{\partial f}{\partial x} \bigg|_{x=x_Q} (x - x_Q) \]

Recall Taylor's Series Expansion of nonlinear function \( f \) at expansion point \( x_0 \)

\[ y = f(x_0) + \sum_{k=1}^{\infty} \left( \frac{1}{k!} \frac{d^k f}{dx^k} \bigg|_{x=x_0} \right) (x-x_0)^k \]

Small-Signal Model:

\[ y = f(x_Q) + \frac{\partial f}{\partial x} \bigg|_{x=x_Q} (x - x_Q) \]

\[ y_{ss} = \frac{\partial f}{\partial x} \bigg|_{x=x_Q} x_{ss} \]

Mathematically, linearized model is simply Taylor's series expansion at the Q-point truncated after first-order terms with notation \( x_Q = x_0 \)
Small-Signal Principle

\[ y = f(x_Q) + \frac{\partial f}{\partial x} \bigg|_{x=x_Q} x_{SS} \]

Quiescent Output

ss Gain

How can a circuit be linearized at an operating point as an alternative to linearizing a nonlinear function at an operating point?
Small-Signal Principle

\[ y = \frac{\partial I}{\partial V} \bigg|_{V=V_Q} \]

\[ i_{ss} = \frac{\partial I}{\partial V} \bigg|_{V=V_Q} \]

\[ v_{ss} \]

\[ i_{ss} = i \]

\[ v_{ss} = V \]

\[ y = \frac{\partial I}{\partial V} \bigg|_{V=V_Q} \]

\[ i = y \cdot V \]

Model of the nonlinear device at the Q-point
The small-signal model of this 2-terminal electrical network is a resistor of value $1/y$.

One small-signal parameter characterizes this one-port but it is dependent on Q-point.
Small-Signal Principle

Goal with small signal model is to predict performance of circuit or device in the vicinity of an operating point.

Operating point is often termed Q-point.

Will be extended to functions of two and three variables.
Solution for the example was based upon solving the nonlinear circuit for $V_{\text{OUT}}$ and then linearizing the solution by doing a Taylor’s series expansion.

- Solution of nonlinear equations very involved with two or more nonlinear devices.
- Taylor’s series linearization can get very tedious if multiple nonlinear devices are present.

**Standard Approach to small-signal analysis of nonlinear networks**

1. Solve nonlinear network
2. Linearize solution

**Alternative Approach to small-signal analysis of nonlinear networks**

1. Linearize nonlinear devices
2. Obtain small-signal model from linearized device models
3. Replace all devices with small-signal equivalent
4. Solve linear small-signal network
Alternative Approach to small-signal analysis of nonlinear networks

1. Linearize nonlinear devices

2. Replace all devices with small-signal equivalent

3. Solve linear small-signal network

• Must only develop linearized model once for any nonlinear device
e.g. once for a MOSFET, once for a JFET, and once for a BJT

Linearized model for nonlinear device termed “small-signal model”

derivation of small-signal model for most nonlinear devices is less complicated than solving even one simple nonlinear circuit

• Solution of linear network much easier than solution of nonlinear network
Standard Approach to small-signal analysis of nonlinear networks

1. Linearize nonlinear devices

2. Replace all devices with small-signal equivalent

3. Solve linear small-signal network
Standard Approach to analysis of nonlinear networks

Nonlinear Network

dc Equivalent Network

Q-point

Values for small-signal parameters

Small-signal equivalent network

Small-signal output

Total output
(good approximation)
Standard Approach to small-signal analysis of nonlinear networks

Nonlinear Network

dc Equivalent Network

Q-point

Values for small-signal parameters

Small-signal equivalent network

Small-signal output

Total output

(good approximation)
Linearized nonlinear devices
Example:

Nonlinear network

Linearized small-signal network
Dc and small-signal equivalent elements

<table>
<thead>
<tr>
<th>Element</th>
<th>ss equivalent</th>
<th>dc equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>dc Voltage Source</td>
<td>$V_{DC}$</td>
<td>$V_{DC}$</td>
</tr>
<tr>
<td>ac Voltage Source</td>
<td>$V_{AC}$</td>
<td>$V_{AC}$</td>
</tr>
<tr>
<td>dc Current Source</td>
<td>$I_{DC}$</td>
<td>$I_{DC}$</td>
</tr>
<tr>
<td>ac Current Source</td>
<td>$I_{AC}$</td>
<td>$I_{AC}$</td>
</tr>
<tr>
<td>Resistor</td>
<td>$R$</td>
<td>$R$</td>
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</table>
Dc and small-signal equivalent elements

