EE 330
Lecture 23

• Small Signal Analysis
  – SS Models for MOSFET
  – SS Models for BJT
Exam Schedule

Exam 2       Friday October 21
Review from Last Lecture

Small-signal and simplified dc equivalent elements

<table>
<thead>
<tr>
<th>Element</th>
<th>ss equivalent</th>
<th>Simplified dc equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>dc Voltage Source</td>
<td>$V_{DC}$</td>
<td>$V_{DC}$</td>
</tr>
<tr>
<td>ac Voltage Source</td>
<td>$V_{AC}$</td>
<td>$V_{AC}$</td>
</tr>
<tr>
<td>dc Current Source</td>
<td>$I_{DC}$</td>
<td>$I_{DC}$</td>
</tr>
<tr>
<td>ac Current Source</td>
<td>$I_{AC}$</td>
<td>$I_{AC}$</td>
</tr>
<tr>
<td>Resistor</td>
<td>$R$</td>
<td>$R$</td>
</tr>
</tbody>
</table>

\[ V_{DC} \quad V_{AC} \quad I_{DC} \quad I_{AC} \quad R \]
Small-signal and simplified dc equivalent elements

- **Capacitors**
  - Large: $C$
  - Small: $C$

- **Inductors**
  - Large: $L$
  - Small: $L$

- **Diodes**

- **MOS transistors**
  - (MOSFET (enhancement or depletion), JFET)

Review from Last Lecture
Small-signal and simplified dc equivalent elements

- **Bipolar Transistors**
  - $V_O = A_V V_{IN}$
  - $I_O = A_I I_{IN}$

- **Dependent Sources (Linear)**
  - $V_O = R_I I_{IN}$
  - $I_O = G_I V_{IN}$

Review from Last Lecture
Small signal model is that which represents the relationship between the small signal voltages and the small signal currents.

Consider 4-terminal network

\[
\begin{align*}
I_1 &= f_1(V_1, V_2, V_3) \\
I_2 &= f_2(V_1, V_2, V_3) \\
I_3 &= f_3(V_1, V_2, V_3)
\end{align*}
\]

Define

\[
\begin{align*}
i_1 &= I_1 - I_{1Q} \\
i_2 &= I_2 - I_{2Q} \\
i_3 &= I_3 - I_{3Q}
\end{align*}
\]

\[
\begin{align*}
u_1 &= V_1 - V_{1Q} \\
u_2 &= V_2 - V_{2Q} \\
u_3 &= V_3 - V_{3Q}
\end{align*}
\]
Review from Last Lecture

**Consider 4-terminal network**

![4-Terminal Device Diagram](image)

\[
\begin{align*}
I_1 &= f_1(V_1, V_2, V_3) \\
I_2 &= f_2(V_1, V_2, V_3) \\
I_3 &= f_3(V_1, V_2, V_3)
\end{align*}
\]

Nonlinear network characterized by 3 functions each functions of 3 variables
Small Signal Model Development

**Nonlinear Model**

\[
I_1 = f_1(V_1, V_2, V_3) \quad \Rightarrow \quad i_1 = y_{11}v_1 + y_{12}v_2 + y_{13}v_3
\]

\[
I_2 = f_2(V_1, V_2, V_3) \quad \Rightarrow \quad i_2 = y_{21}v_1 + y_{22}v_2 + y_{23}v_3
\]

\[
I_3 = f_3(V_1, V_2, V_3) \quad \Rightarrow \quad i_3 = y_{31}v_1 + y_{32}v_2 + y_{33}v_3
\]

**Linear Model at** \( \tilde{V}_q \) (alt. small signal model)

where

\[
y_{ij} = \left. \frac{\partial f_i(V_1, V_2, V_3)}{\partial V_j} \right|_{\tilde{V} = \tilde{V}_q}
\]
Small Signal Model

\[ i_1 = y_{11}v_1 + y_{12}v_2 + y_{13}v_3 \]
\[ i_2 = y_{21}v_1 + y_{22}v_2 + y_{23}v_3 \]
\[ i_3 = y_{31}v_1 + y_{32}v_2 + y_{33}v_3 \]

where

\[ y_{ij} = \left. \frac{\partial f_i(V_1, V_2, V_3)}{\partial V_j} \right|_{\vec{V} = \bar{V}_Q} \]

