EE 330
Lecture 24

Small Signal Models and Analysis
Due Wednesday at **beginning** of class

All of the work on this exam is my own and I did not discuss it with anyone beyond the instructor or Teaching Assistant for this course.
Operation with Small-Signal Inputs

- Analysis procedure for these simple circuits was very tedious
- This approach will be unmanageable for even modestly more complicated circuits
- Faster analysis method is needed!
Operation with Small-Signal Inputs

Why was this analysis so tedious?

Because of the nonlinearity in the device models

What was the key technique in the analysis that was used to obtain a simple expression for the output?

\[ V_{\text{OUT}} = V_{\text{CC}} - J_S A E R_1 \left( \frac{-V_{EE}}{V_i} + \frac{V_M \sin(\omega t)}{V_i} \right) \]

\[ V_{\text{OUT}} \approx \left[ V_{\text{CC}} - I_{CQ} R_1 \right] - \left( \frac{I_{CQ} R_1}{V_i} \right) V_M \sin(\omega t) \]

Linearization of the nonlinear output expression at the operating point
Operation with Small-Signal Inputs

Quiescent Output

\[ V_{\text{OUT}} \approx \left[ V_{\text{cc}} - I_{\text{cq}} R_1 \right] - \left( \frac{I_{\text{cq}} R_1}{V_t} \right) V_m \sin(\omega t) \]

Small-signal analysis strategy

1. Obtain Quiescent Output (Q-point)
2. Linearize circuit at Q-point instead of linearize the nonlinear solution
3. Analyze linear “small-signal” circuit
4. Add quiescent and small-signal outputs to obtain good approximation to actual output
Small-Signal Principle

Relationship is nearly linear in a small enough region around Q-point.
Region of linearity is often quite large.
Linear relationship may be different for different Q-points.
Small-Signal Principle

Review from last time

\[ y - y_Q = \frac{\partial f}{\partial x} \bigg|_{x=x_Q} (x - x_Q) \]

\[ y = \left[ \frac{\partial f}{\partial x} \bigg|_{x=x_Q} \right] x + \left[ y_Q - x_Q \frac{\partial f}{\partial x} \bigg|_{x=x_Q} \right] \]

\[ m = \left[ \frac{\partial f}{\partial x} \bigg|_{x=x_Q} \right] \quad b = \left[ y_Q - x_Q \frac{\partial f}{\partial x} \bigg|_{x=x_Q} \right] \]
Small-Signal Principle

Review from last time

Small-Signal Model:

\[ y_{ss} = \frac{\partial f}{\partial x} \bigg|_{x=x_Q} x_{ss} \]

- Linearized model for the nonlinear function \( y = f(x) \)
- Valid in the region of the Q-point
- Will show the small signal model is simply Taylor’s series expansion at the Q-point truncated after first-order terms
Small-Signal Principle

**Observe:**

\[ y - y_Q = \frac{\partial f}{\partial x}|_{x=x_Q} (x - x_Q) \]

\[ y = y_Q + \frac{\partial f}{\partial x}|_{x=x_Q} (x - x_Q) \]

Recall Taylor's Series Expansion of nonlinear function \( f \) at expansion point \( x_0 \)

\[ y = f(x_0) + \sum_{k=1}^{\infty} \frac{1}{k! \frac{\partial f}{\partial x}|_{x=x_0}} (x-x_0)^k \]

**Small-Signal Model:**

\[ y = f(x_Q) + \frac{\partial f}{\partial x}|_{x=x_Q} (x - x_Q) \]

\[ y_{ss} = \frac{\partial f}{\partial x}|_{x=x_Q} x_{ss} \]

**Mathematically, linearized model is simply Taylor’s series expansion at the Q-point truncated after first-order terms with notation** \( x_Q = x_0 \)
Small-Signal Principle

\[ y = f(x_Q) + \left. \frac{\partial f}{\partial x} \right|_{x=x_Q} x_{ss} \]

Quiescent Output

ss Gain

How can a circuit be linearized at an operating point as an alternative to linearizing a nonlinear function at an operating point?
Small-Signal Principle

\[ i_{SS} = \frac{\partial I}{\partial V} \bigg|_{V=V_Q} \]

\[ \nu_{SS} = \nu \]

Model of the nonlinear device at the Q-point

\[ i = y \nu \]
Small-Signal Principle

A Small Signal Equivalent Circuit

The small-signal model of this 2-terminal electrical network is a resistor of value $1/y$

One small-signal parameter characterizes this one-port but it is dependent on Q-point
Small-Signal Principle

Goal with small signal model is to predict performance of circuit or device in the vicinity of an operating point

Operating point is often termed Q-point

Will be extended to functions of two and three variables
Solution for the example was based upon solving the nonlinear circuit for $V_{OUT}$ and then linearizing the solution by doing a Taylor’s series expansion.