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<thead>
<tr>
<th>Element</th>
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<th>dc equivalent</th>
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</thead>
<tbody>
<tr>
<td>Capacitors</td>
<td>C Large</td>
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<td>Inductors</td>
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<td>Diodes</td>
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<td>MOS transistors</td>
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Dc and small-signal equivalent elements

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</table>
| Bipolar Transistors
| ![Bipolar Transistor Diagram](image1)
| ![Simplified Bipolar Transistor Diagram](image2)
| ![Simplified Bipolar Transistor Diagram](image3)
| Dependent Sources
| ![Dependent Source Diagram](image4)
| ![Dependent Source Diagram](image5)
| ![Dependent Source Diagram](image6)
How is the small-signal equivalent circuit obtained from the nonlinear circuit?

What is the small-signal equivalent of the MOSFET, BJT, and diode?
Small-Signal Diode Model

A Small Signal Equivalent Circuit

Thus, for the diode

\[ R_d = \left( \frac{\partial |D|}{\partial V_D} \right)^{-1}_Q \]
Small-Signal Diode Model

For the diode

\[ R_d = \left( \frac{\partial I_D}{\partial V_D} \right)_Q^{-1} \]

\[ I_D = I_{Se} \frac{V_D}{V_t} \]

\[ \left. \frac{\partial I_D}{\partial V_t} \right|_Q = \left[ \left( I_{Se} \frac{V_D}{V_t} \right) \frac{1}{V_t} \right]_Q = \frac{I_{DQ}}{V_t} \]

\[ R_d = \frac{V_t}{I_{DQ}} \]
End of Lecture 23
Small-Signal Model

Consider 4-terminal network

\[ I_1 = f_1(V_1, V_2, V_3) \]
\[ I_2 = f_2(V_1, V_2, V_3) \]
\[ I_3 = f_3(V_1, V_2, V_3) \]

Define

\[ i_1 = I_1 - I_{1Q} \]
\[ i_2 = I_2 - I_{2Q} \]
\[ i_3 = I_3 - I_{3Q} \]

\[ v_1 = V_1 - V_{1Q} \]
\[ v_2 = V_2 - V_{2Q} \]
\[ v_3 = V_3 - V_{3Q} \]

Small signal model is that which represents the relationship between the small signal voltages and the small signal currents.
Small-Signal Model

Consider 4-terminal network

Small signal model is that which represents the relationship between the small signal voltages and the small signal currents

For small signals, this relationship should be linear

Can be thought of as a change in coordinate systems from the large signal coordinate system to the small-signal coordinate system

\[
\begin{align*}
  i_1 &= g_1(v_1, v_2, v_3) \\
  i_2 &= g_2(v_1, v_2, v_3) \\
  i_3 &= g_3(v_1, v_2, v_3)
\end{align*}
\]
Recall for a function of one variable

\[ y = f(x) \]

Taylor’s Series Expansion about the point \( x_0 \)

\[ y = f(x) = f(x_0) + \frac{\partial f}{\partial x} \bigg|_{x=x_0} (x-x_0) + \frac{\partial^2 f}{\partial x^2} \bigg|_{x=x_0} \frac{1}{2!} (x-x_0)^2 + ... \]

If \( x-x_0 \) is small

\[ y \approx f(x_0) + \frac{\partial f}{\partial x} \bigg|_{x=x_0} (x-x_0) \]

\[ y \approx y_0 + \frac{\partial f}{\partial x} \bigg|_{x=x_0} (x-x_0) \]
Recall for a function of one variable

\[ y = f(x) \]

If \( x-x_0 \) is small

\[ y \approx y_0 + \frac{\partial f}{\partial x}ight|_{x=x_0} (x-x_0) \]

\[ y - y_0 = \frac{\partial f}{\partial x}ight|_{x=x_0} (x-x_0) \]

If we define the small signal variables as

\[ y = y - y_0 \]

\[ \chi = x - x_0 \]
Recall for a function of one variable

\[ y = f(x) \]

If \(x-x_0\) is small

\[ y - y_0 = \left. \frac{\partial f}{\partial x} \right|_{x=x_0} (x - x_0) \]

If we define the small signal variables as

\[ y = y - y_0 \]

\[ x = x - x_0 \]