- This is a small-signal model of a 4-terminal network and it is linear
- 9 small-signal parameters characterize the linear 4-terminal network
- Small-signal model parameters dependent upon Q-point!
Review from Last Lecture

4-terminal small-signal network summary

$$I_1 = f_1(V_1, V_2, V_3)$$
$$I_2 = f_2(V_1, V_2, V_3)$$
$$I_3 = f_3(V_1, V_2, V_3)$$

Small signal model:

$$i_1 = y_{11}v_1 + y_{12}v_2 + y_{13}v_3$$
$$i_2 = y_{21}v_1 + y_{22}v_2 + y_{23}v_3$$
$$i_3 = y_{31}v_1 + y_{32}v_2 + y_{33}v_3$$

$$y_{ij} = \frac{\partial f_i(V_1, V_2, V_3)}{\partial V_j} \bigg|_{\bar{v} = \bar{v}_q}$$
Small-Signal Model

Review from Last Lecture
Consider 2-terminal network

\[ i_1 = g_1(v_1, v_2, v_3) \]
\[ i_2 = g_2(v_1, v_2, v_3) \]
\[ i_3 = g_3(v_1, v_2, v_3) \]

\[ i_1 = y_{11}v_1 + y_{12}v_2 + y_{13}v_3 \]
\[ i_2 = y_{21}v_1 + y_{22}v_2 + y_{23}v_3 \]
\[ i_3 = y_{31}v_1 + y_{32}v_2 + y_{33}v_3 \]

\[ y_{ij} = \left. \frac{\partial f_i(v_1, v_2, v_3)}{\partial v_j} \right|_{\bar{v}=\bar{v}_q} \]
Small Signal Model of MOSFET

Large Signal Model

\[ I_G = 0 \]

\[ I_D = \begin{cases} 
0 & V_{GS} \leq V_T \\
\mu C_{ox} \frac{W}{L} \left( V_{GS} - V_T - \frac{V_{DS}}{2} \right) V_{DS} & V_{GS} > V_T, V_{DS} < V_{GS} - V_T \\
\mu C_{ox} \frac{W}{2L} \left( V_{GS} - V_T \right)^2 (1 + \lambda V_{DS}) & V_{GS} > V_T, V_{DS} \geq V_{GS} - V_T 
\end{cases} \]

MOSFET is usually operated in saturation region in linear applications where a small-signal model is needed so will develop the small-signal model in the saturation region.
Small Signal Model of MOSFET

\[ I_D = \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS}) \]

\[ y_{11} = 0 \]
\[ y_{12} = 0 \]
\[ y_{21} = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \]
\[ y_{22} = \lambda I_{DQ} \]

\[ i_G = y_{11} V_{GS} + y_{12} V_{DS} \]
\[ i_D = y_{21} V_{GS} + y_{22} V_{DS} \]
Small Signal Model of MOSFET

by convention, \( y_{21} = g_m, \ y_{22} = g_0 \)

\[
delimsn \therefore y_{21} \simeq g_m = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \\
y_{22} = g_o \simeq \lambda I_{DQ}
\]

Note: \( g_o \) vanishes when \( \lambda = 0 \)
Small Signal Model of MOSFET

$$g_m = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T)$$

$$g_o \equiv I_{DQ}$$

Alternate equivalent expressions for $g_m$:

$$I_{DQ} = \mu C_{ox} \frac{W}{2L} (V_{GSQ} - V_T)^2 (1 + \lambda V_{DSQ}) \approx \mu C_{ox} \frac{W}{2L} (V_{GSQ} - V_T)^2$$

$$g_m = \sqrt{2\mu C_{ox} \frac{W}{L} \cdot \sqrt{I_{DQ}}}$$

$$g_m = \frac{2I_{DQ}}{V_{GSQ} - V_T}$$
Consider again:

**Small signal analysis example**

\[ v_{IN} = V_M \sin \omega t \]

\[ v_{OUT} \]

\[ V_{DD} \]

\[ R \]

\[ M_1 \]

\[ V_{SS} \]

\[ A_v = \frac{2I_{DQ} R}{[V_{SS} + V_T]} \]

Derived for \( \lambda = 0 \)

\[ I_D = \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 \]

Recall the derivation was very tedious and time consuming!
Consider again:

Small signal analysis example

\[ A_v = \frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{g_m}{g_o + 1/R} \]