- Solution of nonlinear equations very involved with two or more nonlinear devices
- Taylor’s series linearization can get very tedious if multiple nonlinear devices are present

**Standard Approach to small-signal analysis of nonlinear networks**

1. Solve nonlinear network
2. Linearize solution

**Alternative Approach to small-signal analysis of nonlinear networks**

1. Linearize nonlinear devices
2. Obtain small-signal model from linearized device models
3. Replace all devices with small-signal equivalent
4. Solve linear small-signal network
Alternative Approach to small-signal analysis of nonlinear networks

1. Linearize nonlinear devices

2. Replace all devices with small-signal equivalent

3. Solve linear small-signal network

- Must only develop linearized model once for any nonlinear device
  
e.g. once for a MOSFET, once for a JFET, and once for a BJT

  Linearized model for nonlinear device termed “small-signal model”

  derivation of small-signal model for most nonlinear devices is less complicated than solving even one simple nonlinear circuit

- Solution of linear network much easier than solution of nonlinear network
Standard Approach to small-signal analysis of nonlinear networks

1. Linearize nonlinear devices

2. Replace all devices with small-signal equivalent

3. Solve linear small-signal network
Standard Approach to analysis of nonlinear networks

Nonlinear Network

$dc$ Equivalent Network

Q-point

Values for small-signal parameters

Small-signal equivalent network

Small-signal output

Total output

(good approximation)
Standard Approach to small-signal analysis of nonlinear networks

Nonlinear Network

dc Equivalent Network

Q-point

Values for small-signal parameters

Small-signal equivalent network

Small-signal output

Total output
(good approximation)
Linearized nonlinear devices
Example:

Nonlinear network

Linearized small-signal network
### Dc and small-signal equivalent elements

<table>
<thead>
<tr>
<th>Element</th>
<th>ss equivalent</th>
<th>dc equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>dc Voltage Source</td>
<td>$V_{DC}$</td>
<td>$V_{DC}$</td>
</tr>
<tr>
<td>ac Voltage Source</td>
<td>$V_{AC}$</td>
<td>$V_{AC}$</td>
</tr>
<tr>
<td>dc Current Source</td>
<td>$I_{DC}$</td>
<td>$I_{DC}$</td>
</tr>
<tr>
<td>ac Current Source</td>
<td>$I_{AC}$</td>
<td>$I_{AC}$</td>
</tr>
<tr>
<td>Resistor</td>
<td>$R$</td>
<td>$R$</td>
</tr>
</tbody>
</table>
Dc and small-signal equivalent elements

<table>
<thead>
<tr>
<th>Element</th>
<th>ss equivalent</th>
<th>dc equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capacitors</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Large C</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Small C</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inductors</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Large L</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Small L</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Diodes</td>
<td></td>
<td>Simplified</td>
</tr>
<tr>
<td>MOS transistors</td>
<td></td>
<td>Simplified</td>
</tr>
</tbody>
</table>
### Dc and small-signal equivalent elements

<table>
<thead>
<tr>
<th>Element</th>
<th>ss equivalent</th>
<th>dc equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bipolar Transistors</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dependent Sources</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
How is the small-signal equivalent circuit obtained from the nonlinear circuit?

What is the small-signal equivalent of the MOSFET, BJT, and diode?
A Small Signal Equivalent Circuit

Thus, for the diode

\[
R_d = \left( \frac{\partial I_D}{\partial V_D} \right)_Q^{-1}
\]
Small-Signal Diode Model

For the diode

\[ R_d = \frac{\partial I_D}{\partial V_D}^{-1} \]

\[ V_D = \frac{I_D}{V_t} \]

\[ \frac{\partial I_D}{\partial V_t} = \left[ \left( \frac{V_D}{V_t} \right) \frac{1}{V_t} \right] Q = \frac{I_{DQ}}{V_t} \]

\[ R_d = \frac{V_t}{I_{DQ}} \]
Example of diode circuit where small-signal diode model is useful

Voltage Reference

Small-signal model of Voltage Reference
(useful for compensation)
Small-Signal Model

