EE 330
Lecture 23

• Small Signal Analysis
  – SS Models for MOSFET
  – SS Models for BJT
Exam Schedule

Exam 2  Friday March 11
Comparison of Gains for MOSFET and BJT Circuits

**BJT**

\[ A_{VB} = - \frac{I_{CQ}}{R_1} \frac{R}{V_t} \]

If \( I_{DQ} R = I_{CQ} R_1 = 2V, V_{SS} + V_T = -1V, V_t = 25mV \)

\[ A_{VB} = - \frac{I_{CQ}}{V_t} R_1 = \frac{2V}{25mV} = -80 \]

**MOSFET**

\[ A_{VM} = \frac{2I_{DQ} R}{V_{SS} + V_T} \]

\[ A_{VM} = \frac{4V}{-1V} = -4 \]

Observe \( A_{VB} >> A_{VM} \)

Due to exponential-law rather than square-law model
Review from Last Lecture

Small-signal and simplified dc equivalent elements

<table>
<thead>
<tr>
<th>Element</th>
<th>ss equivalent</th>
<th>Simplified dc equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>dc Voltage Source</td>
<td>$V_{DC}$</td>
<td>$V_{DC}$</td>
</tr>
<tr>
<td>ac Voltage Source</td>
<td>$V_{AC}$</td>
<td>$V_{AC}$</td>
</tr>
<tr>
<td>dc Current Source</td>
<td>$I_{DC}$</td>
<td>$I_{DC}$</td>
</tr>
<tr>
<td>ac Current Source</td>
<td>$I_{AC}$</td>
<td>$I_{AC}$</td>
</tr>
<tr>
<td>Resistor</td>
<td>$R$</td>
<td>$R$</td>
</tr>
</tbody>
</table>


dc Voltage Source

- Symbol: $V_{DC}$
- Diagram: \( V_{DC} \)
- Simplified: \( V_{DC} \)

ac Voltage Source

- Symbol: $V_{AC}$
- Diagram: \( V_{AC} \)
- Simplified: \( V_{AC} \)

dc Current Source

- Symbol: $I_{DC}$
- Diagram: \( I_{DC} \)
- Simplified: \( I_{DC} \)

ac Current Source

- Symbol: $I_{AC}$
- Diagram: \( I_{AC} \)
- Simplified: \( I_{AC} \)

Resistor

- Symbol: $R$
- Diagram: \( R \)
- Simplified: \( R \)
Small-signal and simplified dc equivalent elements

Element | ss equivalent | Simplified dc equivalent

Capacitors
- Large Capacitors
- Small Capacitors

Inductors
- Large Inductors
- Small Inductors

Diodes

MOS transistors
- (MOSFET (enhancement or depletion), JFET)

Review from Last Lecture
Review from Last Lecture

Small-signal and simplified dc equivalent elements

Element | ss equivalent | Simplified dc equivalent
---|---|---

Bipolar Transistors

Dependent Sources (Linear)

\[ V_O = A_V V_{IN} \]
\[ I_O = A_I I_{IN} \]
\[ V_O = R_T I_{IN} \]
\[ I_O = G_T V_{IN} \]
Small signal model is that which represents the relationship between the small signal voltages and the small signal currents.

Consider 4-terminal network:

\[
\begin{align*}
I_1 &= f_1(V_1, V_2, V_3) \\
I_2 &= f_2(V_1, V_2, V_3) \\
I_3 &= f_3(V_1, V_2, V_3)
\end{align*}
\]

Define:

\[
\begin{align*}
i_1 &= I_1 - I_{1Q} \\
i_2 &= I_2 - I_{2Q} \\
i_3 &= I_3 - I_{3Q}
\end{align*}
\]

\[
\begin{align*}
u_1 &= V_1 - V_{1Q} \\
u_2 &= V_2 - V_{2Q} \\
u_3 &= V_3 - V_{3Q}
\end{align*}
\]

Review from Last Lecture:

Small-Signal Model of BJT and MOSFET
Consider 4-terminal network

\[ I_1 = f_1(V_1, V_2, V_3) \]
\[ I_2 = f_2(V_1, V_2, V_3) \]
\[ I_3 = f_3(V_1, V_2, V_3) \]

Nonlinear network characterized by 3 functions each functions of 3 variables
Small Signal Model Development

Nonlinear Model

\[ I_1 = f_1(V_1, V_2, V_3) \]
\[ I_2 = f_2(V_1, V_2, V_3) \]
\[ I_3 = f_3(V_1, V_2, V_3) \]