Then

\[ y = \left. \frac{\partial f}{\partial x} \right|_{x=x_0} x \]

This relationship is linear!
Consider 4-terminal network

Nonlinear network characterized by 3 functions each functions of 3 variables
Consider now 3 functions each functions of 3 variables

\[
\begin{align*}
I_1 &= f_1(V_1, V_2, V_3) \\
I_2 &= f_2(V_1, V_2, V_3) \\
I_3 &= f_3(V_1, V_2, V_3)
\end{align*}
\]

Define

\[
\tilde{V}_Q = \begin{bmatrix} V_{1Q} \\ V_{2Q} \\ V_{3Q} \end{bmatrix}
\]
Consider now 3 functions each functions of 3 variables

\[
\begin{align*}
    I_1 &= f_1(V_1, V_2, V_3) \\
    I_2 &= f_2(V_1, V_2, V_3) \\
    I_3 &= f_3(V_1, V_2, V_3)
\end{align*}
\]

Define

\[
\bar{V}_Q = \begin{bmatrix} V_{1Q} \\ V_{2Q} \\ V_{3Q} \end{bmatrix}
\]

\[
\begin{align*}
    I_1 &= f_1(V_1, V_2, V_3) = f_1(V_{1Q}, V_{2Q}, V_{3Q}) + \\
    &\quad \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_1} \right|_{V=V_a} (V_1 - V_{1Q}) + \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_2} \right|_{V=V_a} (V_2 - V_{2Q}) + \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_3} \right|_{V=V_a} (V_3 - V_{3Q}) \\
    I_1 - I_{1Q} &= \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_1} \right|_{V=V_a} (V_1 - V_{1Q}) + \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_2} \right|_{V=V_a} (V_2 - V_{2Q}) + \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_3} \right|_{V=V_a} (V_3 - V_{3Q})
\end{align*}
\]
Consider now 3 functions each functions of 3 variables

\[ I_1 - I_{1Q} = \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_1} \right|_{\bar{V} = \bar{V}_Q} (V_1 - V_{1Q}) + \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_2} \right|_{\bar{V} = \bar{V}_Q} (V_2 - V_{2Q}) + \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_3} \right|_{\bar{V} = \bar{V}_Q} (V_3 - V_{3Q}) \]

\[ y_{11} = \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_1} \right|_{\bar{V} = \bar{V}_Q} \]

\[ i_1 = I_1 - I_{1Q} \]

\[ y_{12} = \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_2} \right|_{\bar{V} = \bar{V}_Q} \]

\[ i_2 = I_2 - I_{2Q} \]

\[ y_{13} = \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_3} \right|_{\bar{V} = \bar{V}_Q} \]

\[ i_3 = I_3 - I_{3Q} \]

\[ y_{21} = \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_1} \right|_{\bar{V} = \bar{V}_Q} \]

\[ v_1 = V_1 - V_{1Q} \]

\[ y_{22} = \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_2} \right|_{\bar{V} = \bar{V}_Q} \]

\[ v_2 = V_2 - V_{2Q} \]

\[ y_{23} = \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_3} \right|_{\bar{V} = \bar{V}_Q} \]

\[ v_3 = V_3 - V_{3Q} \]
Consider now 3 functions each functions of 3 variables

\[ I_1 - I_{1Q} = \left. \frac{\partial f_1(v_1, v_2, v_3)}{\partial v_1} \right|_{v = v_q} (v_1 - v_{1Q}) + \left. \frac{\partial f_1(v_1, v_2, v_3)}{\partial v_2} \right|_{v = v_q} (v_2 - v_{2Q}) + \left. \frac{\partial f_1(v_1, v_2, v_3)}{\partial v_3} \right|_{v = v_q} (v_3 - v_{3Q}) \]

\[ i_1 = y_{11} v_1 + y_{12} v_2 + y_{13} v_3 \]

This is now a linear relationship between the small signal electrical variables
Consider now 3 functions each functions of 3 variables

\[ i_1 = y_{11} v_1 + y_{12} v_2 + y_{13} v_3 \]

Let's now extend this to \( I_2 \) and \( I_3 \)

Define

\[ y_{ij} = \left. \frac{\partial f_i(v_1, v_2, v_3)}{\partial v_j} \right|_{v=v_q} \]

\[ i_1 = y_{11} v_1 + y_{12} v_2 + y_{13} v_3 \]

\[ i_2 = y_{21} v_1 + y_{22} v_2 + y_{23} v_3 \]

\[ i_3 = y_{31} v_1 + y_{32} v_2 + y_{33} v_3 \]