Consider 4-terminal network

\[ I_1 = f_1(V_1, V_2, V_3) \]
\[ I_2 = f_2(V_1, V_2, V_3) \]
\[ I_3 = f_3(V_1, V_2, V_3) \]

Define

\[ i_1 = I_1 - I_{1Q} \]
\[ i_2 = I_2 - I_{2Q} \]
\[ i_3 = I_3 - I_{3Q} \]
\[ v_1 = V_1 - V_{1Q} \]
\[ v_2 = V_2 - V_{2Q} \]
\[ v_3 = V_3 - V_{3Q} \]

Small signal model is that which represents the relationship between the small signal voltages and the small signal currents.
Small-Signal Model

Consider 4-terminal network

\[
\begin{align*}
i_1 &= g_1(v_1, v_2, v_3) \\
i_2 &= g_2(v_1, v_2, v_3) \\
i_3 &= g_3(v_1, v_2, v_3)
\end{align*}
\]

Small signal model is that which represents the relationship between the small signal voltages and the small signal currents.

For small signals, this relationship should be linear.

Can be thought of as a change in coordinate systems from the large signal coordinate system to the small-signal coordinate system.
Recall for a function of one variable

\[ y = f(x) \]

Taylor's Series Expansion about the point \( x_0 \)

\[ y = f(x) = f(x)\bigg|_{x=x_0} + \frac{\partial f}{\partial x}\bigg|_{x=x_0} (x-x_0) + \frac{\partial^2 f}{\partial x^2}\bigg|_{x=x_0} \frac{1}{2!} (x-x_0)^2 + \ldots \]

If \( x-x_0 \) is small

\[ y \approx f(x)\bigg|_{x=x_0} + \frac{\partial f}{\partial x}\bigg|_{x=x_0} (x-x_0) \]

\[ y \approx y_0 + \frac{\partial f}{\partial x}\bigg|_{x=x_0} (x-x_0) \]
Recall for a function of one variable

\[ y = f(x) \]

If \( x-x_0 \) is small

\[ y \approx y_0 + \left. \frac{\partial f}{\partial x} \right|_{x=x_0} (x-x_0) \]

\[ y - y_0 = \left. \frac{\partial f}{\partial x} \right|_{x=x_0} (x-x_0) \]

If we define the small signal variables as

\[ y = y - y_0 \]

\[ x = x - x_0 \]
Recall for a function of one variable

\[ y = f(x) \]

If \( x - x_0 \) is small

\[ y - y_0 = \left. \frac{\partial f}{\partial x} \right|_{x=x_0} (x - x_0) \]

If we define the small signal variables as

\[ y = y - y_0 \]

\[ \delta x = x - x_0 \]

Then

\[ y = \left. \frac{\partial f}{\partial x} \right|_{x=x_0} \delta x \]

This relationship is linear!
Consider now a function of \( n \) variables

\[
y = f(x_1, \ldots x_n) = f(\bar{x})
\]

If we define the small signal variables as \( \bar{X}_0 = \{x_{10}, x_{20}, \ldots x_{n0}\} \)

The multivariate Taylor's series expansion around the point \( \bar{X}_0 \) is given by

\[
y = f(\bar{x}) = f(x)\big|_{\bar{x}=x_0} + \sum_{k=1}^{n} \left( \frac{\partial f}{\partial x_k} \bigg|_{\bar{x}=x_0} (x_k - x_{k0}) \right)
\]

\[
+ \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k} \bigg|_{\bar{x}=x_0} \frac{1}{2!} (x_j - x_{j0})(x_k - x_{k0}) + \ldots \text{(H.O.T.)}
\]

Truncating after first-order terms, we obtain the approximation

\[
y - y_0 \approx \sum_{k=1}^{n} \left( \frac{\partial f}{\partial x_k} \bigg|_{\bar{x}=x_0} (x_k - x_{k0}) \right)
\]

where \( y_0 = f(x)\big|_{\bar{x}=x_0} \)
Multivariate Taylors Series Expansion

\[ y = f(x_1, \ldots, x_n) = f(\bar{x}) \]

Linearized approximation

\[ y - y_0 \approx \sum_{k=1}^{n} \left( \frac{\partial f}{\partial x_k} \bigg|_{x=x_0} (x_k - x_{k0}) \right) \]