Linear Model at \( \tilde{V}_\alpha \) (alt. small signal model)

\[ i_1 = y_{11} v_1 + y_{12} v_2 + y_{13} v_3 \]
\[ i_2 = y_{21} v_1 + y_{22} v_2 + y_{23} v_3 \]
\[ i_3 = y_{31} v_1 + y_{32} v_2 + y_{33} v_3 \]

where

\[ y_{ij} = \left. \frac{\partial f_i(V_1, V_2, V_3)}{\partial V_j} \right|_{\hat{v} = \tilde{v}_\alpha} \]
Small Signal Model

\[
\begin{align*}
    i_1 &= y_{11}v_1 + y_{12}v_2 + y_{13}v_3 \\
    i_2 &= y_{21}v_1 + y_{22}v_2 + y_{23}v_3 \\
    i_3 &= y_{31}v_1 + y_{32}v_2 + y_{33}v_3
\end{align*}
\]

where

\[
y_{ij} = \left. \frac{\partial f_i(V_1, V_2, V_3)}{\partial V_j} \right|_{\bar{V} = \bar{V}_Q}
\]

- This is a small-signal model of a 4-terminal network and it is linear
- 9 small-signal parameters characterize the linear 4-terminal network
- Small-signal model parameters dependent upon Q-point!
Review from Last Lecture

4-terminal small-signal network summary

Small signal model:

\[
\begin{align*}
    i_1 &= y_{11}v_1 + y_{12}v_2 + y_{13}v_3 \\
    i_2 &= y_{21}v_1 + y_{22}v_2 + y_{23}v_3 \\
    i_3 &= y_{31}v_1 + y_{32}v_2 + y_{33}v_3
\end{align*}
\]

\[
y_{ij} = \left. \frac{\partial f_i(V_1, V_2, V_3)}{\partial V_j} \right|_{\bar{V}=\bar{V}_q}
\]
Consider 2-terminal network

```
\[ i_1 = y_{11} V_1 + y_{12} V_2 + y_{13} V_3 \]
\[ i_2 = y_{21} V_1 + y_{22} V_2 + y_{23} V_3 \]
\[ i_3 = y_{31} V_1 + y_{32} V_2 + y_{33} V_3 \]
```

\[ y_{ij} = \frac{\partial f_i(V_1, V_2, V_3)}{\partial V_j} \bigg|_{\tilde{V}=\tilde{V}_q} \]
Small Signal Model of MOSFET

Large Signal Model

\[ I_G = 0 \]

\[ I_D = \begin{cases} 
0 & V_{GS} \leq V_T \\
\mu C_{ox} \frac{W}{L} \left( V_{GS} - V_T - \frac{V_{DS}}{2} \right) V_{DS} & V_{GS} \geq V_T, V_{DS} < V_{GS} - V_T \\
\mu C_{ox} \frac{W}{2L} \left( V_{GS} - V_T \right)^2 \left( 1 + \lambda V_{DS} \right) & V_{GS} \geq V_T, V_{DS} \geq V_{GS} - V_T 
\end{cases} \]

3-terminal device

MOSFET is usually operated in saturation region in linear applications where a small-signal model is needed so will develop the small-signal model in the saturation region.
Small Signal Model of MOSFET

\[ I_G = 0 \]

\[ I_D = \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS}) \]

\[
\begin{align*}
y_{12} &= 0 \\
y_{11} &= 0 \\
y_{21} &= \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \\
y_{22} &= \lambda I_{DQ}
\end{align*}
\]

\[
\begin{align*}
i_G &= y_{11} V_{GS} + y_{12} V_{DS} \\
i_D &= y_{21} V_{GS} + y_{22} V_{DS}
\end{align*}
\]
Small Signal Model of MOSFET

by convention, \( y_{21} = g_m, \quad y_{22} = g_0 \)

\[
\therefore y_{21} \approx g_m = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \\
y_{22} = g_0 \approx \lambda I_{DQ}
\]

Note: \( g_0 \) vanishes when \( \lambda = 0 \)
Small Signal Model of MOSFET

\[ g_m = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \]

\[ g_o \equiv \lambda I_{DQ} \]

Alternate equivalent expressions for \( g_m \):

\[ I_{DQ} = \mu C_{ox} \frac{W}{2L} (V_{GSQ} - V_T)^2 \left(1 + \lambda V_{DSQ}\right) \approx \mu C_{ox} \frac{W}{2L} (V_{GSQ} - V_T)^2 \]

\[ g_m = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \]

\[ g_m = \sqrt{2\mu C_{ox}} \frac{W}{L} \cdot \sqrt{I_{DQ}} \]

\[ g_m = \frac{2I_{DQ}}{V_{GSQ} - V_T} \]
Consider again:

Small signal analysis example

\[ v_{IN} = V_M \sin(\omega t) \]

\[ A_v = \frac{2I_{DQ} R}{[V_{SS} + V_T]} \]

Derived for \( \lambda = 0 \)

\[ I_D = \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 \]

Recall the derivation was very tedious and time consuming!
Consider again:

Small signal analysis example

\[ A_v = \frac{V_{out}}{V_{in}} = -\frac{g_m}{g_o + 1/R} \]

For \( \lambda=0 \), \( g_o = \lambda I_{DQ} = 0 \)

\[ A_v = \frac{V_{out}}{V_{in}} = -g_m R \]

but \( g_m = \frac{2I_{DQ}}{V_{GSQ} - V_T} \)

\( V_{GSQ} = -V_{SS} \)

thus

\[ A_v = \frac{2I_{DQ} R}{V_{SS} + V_T} \]
Consider again:

Small signal analysis example

For $\lambda = 0$, $g_o = \lambda I_{DQ} = 0$

More accurate gain can be obtained if $\lambda$ effects are included and does not significantly increase complexity of small signal analysis.
Small Signal Model of BJT

3-terminal device

Forward Active Model:

\[ I_C = J_S A_E e^{\frac{V_{BE}}{V_t}} \left( 1 + \frac{V_{CE}}{V_{AF}} \right) \]

\[ I_B = \frac{J_S A_E}{\beta} e^{\frac{V_{BE}}{V_t}} \]

Usually operated in Forward Active Region when small-signal model is needed
Small Signal Model of BJT

**Nonlinear model:**

\[
\begin{align*}
I_1 & = f_1 (V_1, V_2) \\
I_2 & = f_2 (V_1, V_2)
\end{align*}
\]

\[
I_B = \frac{J_S A_E}{\beta} e^{\frac{V_{BE}}{V_t}}
\]

\[
I_C = J_S A_E e^{\frac{V_{BE}}{V_t}} \left(1 + \frac{V_{CE}}{V_{AF}}\right)
\]

**Small-signal model:**

\[
\begin{align*}
i_B & = y_{11} V_{BE} + y_{12} V_{CE} \\
i_C & = y_{21} V_{BE} + y_{22} V_{CE}
\end{align*}
\]

\[
y_{ij} = \left. \frac{\partial f_i (V_1, V_2)}{\partial V_j} \right|_{V=V_0}
\]

\[
y_{11} = g_m = \left. \frac{\partial I_B}{\partial V_{BE}} \right|_{V=V_0}
\]

\[
y_{21} = g_m = \left. \frac{\partial I_C}{\partial V_{BE}} \right|_{V=V_0}
\]

\[
y_{12} = \left. \frac{\partial I_B}{\partial V_{CE}} \right|_{V=V_0}
\]

\[
y_{22} = g_o = \left. \frac{\partial I_C}{\partial V_{CE}} \right|_{V=V_0}
\]

Note: \(g_m, g_\pi\) and \(g_o\) used for notational consistency with legacy terminology.
Small Signal Model of BJT

Nonlinear model:

\[ I_B = \frac{J_S A_E}{\beta} e^{\frac{V_{BE}}{V_t}} \]

\[ I_C = J_S A_E e^{\frac{V_{BE}}{V_t}} \left(1 + \frac{V_{CE}}{V_{AF}}\right) \]

Small-signal model:

\[ i_B = y_{11} V_{BE} + y_{12} V_{CE} \]

\[ i_C = y_{21} V_{BE} + y_{22} V_{CE} \]

\[ y_{ij} = \frac{\partial f_i(V_1, V_2)}{\partial V_j} \bigg|_{V_1=V_2=V_Q} \]

\[ y_{11} = \frac{\partial I_B}{\partial V_{BE}} \bigg|_{V=V_Q} = ? \]

\[ y_{12} = \frac{\partial I_B}{\partial V_{CE}} \bigg|_{V=V_Q} = ? \]

\[ y_{21} = \frac{\partial I_C}{\partial V_{BE}} \bigg|_{V=V_Q} = ? \]

\[ y_{22} = \frac{\partial I_C}{\partial V_{CE}} \bigg|_{V=V_Q} = ? \]
Small Signal Model of BJT

Nonlinear model:

\[ I_B = \frac{J_S A E}{\beta} e^{\frac{V_{BE}}{V_t}} \]
\[ I_C = J_S A E e^{\frac{V_{BE}}{V_t}} \left(1 + \frac{V_{CE}}{V_{AF}}\right) \]

\[ i_B = y_{11} v_{BE} + y_{12} v_{CE} \]
\[ i_C = y_{21} v_{BE} + y_{22} v_{CE} \]