This is a small-signal model of a 4-terminal network and it is linear
9 small-signal parameters characterize the linear 4-terminal network
Small-signal model parameters dependent upon Q-point!
A small-signal equivalent circuit of a 4-terminal nonlinear network

\[ y_{ij} = \left. \frac{\partial f_i(V_1, V_2, V_3)}{\partial V_j} \right|_{V=\tilde{V}_Q} \]

Equivalent circuit is not unique
4-terminal small-signal network summary

\[ I_1 = f_1(V_1, V_2, V_3) \]
\[ I_2 = f_2(V_1, V_2, V_3) \]
\[ I_3 = f_3(V_1, V_2, V_3) \]

Small signal model:

\[ i_1 = y_{11} u_1 + y_{12} u_2 + y_{13} u_3 \]
\[ i_2 = y_{21} u_1 + y_{22} u_2 + y_{23} u_3 \]
\[ i_3 = y_{31} u_1 + y_{32} u_2 + y_{33} u_3 \]

\[ y_{ij} = \left. \frac{\partial f_i(V_1, V_2, V_3)}{\partial V_j} \right|_{V=V_Q} \]
Consider 3-terminal network

Small-Signal Model

\[
\begin{align*}
\mathbf{i}_1 &= g_1(v_1, v_2, v_3) \\
\mathbf{i}_2 &= g_2(v_1, v_2, v_3) \\
\mathbf{i}_3 &= g_3(v_1, v_2, v_3)
\end{align*}
\]

\[
\begin{align*}
\mathbf{i}_1 &= y_{11} v_1 + y_{12} v_2 + y_{13} v_3 \\
\mathbf{i}_2 &= y_{21} v_1 + y_{22} v_2 + y_{23} v_3 \\
\mathbf{i}_3 &= y_{31} v_1 + y_{32} v_2 + y_{33} v_3
\end{align*}
\]

\[
y_{ij} = \left. \frac{\partial f_i(v_1, v_2, v_3)}{\partial v_j} \right|_{v=v_q}
\]
Consider 3-terminal network

Small-Signal Model

\[
\begin{align*}
I_1 &= f_1(V_1, V_2) \\
I_2 &= f_2(V_1, V_2)
\end{align*}
\]

Define

\[
\begin{align*}
I_{1} &= I_1 - I_{1Q} \\
I_{2} &= I_2 - I_{2Q} \\
V_1 &= V_1 - V_{1Q} \\
V_2 &= V_2 - V_{2Q}
\end{align*}
\]

Small signal model is that which represents the relationship between the small signal voltages and the small signal currents.
Consider 3-terminal network

Small-Signal Model

\[ i_1 = y_{11} V_1 + y_{12} V_2 \]
\[ i_2 = y_{21} V_1 + y_{22} V_2 \]

A Small Signal Equivalent Circuit

4 small-signal parameters characterize this 3-terminal (two-port) linear network
Small signal parameters dependent upon Q-point
3-terminal small-signal network summary

\begin{align*}
I_1 &= f_1(V_1, V_2) \\
I_2 &= f_2(V_1, V_2)
\end{align*}

Small signal model:

\begin{align*}
i_1 &= y_{11}V_1 + y_{12}V_2 \\
i_2 &= y_{21}V_1 + y_{22}V_2
\end{align*}

\[y_{ij} = \left. \frac{\partial f_i(V_1, V_2)}{\partial V_j} \right|_{V = \bar{V}_Q} \]
Consider 2-terminal network

**Small-Signal Model**

\[
\begin{align*}
i_1 &= g_1(v_1, v_2, v_3) \\
i_2 &= g_2(v_1, v_2, v_3) \\
i_3 &= g_3(v_1, v_2, v_3)
\end{align*}
\]

\[\mathbf{y}_{ij} = \left. \frac{\partial f_i(v_1, v_2, v_3)}{\partial v_j} \right|_{v_1=v_1, v_2=v_2, v_3=v_3}\]
Consider 2-terminal network

Small-Signal Model

\[ I_1 = f_1(V_1) \]

Define

\[ i_1 = I_1 - I_{1Q} \]
\[ v_1 = V_1 - V_{1Q} \]

Small signal model is that which represents the relationship between the small signal voltages and the small signal currents.
Consider 2-terminal network

Small-Signal Model

\[ i_1 = y_{11} V_1 \]

\[ y_{11} = \frac{\partial f_1 (V_1)}{\partial V_1} \bigg|_{V=V_0} \]

\[ \tilde{V} = V_{1Q} \]

A Small Signal Equivalent Circuit
Small Signal Model of MOSFET

MOSFET is actually a 4-terminal device but for many applications acceptable predictions of performance can be obtained by treating it as a 3-terminal device by neglecting the bulk terminal.