For \( \lambda = 0 \), \( g_o = \lambda I_{DQ} = 0 \)

\[ A_v = \frac{V_{\text{out}}}{V_{\text{in}}} = -g_m R \]

but

\[ g_m = \frac{2I_{DQ}}{V_{GSQ} - V_T} \]

\[ V_{GSQ} = -V_{SS} \]

thus

\[ A_v = \frac{2I_{DQ} R}{V_{SS} + V_T} \]
Consider again:

Small signal analysis example

For $\lambda=0$, $g_O = \lambda I_{DQ} = 0$

More accurate gain can be obtained if $\lambda$ effects are included and does not significantly increase complexity of small signal analysis.
Small Signal Model of BJT

3-terminal device

Forward Active Model:

\[ I_C = J_S A_E e^{\frac{V_{BE}}{V_t}} \left( 1 + \frac{V_{CE}}{V_{AF}} \right) \]

\[ I_B = \frac{J_S A_E}{\beta} e^{\frac{V_{BE}}{V_t}} \]

Usually operated in Forward Active Region when small-signal model is needed
Small Signal Model of BJT

**Nonlinear model:**

\[ I_1 = f_1(V_1, V_2) \quad \longleftrightarrow \quad I_B = \frac{J_SA_E}{\beta} e^{\frac{V_{BE}}{V_t}} \]

\[ I_2 = f_2(V_1, V_2) \quad \longleftrightarrow \quad I_C = J_SA_E e^{\frac{V_{BE}}{V_t}} \left(1 + \frac{V_{CE}}{V_{AF}}\right) \]

**Small-signal model:**

\[ i_B = y_{11} V_{BE} + y_{12} V_{CE} \]

\[ i_C = y_{21} V_{BE} + y_{22} V_{CE} \]

\[ y_{ij} = \left. \frac{\partial f_i(V_1, V_2)}{\partial V_j} \right|_{V=V_0} \]

\[ y_{11} = g_{\pi} = \left. \frac{\partial I_B}{\partial V_{BE}} \right|_{V=V_0} \]

\[ y_{12} = \left. \frac{\partial I_B}{\partial V_{CE}} \right|_{V=V_0} \]

\[ y_{21} = g_m = \left. \frac{\partial I_C}{\partial V_{BE}} \right|_{V=V_0} \]

\[ y_{22} = g_o = \left. \frac{\partial I_C}{\partial V_{CE}} \right|_{V=V_0} \]

Note: \( g_m, g_{\pi} \) and \( g_o \) used for notational consistency with legacy terminology
**Small Signal Model of BJT**

**Nonlinear model:**

\[
I_B = \frac{J_S A_E}{\beta} e^{\frac{V_{BE}}{V_t}}
\]

\[
I_C = J_S A_E e^{\frac{V_{BE}}{V_t}} \left( 1 + \frac{V_{CE}}{V_{AF}} \right)
\]

**Small-signal model:**

\[
i_B = y_{11} V_{BE} + y_{12} V_{CE}
\]

\[
i_C = y_{21} V_{BE} + y_{22} V_{CE}
\]

\[
y_{ij} = \frac{\partial f_i(V_1, V_2)}{\partial V_j} \bigg|_{V_i = V_o}
\]

\[
y_{11} = g_x = \frac{\partial I_B}{\partial V_{BE}} \bigg|_{V_i = V_o} = ?
\]

\[
y_{12} = \frac{\partial I_B}{\partial V_{CE}} \bigg|_{V_i = V_o} = ?
\]

\[
y_{21} = g_m = \frac{\partial I_C}{\partial V_{BE}} \bigg|_{V_i = V_o} = ?
\]

\[
y_{22} = g_o = \frac{\partial I_C}{\partial V_{CE}} \bigg|_{V_i = V_o} = ?
\]
Small Signal Model of BJT

Nonlinear model:

\[ I_B = \frac{J_S A_E}{\beta} e^{\frac{v_{BE}}{V_t}} \]
\[ I_C = J_S A_E e^{\frac{v_{BE}}{V_t}} \left(1 + \frac{v_{CE}}{V_{AF}}\right) \]

\[ i_B = y_{11} v_{BE} + y_{12} v_{CE} \]
\[ i_C = y_{21} v_{BE} + y_{22} v_{CE} \]