Small-signal model:

\[ y_{11} = g_r = \left. \frac{\partial I_B}{\partial V_{BE}} \right|_{v=V_Q} = \frac{1}{V_t} \frac{J_S A E}{\beta} e^{\frac{V_{BE}}{V_t}} \bigg|_{v=V_Q} \approx \frac{I_{EQ}}{\beta V_t} \]

\[ y_{21} = g_m = \left. \frac{\partial I_C}{\partial V_{BE}} \right|_{v=V_Q} = \frac{1}{V_t} J_S A E e^{\frac{V_{BE}}{V_t}} \left(1 + \frac{V_{CE}}{V_{AF}}\right) \bigg|_{v=V_Q} = \frac{I_{EQ}}{V_t} \]

\[ y_{12} = \left. \frac{\partial I_B}{\partial V_{CE}} \right|_{v=V_Q} = 0 \]

\[ y_{22} = g_o = \left. \frac{\partial I_C}{\partial V_{CE}} \right|_{v=V_Q} = \frac{J_S A E}{V_{AF}} \bigg|_{v=V_Q} \approx \frac{I_{EQ}}{V_{AF}} \]

Note: usually prefer to express in terms of \( I_{EQ} \)
Small Signal Model of BJT

\[ i_B = y_{11} v_{BE} + y_{12} v_{CE} \]
\[ i_C = y_{21} v_{BE} + y_{22} v_{CE} \]

\[ g_\pi = \frac{I_{CQ}}{\beta V_t} \]
\[ g_m = \frac{I_{CQ}}{V_t} \]
\[ g_o = \frac{I_{CQ}}{V_{AF}} \]

An equivalent circuit
Consider again:

**Small signal analysis example**

\[ V_{IN} = V_M \sin \omega t \]

Recall the derivation was very tedious and time consuming!

Derived for \( V_{AF} = 0 \)

\[ A_{VB} = - \frac{I_{CQ} R}{V_t} \]
Neglect $V_{AF}$ effects (i.e. $V_{AF}=\infty$) to be consistent with earlier analysis.

\[ g_o = \frac{I_{CQ}}{V_{AF}} \quad V_{AF}=\infty \quad 0 \]

\[ V_{OUT} = -g_m R V_{BE} \quad V_{IN} = V_{BE} \]

\[ A_V = \frac{V_{OUT}}{V_{IN}} = -g_m R \]

\[ g_m = \frac{I_{CQ}}{V_t} \]

\[ A_V = -\frac{I_{CQ} R}{V_t} \]

Note this is identical to what was obtained with the direct nonlinear analysis.
Small Signal BJT Model – alternate representation

Observe:

\[ g_{\pi} v_{be} = i_b \]

\[ g_m v_{be} = i_b \frac{g_m}{g_{\pi}} \]

\[ g_m = \frac{I_Q}{V_t} \]

\[ g_{\pi} = \frac{I_Q}{\beta V_t} \]

\[ g_o \approx \frac{I_Q}{V_{AF}} \]

Can replace the voltage dependent current source with a current dependent current source
Small Signal BJT Model – alternate representation

\[ g_m = \frac{I_{CQ}}{V_t} \]
\[ g_\pi = \frac{I_{CQ}}{\beta V_t} \]
\[ g_o \approx \frac{I_{CQ}}{V_{AF}} \]

Alternate equivalent small signal model

\[ g_\pi = \frac{I_{CQ}}{\beta V_t} \]
\[ g_o \approx \frac{I_{CQ}}{V_{AF}} \]
Small-Signal Model Representations

(3-terminal network – also relevant with 4-terminal networks)

- Bi-Linear Relationship between $i_1$, $i_2$, $v_1$, $v_2$
- Linear Two Port

Have developed small-signal models for the MOSFET and BJT

Models have been based upon arbitrary assumption that $v_1$, $v_2$ are independent variables

Have already seen some alternatives for parameter definitions in these models

Alternative representations are sometimes used
Small-Signal Model Representations

The good, the bad, and the unnecessary!!

what we have developed:

The hybrid parameters:
Small-Signal Model Representations

The $z$-parameters

$$v_1 = z_{11}i_1 + z_{12}i_2$$
$$v_2 = z_{21}i_1 + z_{22}i_2$$

The ABCD parameters:

$$v_1 = Av_2 - Bi_2$$
$$i_1 = Cv_2 - Di_2$$
Small-Signal Model Representations

Bi-Linear Relationship between $i_1, i_2, v_1, v_2$

The $S$-parameters

$S$-parameters (embedded with source and load impedances)

The $T$ parameters:

$T$-parameters (embedded with source and load impedances)
Small-Signal Model Representations

The good, the bad, and the unnecessary!!