In this course, we have been treating it as a 3-terminal device and in this lecture will develop the small-signal model by treating it as a 3-terminal device.

If considered as a 4-terminal device, the bulk voltage must also be considered and it introduces one additional term to the small signal model which is often either negligibly small or has no effect on circuit performance.
Small Signal Model of MOSFET

Large Signal Model

\[ I_C = 0 \]

3-terminal device

\[ I_D = \begin{cases} 0 & V_{GS} \leq V_T \\ \mu C_{OX} \frac{W}{L} \left( V_{GS} - V_T - \frac{V_{DS}}{2} \right) V_{DS} & V_{GS} \geq V_T, V_{DS} < V_{GS} - V_T \\ \mu C_{OX} \frac{W}{2L} \left( V_{GS} - V_T \right)^2 \left( 1 + \lambda V_{DS} \right) & V_{GS} \geq V_T, V_{DS} \geq V_{GS} - V_T \end{cases} \]

MOSFET is usually operated in saturation region in linear applications where a small-signal model is needed so will develop the small-signal model in the saturation region.
Small Signal Model of MOSFET

\[ I_1 = f_1 (V_1, V_2) \quad \rightarrow \quad I_G = 0 \]

\[ I_2 = f_2 (V_1, V_2) \quad \rightarrow \quad I_D = \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS}) \]

Small-signal model:

\[ y_{ij} = \left. \frac{\partial f_i (V_1, V_2)}{\partial V_j} \right|_{V_i = V_Q} \]

\[ y_{11} = \left. \frac{\partial I_G}{\partial V_{GS}} \right|_{V = V_Q} \]

\[ y_{12} = \left. \frac{\partial I_G}{\partial V_{DS}} \right|_{V = V_Q} \]

\[ y_{21} = \left. \frac{\partial I_D}{\partial V_{GS}} \right|_{V = V_Q} \]

\[ y_{22} = \left. \frac{\partial I_D}{\partial V_{DS}} \right|_{V = V_Q} \]
Small Signal Model of MOSFET

\[ I_1 = f_1(V_1, V_2) \quad \iff \quad I_G = 0 \]

\[ I_2 = f_2(V_1, V_2) \quad \iff \quad I_D = \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS}) \]

**Small-signal model:**

\[
y_{11} = \left. \frac{\partial I_G}{\partial V_{GS}} \right|_{V=V_Q} = 0
\]

\[
y_{12} = \left. \frac{\partial I_G}{\partial V_{DS}} \right|_{V=V_Q} = 0
\]

\[
y_{21} = \left. \frac{\partial I_D}{\partial V_{GS}} \right|_{V=V_Q} = 2\mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^1 (1 + \lambda V_{DS}) \]

\[
y_{21} \approx \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T)
\]

\[
y_{22} = \left. \frac{\partial I_D}{\partial V_{DS}} \right|_{V=V_Q} = \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 \lambda \approx \lambda I_{DQ}
\]
Small Signal Model of MOSFET

\[ I_G = 0 \]

\[ I_D = \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS}) \]

\[ y_{12} = 0 \]
\[ y_{11} = 0 \]
\[ y_{21} \approx \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \]
\[ y_{22} \approx \lambda I_{DQ} \]

\[ i_G = y_{11} V_{GS} + y_{12} V_{DS} \]
\[ i_D = y_{21} V_{GS} + y_{22} V_{DS} \]
Small Signal Model of MOSFET

by convention, \( y_{21} = g_m, y_{22} = g_0 \)

\[ y_{21} \cong g_m = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_t) \]

\[ y_{22} = g_o \cong \lambda I_{DQ} \]
Small Signal Model of MOSFET

\[ g_m = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \]

\[ g_o \approx \lambda I_{DQ} \]