Small-signal model:

\[ y_{11} = g_e = \left. \frac{\partial I_B}{\partial v_{BE}} \right|_{v_{BE}=0} = \frac{J_S A_E}{\beta} e^{\frac{v_{BE}}{V_t}} \approx \frac{I_{BO}}{V_t} \]
\[ y_{21} = g_m = \left. \frac{\partial I_C}{\partial v_{BE}} \right|_{v_{BE}=0} = \frac{1}{V_t} J_S A_E e^{\frac{v_{BE}}{V_t}} \left(1 + \frac{v_{CE}}{V_{AF}}\right) \bigg|_{v_{BE}=0} = \frac{I_{CO}}{V_t} \]

\[ y_{12} = \left. \frac{\partial I_B}{\partial v_{CE}} \right|_{v_{CE}=0} = 0 \]
\[ y_{22} = g_o = \left. \frac{\partial I_C}{\partial v_{CE}} \right|_{v_{CE}=0} = \frac{J_S A_E e^{\frac{v_{BE}}{V_t}}}{V_{AF}} \bigg|_{v_{CE}=V_o} = \frac{I_{CO}}{V_{AF}} \]

Note: usually prefer to express in terms of \( I_{CO} \)
Small Signal Model of BJT

\[ i_B = y_{11} V_{BE} + y_{12} V_{CE} \]
\[ i_C = y_{21} V_{BE} + y_{22} V_{CE} \]

\[ g_\pi = \frac{I_{CQ}}{\beta V_t} \quad g_m = \frac{I_{CQ}}{V_t} \quad g_o = \frac{I_{CQ}}{V_{AF}} \]

An equivalent circuit
Consider again:

**Small signal analysis example**

\[ V_{IN} = V_M \sin \omega t \]

\[ V_{OUT} = \frac{I_C Q R}{V_T} \]

Derived for \( V_{AF} = 0 \)

Recall the derivation was very tedious and time consuming!
Neglect $V_{AF}$ effects (i.e. $V_{AF}=\infty$) to be consistent with earlier analysis

\[
g_o = \frac{I_{CQ}}{V_{AF}} = \frac{0}{V_{AF}} = 0
\]

\[
A_V = \frac{V_{OUT}}{V_{IN}} = -g_mR
\]

\[
g_m = \frac{I_{CQ}}{V_t}
\]

\[
A_V = -\frac{I_{CQ}R}{V_t}
\]

Note this is identical to what was obtained with the direct nonlinear analysis
Small Signal BJT Model – alternate representation

\[ g_m = \frac{I_{CQ}}{V_t} \]

\[ g_\pi = \frac{I_{CQ}}{\beta V_t} \]

\[ g_o \approx \frac{I_{CQ}}{V_{AF}} \]

Observe:

\[ g_\pi v_{be} = i_b \]

\[ g_m v_{be} = i_b \frac{g_m}{g_\pi} \]

\[ g_m = \left[ \begin{array}{c} \frac{I_Q}{V_t} \\ \frac{I_Q}{\beta V_t} \end{array} \right] = \beta \]

Can replace the voltage dependent current source with a current dependent current source
Small Signal BJT Model – alternate representation

Alternate equivalent small signal model

\[ g_m = \frac{I_{CQ}}{V_t} \]
\[ g_\pi = \frac{I_{CQ}}{\beta V_t} \]
\[ g_o \approx \frac{I_{CQ}}{V_{AF}} \]
Small-Signal Model Representations

(3-terminal network – also relevant with 4-terminal networks)

Have developed small-signal models for the MOSFET and BJT

Models have been based upon arbitrary assumption that $v_1, v_2$ are independent variables

Have already seen some alternatives for parameter definitions in these models

Alternative representations are sometimes used
Small-Signal Model Representations

The good, the bad, and the unnecessary!!

what we have developed:

The hybrid parameters:
Small-Signal Model Representations

The z-parameters

\[ v_1 = z_{11}i_1 + z_{12}i_2 \]
\[ v_2 = z_{21}i_1 + z_{22}i_2 \]

The ABCD parameters:

\[ v_1 = Av_2 - Bi_2 \]
\[ i_1 = Cv_2 - Di_2 \]
Small-Signal Model Representations

The S-parameters

The T parameters:
Small-Signal Model Representations

Bi-Linear Relationship between $i_1, i_2, v_1, v_2$

Linear Two Port

The good, the bad, and the **unnecessary** !!

- Equivalent circuits often given for each representation
- All provide identical characterization
- Easy to move from any one to another
Small-Signal Model Representations

The good, the bad, and the unnecessary!!
Small-Signal Model Representations

The good, the bad, and the unnecessary!!