- Equivalent circuits often given for each representation
- All provide identical characterization
- Easy to move from any one to another
Small-Signal Model Representations

The good, the bad, and the unnecessary!!

IEEE TRANSACTIONS ON MICROWAVE THEORY AND TECHNIQUES, VOL. 42, NO. 2, FEBRUARY 1994

Conversions Between $S$, $Z$, $Y$, $h$, $ABCD$, and $T$ Parameters which are Valid for Complex Source and Load Impedances

Dean A. Frickey, Member, IEEE
Small-Signal Model Representations

The good, the bad, and the unnecessary !!

Conversions Between $S$, $Z$, $Y$, $h$, $ABCD$, and $T$ Parameters which are Valid for Complex Source and Load Impedances

Dean A. Frickey, Member, IEEE
Active Device Model Summary

<table>
<thead>
<tr>
<th>Element</th>
<th>ss equivalent</th>
<th>dc equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diodes</td>
<td></td>
<td>Simplified</td>
</tr>
<tr>
<td>MOS transistors</td>
<td></td>
<td>Simplified</td>
</tr>
<tr>
<td>Bipolar Transistors</td>
<td></td>
<td>Simplified</td>
</tr>
</tbody>
</table>

What are the simplified dc equivalent models?
What are the simplified dc equivalent models?

\[ \mu C_{OX} W \left( V_{GSQ} - V_{Tn} \right)^2 / 2L \]

\[ \mu C_{OX} W \left( V_{GSQ} - V_{Tp} \right)^2 / 2L \]
Small-signal Operation of Nonlinear Circuits

- Small-signal principles
- Example Circuit
- Small-Signal Models

Small-Signal Analysis of Nonlinear Circuits
Recall:

Alternative Approach to small-signal analysis of nonlinear networks

1. Linearize nonlinear devices
   (have small-signal model for key devices!)

2. Replace all devices with small-signal equivalent

3. Solve linear small-signal network

Remember that the small-signal model is operating point dependent!

Thus need Q-point to obtain values for small signal parameters
Example:

Determine the small signal voltage gain $A_V = V_{OUT}/V_{IN}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$. 

---

$V_{DD}$

$M_2$

$V_{OUT}$

$V_{IN}$

$M_1$

$V_{SS}$
Example: Determine the small signal voltage gain $A_V = \frac{V_{OUT}}{V_{IN}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$.

Small-signal circuit
Example: Determine the small signal voltage gain $A_V = \frac{v_{OUT}}{v_{IN}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$. 

Small-signal circuit:

Small-signal MOSFET model for $\lambda = 0$
Example: Determine the small signal voltage gain $A_V = \frac{V_{\text{OUT}}}{V_{\text{IN}}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$. 

Small-signal circuit
Example:

Small-signal circuit

Analysis:

By KCL

\[ g_{m1} \varepsilon_{GS1} = g_{m2} \varepsilon_{GS2} \]

but

\[ \varepsilon_{GS1} = \varepsilon_{IN} \]

\[ -\varepsilon_{GS2} = \varepsilon_{OUT} \]

thus:

\[ A_v = \frac{\varepsilon_{OUT}}{\varepsilon_{IN}} = -\frac{g_{m1}}{g_{m2}} \]
Example:

Small-signal circuit

Analysis:

\[ A_v = \frac{V_{OUT}}{V_{IN}} = -\frac{g_{m1}}{g_{m2}} \]

Recall:

\[ g_m = -\sqrt{2I_D \mu C_{ox}} \sqrt{\frac{W_1}{L_1}} \]

\[ A_v = -\sqrt{\frac{2I_D \mu C_{ox}}{W_2} \frac{W_1}{L_1}} = -\sqrt{\frac{W_1}{W_2}} \sqrt{\frac{L_2}{L_1}} \]
Example:

Small-signal circuit

\[ A_v = \frac{V_{OUT}}{V_{IN}} = -\frac{g_{m1}}{g_{m2}} \]

Recall:

\[ A_v = -\frac{\sqrt{W_1}}{\sqrt{W_2}} \frac{\sqrt{L_2}}{\sqrt{L_1}} \]

If \( L_1 = L_2 \), obtain

\[ A_v = -\frac{W_1}{W_2} \frac{L_2}{L_1} = -\frac{W_1}{W_2} \]

The width and length ratios can be accurately set when designed in a standard CMOS process
End of Lecture 23