Alternate equivalent expressions:

\[ I_{DQ} = \mu C_{ox} \frac{W}{2L} (V_{GSQ} - V_T)^2 \left(1 + \lambda V_{DSQ}\right) \approx \mu C_{ox} \frac{W}{2L} (V_{GSQ} - V_T)^2 \]

\[ g_m = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \]

\[ g_m = \sqrt{2 \mu C_{ox} \frac{W}{L}} \cdot \sqrt{I_{DQ}} \]

\[ g_m = \frac{2I_{DQ}}{V_{GSQ} - V_T} \]
Consider again:

**Small signal analysis example**

\[ V_{IN} = V_M \sin \omega t \]

\[ A_v = \frac{2I_{DQ} R}{[V_{SS} + V_T]} \]

Derived for \( \lambda = 0 \)

\[ I_{DQ} = \mu C_{ox} \frac{W}{2L} (V_{GSQ} - V_T)^2 \]
Consider again:

**Small signal analysis example**

\[
A_v = \frac{V_{\text{OUT}}}{V_{\text{IN}}} = -\frac{g_m}{g_o + 1/R}
\]

For \(\lambda=0\), \(g_o = \lambda_{\text{IDQ}} = 0\)

\[
A_v = \frac{V'_{\text{OUT}}}{V'_{\text{IN}}} = -g_m R
\]

but

\[
g_m = \frac{2I_{\text{DQ}}}{V_{\text{GSQ}} - V_T}
\]

\(V_{\text{GSQ}} = -V_{\text{SS}}\)

thus

\[
A_v = \frac{2I_{\text{DQ}} R}{\left[ V_{\text{SS}} + V_T \right]}
\]
Consider again:

Small signal analysis example

For $\lambda = 0$, $g_o = \lambda_{IDQ} = 0$

$$A_v = \frac{V_{OUT}}{V_{IN}} = -\frac{g_m}{g_o + 1/R}$$

Same expression as derived before

More accurate gain can be obtained if $\lambda$ effects are included and does not significantly increase complexity of small signal analysis
Small Signal Model of BJT

3-terminal device

Forward Active Model:

\[ I_C = J_S A_E \frac{V_{BE}}{V_t} e^{\frac{V_{CE}}{V_{AF}}} \left(1 + \frac{V_{CE}}{V_{AF}}\right) \]

\[ I_B = \frac{J_S A_E}{\beta} e^{\frac{V_{BE}}{V_t}} \]

Usually operated in Forward Active Region when small-signal model is needed
Small Signal Model of BJT

\[ I_1 = f_1(V_1, V_2) \quad \leftrightarrow \quad I_B = \frac{J_S A_E}{\beta} e^{\frac{V_{BE}}{V_t}} \]

\[ I_2 = f_2(V_1, V_2) \quad \leftrightarrow \quad I_C = J_S A_E e^{\frac{V_{BE}}{V_t}} \left( 1 + \frac{V_{CE}}{V_{AF}} \right) \]

Small-signal model:

\[ y_{ij} = \left. \frac{\partial f_i(V_1, V_2)}{\partial V_j} \right|_{V=V_Q} \]

\[ y_{11} = g_m = \left. \frac{\partial I_B}{\partial V_{BE}} \right|_{V=V_Q} \]

\[ y_{12} = \left. \frac{\partial I_B}{\partial V_{CE}} \right|_{V=V_Q} \]

\[ y_{21} = g_m = \left. \frac{\partial I_C}{\partial V_{BE}} \right|_{V=V_Q} \]

\[ y_{22} = g_o = \left. \frac{\partial I_C}{\partial V_{CE}} \right|_{V=V_Q} \]
Small Signal Model of BJT

\[ I_B = \frac{J_s A_E}{\beta} e^{\frac{V_{BE}}{V_t}} \]

\[ I_C = J_s A_E e^{\frac{V_{BE}}{V_t}} \left( 1 + \frac{V_{CE}}{V_{AF}} \right) \]

**Small-signal model:**

\[ g_n = \frac{\partial I_B}{\partial V_{BE}} \bigg|_{V=V_o} = \frac{1}{V} \frac{J_s A_E}{\beta} e^{\frac{V_{BE}}{V_t}} \bigg|_{V=V_o} = \frac{I_{BO}}{V_t} \approx \frac{I_{CO}}{\beta V_t} \]

\[ y_{12} = \frac{\partial I_B}{\partial V_{CE}} \bigg|_{V=V_o} = 0 \]

\[ y_{21} = g_m = \frac{\partial I_C}{\partial V_{BE}} \bigg|_{V=V_o} = \frac{1}{V} J_s A_E e^{\frac{V_{BE}}{V_t}} \left( 1 + \frac{V_{CE}}{V_{AF}} \right) \bigg|_{V=V_o} = \frac{I_{CO}}{V_t} \]