IEEE TRANSACTIONS ON MICROWAVE THEORY AND TECHNIQUES, VOL. 42, NO. 2, FEBRUARY 1994

Conversions Between S, Z, Y, h, ABCD, and T Parameters which are Valid for Complex Source and Load Impedances

Dean A. Frickey, Member, IEEE
## Active Device Model Summary

<table>
<thead>
<tr>
<th>Element</th>
<th>ss equivalent</th>
<th>dc equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diodes</td>
<td></td>
<td>Simplified</td>
</tr>
<tr>
<td>MOS Transistors</td>
<td></td>
<td>Simplified</td>
</tr>
<tr>
<td>Bipolar Transistors</td>
<td></td>
<td>Simplified</td>
</tr>
</tbody>
</table>

What are the simplified dc equivalent models?
Active Device Model Summary

What are the simplified dc equivalent models?

dc equivalent

\[ \text{Simplified} \]

\[ \text{Simplified} \]

\[ \text{Simplified} \]

\[ \text{Simplified} \]

\[ \text{Simplified} \]

\[ \text{Simplified} \]

\[ \text{Simplified} \]

\[ \text{Simplified} \]

\[ \text{Simplified} \]
Small-signal Operation of Nonlinear Circuits

- Small-signal principles
- Example Circuit
- Small-Signal Models

Small-Signal Analysis of Nonlinear Circuits
Recall:

**Alternative Approach to small-signal analysis of nonlinear networks**

1. **Linearize nonlinear devices**  
   
   *(have small-signal model for key devices!)*

2. **Replace all devices with small-signal equivalent**

3. **Solve linear small-signal network**

Remember that the small-signal model is operating point dependent!

Thus need Q-point to obtain values for small signal parameters.
Example:

Determine the small signal voltage gain $A_V = V_{OUT}/V_{IN}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$.
Example:  

Determine the small signal voltage gain \( A_V = \frac{V_{OUT}}{V_{IN}} \). Assume \( M_1 \) and \( M_2 \) are operating in the saturation region and that \( \lambda = 0 \).

Small-signal circuit
Example: Determine the small signal voltage gain $A_V = \frac{V_{OUT}}{V_{IN}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$.
Example: Determine the small signal voltage gain $A_v = \frac{v_{\text{OUT}}}{v_{\text{IN}}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$.
Example:

Small-signal circuit

Analysis:

By KCL

\[ g_{m1} \cdot V_{GS1} = g_{m2} \cdot V' \]

but

\[ V_{GS1} = V_{IN} \]

\[ -V_{GS2} = V'_{OUT} \]

thus:

\[ A_v = \frac{V'_{OUT}}{V_{IN}} = -\frac{g_{m1}}{g_{m2}} \]
Example:

**Small-signal circuit**

\[ A_v = \frac{V_{OUT}}{V_{IN}} = -\frac{g_{m1}}{g_{m2}} \]

**Analysis:**

Recall:

\[ g_m = -\sqrt{2I_D\mu C_{ox}} \sqrt{\frac{W_1}{L_1}} \]

\[ A_v = -\sqrt{\frac{2I_D\mu C_{ox} \frac{W_1}{L_1}}{\sqrt{2I_D\mu C_{ox} \frac{W_2}{L_2}}}} = -\sqrt{\frac{W_1}{W_2} \sqrt{\frac{L_2}{L_1}}} \]
Example:

\[ \begin{align*}
V_{IN} & \quad \Rightarrow \quad V_{GS1} \quad \Rightarrow \quad g_{m1}V_{GS1} \quad \Rightarrow \quad V_{GS2} \quad \Rightarrow \quad g_{m2}V_{GS2} \quad \Rightarrow \quad V_{OUT}
\end{align*} \]

Analysis:

Small-signal circuit

\[ A_v = \frac{V_{OUT}}{V_{IN}} = -\frac{g_{m1}}{g_{m2}} \]

Recall:

\[ A_v = -\sqrt{\frac{W_1}{W_2}} \sqrt{\frac{L_2}{L_1}} \]

If \( L_1 = L_2 \), obtain

\[ A_v = -\sqrt{\frac{W_1}{W_2}} \sqrt{\frac{L_2}{L_1}} = -\sqrt{\frac{W_1}{W_2}} \]

The width and length ratios can be accurately set when designed in a standard CMOS process.
End of Lecture 23