This can be expressed as

\[ y_{ss} \approx \sum_{k=1}^{n} a_k x_{ss} \]

where

\[ y_{ss} = y - y_0 \]

\[ x_{kss} = x_k - x_{k0} \]

\[ a_k = \frac{\partial f}{\partial x_k} \bigg|_{x=x_0} \]
In the more general form\(^1\), the multivariate Taylor’s series expansion can be expressed as

\[
f(x_1, \ldots, x_n) = \alpha_o + \sum_{m=1}^{\infty} \sum_{k_1, \ldots, k_n}^{\sum k_j = m} \alpha_{k_1, \ldots, k_n, m} (x_1 - x_{1,0})^{k_1} \cdots (x_n - x_{n,0})^{k_n}
\]

(7)

\[
\alpha_o = f(x_{10}, \ldots, x_{n0})
\]

\[
\alpha_{k_1, \ldots, k_n, m} = \frac{1}{k_1! \cdots k_n!} \left. \frac{\partial^m f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \right|_{x_{10}, \ldots, x_{n0}}
\]

(8)

\(^1\) http://www.chem.mtu.edu/~tbco/cm416/taylor.html
Consider 4-terminal network

Nonlinear network characterized by 3 functions each functions of 3 variables

\[
\begin{align*}
I_1 &= f_1(V_1, V_2, V_3) \\
I_2 &= f_2(V_1, V_2, V_3) \\
I_3 &= f_3(V_1, V_2, V_3)
\end{align*}
\]
Consider now 3 functions each functions of 3 variables

\[
\begin{align*}
I_1 &= f_1(V_1, V_2, V_3) \\
I_2 &= f_2(V_1, V_2, V_3) \\
I_3 &= f_3(V_1, V_2, V_3)
\end{align*}
\]

Define

\[
\vec{V}_Q = \begin{bmatrix} V_{1Q} \\ V_{2Q} \\ V_{3Q} \end{bmatrix}
\]

In what follows, we will use \( \vec{V}_Q \) as an expansion point in a Taylor’s series expansion.
Consider now 3 functions each functions of 3 variables

\[ I_1 = f_1(V_1, V_2, V_3) \]
\[ I_2 = f_2(V_1, V_2, V_3) \]
\[ I_3 = f_3(V_1, V_2, V_3) \]

Define
\[ \bar{V}_Q = \begin{bmatrix} V_{1Q} \\ V_{2Q} \\ V_{3Q} \end{bmatrix} \]

Consider first the function \( I_1 \)

The multivariate Taylors Series expansion of \( I_1 \), around the operating point \( \bar{V}_Q \), when truncated after first-order terms, can be expressed as:

\[
I_1 = f_1(V_1, V_2, V_3) \approx f_1(V_{1Q}, V_{2Q}, V_{3Q}) + \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_1} \right|_{V=V_Q} (V_1 - V_{1Q}) + \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_2} \right|_{V=V_Q} (V_2 - V_{2Q}) + \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_3} \right|_{V=V_Q} (V_3 - V_{3Q})
\]

or equivalently as:

\[
I_1 - I_{1Q} = \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_1} \right|_{V=V_Q} (V_1 - V_{1Q}) + \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_2} \right|_{V=V_Q} (V_2 - V_{2Q}) + \left. \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_3} \right|_{V=V_Q} (V_3 - V_{3Q})
\]
repeating from previous slide:

\[ I_1 - I_{1Q} = \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_1} (V_1 - V_{1Q}) + \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_2} (V_2 - V_{2Q}) + \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_3} (V_3 - V_{3Q}) \]

Make the following definitions

\[ y_{11} = \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_1} \bigg|_{V = \bar{v}_Q} \]

\[ y_{12} = \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_2} \bigg|_{V = \bar{v}_Q} \]

\[ y_{13} = \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_3} \bigg|_{V = \bar{v}_Q} \]

It thus follows that

\[ i_1 = y_{11} u_1 + y_{12} u_2 + y_{13} u_3 \]

This is a linear relationship between the small signal electrical variables
Consider now 3 functions, each a function of 3 variables

Extending this approach to the two nonlinear functions $I_2$ and $I_3$

\[ i_1 = y_{11} v_1 + y_{12} v_2 + y_{13} v_3 \]
\[ i_2 = y_{21} v_1 + y_{22} v_2 + y_{23} v_3 \]
\[ i_3 = y_{31} v_1 + y_{32} v_2 + y_{33} v_3 \]

where

\[ y_{ij} = \left. \frac{\partial f_i(v_1, v_2, v_3)}{\partial v_j} \right|_{v = \bar{v}_Q} \]

This is a small-signal model of a 4-terminal network and it is linear
9 small-signal parameters characterize the linear 4-terminal network
Small-signal model parameters dependent upon Q-point!
End of Lecture 24