\[ y_{22} = g_o = \frac{\partial I_C}{\partial V_{CE}} \bigg|_{V=V_o} = \frac{J_s A_E}{V_{AF}} e^{\frac{V_{BE}}{V_t}} \bigg|_{V=V_o} \approx \frac{I_{CO}}{V_{AF}} \]
Small Signal Model of BJT

\[ i_B = y_{11} V_{BE} + y_{12} V_{CE} \]
\[ i_C = y_{21} V_{BE} + y_{22} V_{CE} \]

\[ i_B = g_\pi V_{BE} \]
\[ i_C = g_m V_{BE} + g_o V_{CE} \]

\[ g_\pi = \frac{I_{CO}}{\beta V_t} \quad g_m = \frac{I_{CO}}{V_t} \quad g_o = \frac{I_{CO}}{V_{AF}} \]
Active Device Model Summary

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What are the simplified dc equivalent models?
Active Device Model Summary

What are the simplified dc equivalent models?

dc equivalent

0.6V

\[ 0.6V \]
Small-signal Operation of Nonlinear Circuits

- Small-signal principles
- Example Circuit
- Small-Signal Models

Small-Signal Analysis of Nonlinear Circuits
Recall:

**Alternative Approach to small-signal analysis of nonlinear networks**

1. **Linearize nonlinear devices**  
   *(have small-signal model for key devices!)*

2. **Replace all devices with small-signal equivalent**

3. **Solve linear small-signal network**
Example:

Determine the small signal voltage gain $A_V = \frac{V_{OUT}}{V_{IN}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$. 
Example: Determine the small signal voltage gain $A_V = \frac{V_{OUT}}{V_{IN}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$
Example: Determine the small signal voltage gain $A_V = \frac{v_{OUT}}{v_{IN}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$. 

Small-signal circuit

Small-signal MOSFET model for $\lambda = 0$
Example: Determine the small signal voltage gain $A_V = \frac{v_{OUT}}{v_{IN}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$. 

Small-signal circuit
Example:

![Small-signal circuit diagram](image)

**Analysis:**

*By KCL*

\[ g_{m1} v_{GS1} = g_{m2} v_{GS2} \]

*but*

\[ v_{GS1} = v_{IN} \]

\[ -v_{GS2} = v_{OUT} \]

*thus:*

\[ A_v = \frac{v_{OUT}}{v_{IN}} = -\frac{g_{m1}}{g_{m2}} \]
Example:

Small-signal circuit

\[ A_v = \frac{\mathbf{v}_{\text{OUT}}}{\mathbf{v}_{\text{IN}}} = -g_{m1} \]

Recall:
\[ g_m = -\sqrt{2I_D\mu C_{ox}} \left( \frac{W_1}{L_1} \right) \]

\[ A_v = -\sqrt{\frac{2I_D\mu C_{ox}}{L_2}} \left( \frac{W_1}{L_1} \right) = -\sqrt{\frac{W_1}{W_2}} \sqrt{\frac{L_2}{L_1}} \]
Example:

\[ V_{IN} \quad \pm \quad V_{GS1} \quad g_{m1} \quad V_{GS1} \quad V_{GS2} \quad g_{m2} \quad V_{GS2} \quad V_{OUT} \]

**Small-signal circuit**

\[ A_v = \frac{V_{OUT}}{V_{IN}} = -\frac{g_{m1}}{g_{m2}} \]

**Recall:**

\[ A_v = -\sqrt{\frac{W_1}{W_2}} \sqrt{\frac{L_2}{L_1}} \]

*If* \( L_1 = L_2 \), *obtain*

\[ A_v = -\sqrt{\frac{W_1}{W_2}} \sqrt{\frac{L_2}{L_1}} = -\sqrt{\frac{W_1}{W_2}} \]

*The width and length ratios can be accurately set when designed in a standard CMOS process.*
Small Signal Model of MOSFET  
(as a 4-terminal device)

\[
\begin{align*}
I_G &= 0 \\
I_B &= 0 \\
I_D &= \begin{cases} 
0 & V_{GS} \leq V_T \\
\mu C_{OX} \frac{W}{L} \left( V_{GS} - V_T - \frac{V_{DS}}{2} \right) V_{DS} & V_{GS} \geq V_T, V_{DS} < V_{GS} - V_T \\
\mu C_{OX} \frac{W}{2L} \left( V_{GS} - V_T \right)^2 \left(1 + \lambda V_{DS}\right) & V_{GS} \geq V_T, V_{DS} \geq V_{GS} - V_T
\end{cases}
\]

\[V_T = V_{T0} + \gamma \left( \sqrt{\phi - V_{BS}} - \sqrt{\phi} \right)\]

MOSFET is usually operated in saturation region in linear applications where a small-signal model is needed so will develop the small-signal model in the saturation region.
Small Signal Model of
MOSFET (as 4-terminal device)

\[ I_1 = f_1(V_1, V_2, V_3) \quad \leftrightarrow \quad I_G = 0 \]

\[ I_2 = f_2(V_1, V_2, V_3) \quad \leftrightarrow \quad I_D = \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DSQ}) \]

\[ I_3 = f_3(V_1, V_2, V_3) \quad \leftrightarrow \quad I_B = 0 \]

\[ V_t = V_{T0} + \gamma(\sqrt{\varphi} - V_{BS} - \sqrt{\varphi}) \]

Small-signal model:

\[ y_{11} = \left. \frac{\partial I_G}{\partial V_{GS}} \right|_{V=V_0} = 0 \quad y_{12} = \left. \frac{\partial I_G}{\partial V_{DS}} \right|_{V=V_0} = 0 \quad y_{13} = \left. \frac{\partial I_G}{\partial V_{BS}} \right|_{V=V_0} = 0 \]

\[ y_{21} = \left. \frac{\partial I_D}{\partial V_{GS}} \right|_{V=V_0} = 2\mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS}) \bigg|_{V=V_0} = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) (1 + \lambda V_{DSQ}) \]

\[ y_{21} \approx \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \]

\[ y_{22} = \left. \frac{\partial I_D}{\partial V_{DS}} \right|_{V=V_0} = \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 \lambda \bigg|_{V=V_0} \approx \lambda I_{DQ} \]
Small Signal Model of MOSFET
(as 4-terminal device)

\[ I_1 = f_1(V_1, V_2, V_3) \leftrightarrow I_G = 0 \]
\[ I_2 = f_2(V_1, V_2, V_3) \leftrightarrow I_D = \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS}) \]
\[ I_3 = f_3(V_1, V_2, V_3) \leftrightarrow I_B = 0 \]

Small-signal model:

\[ V_t = V_{T0} + \gamma \left( \sqrt{\varphi - V_{BS}} - \sqrt{\varphi} \right) \]

\[ y_{23} = \left. \frac{\partial I_D}{\partial V_{BS}} \right|_{V = V_o} = \mu C_{ox} \frac{W}{2L} 2 (V_{GS} - V_T) \left. \frac{\partial V_T}{\partial V_{BS}} (1 + \lambda V_{DS}) \right|_{V = V_o} \]
\[ y_{23} = \left. \frac{\partial I_D}{\partial V_{BS}} \right|_{V = V_o} = \mu C_{ox} \frac{W}{2L} 2 (V_{GS} - V_T) \left( \gamma \frac{1}{2} (\varphi - V_{BS})^{\frac{1}{2}} (-1) \right) (1 + \lambda V_{DS}) \]

\[ y_{23} = \left. \frac{\partial I_D}{\partial V_{BS}} \right|_{V = V_o} = y_{21} \left( \frac{\gamma}{2 \sqrt{\varphi - V_{BS}}} \right) \]

\[ y_{31} = \left. \frac{\partial I_B}{\partial V_{GS}} \right|_{V = V_o} = 0 \]
\[ y_{32} = \left. \frac{\partial I_B}{\partial V_{DS}} \right|_{V = V_o} = 0 \]
\[ y_{33} = \left. \frac{\partial I_B}{\partial V_{BS}} \right|_{V = V_o} = 0 \]
Small Signal Model of MOSFET
(as a 4-terminal device)

by convention, \( y_{21} = g_m, \ y_{22} = g_0, \ y_{23} = g_{mb} \)

\[
\therefore \quad y_{21} \approx g_m = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_I) \\
\quad y_{22} = g_o \approx \lambda I_{DQ} \\
y_{23} = g_{mb} = g_m \left( \frac{\gamma}{2\sqrt{\varphi - V_{BSQ}}} \right)
\]

but often either \( g_{mb} \) is small or \( V_{BS} = 0 \) or both