EE 330
Lecture 24

• Small Signal Analysis
• Graphical Small Signal Analysis
• Model Extensions and Simplifications
Exam Schedule

Exam 1          Monday Oct 5
Exam 2          Monday  October 26
Exam 3          Friday November 20
Small-Signal Model

\[
\begin{align*}
i_1 &= g_1(v_1, v_2, v_3) \\
i_2 &= g_2(v_1, v_2, v_3) \\
i_3 &= g_3(v_1, v_2, v_3)
\end{align*}
\]

\[
\begin{align*}
\mathbf{y}_{ij} &= \left. \frac{\partial f_i(v_1, v_2, v_3)}{\partial v_j} \right|_{\bar{v} = \bar{v}_0}
\end{align*}
\]
Small Signal Model of MOSFET

\[ g_m = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \]

\[ g_o \approx I_{dQ} \]

Alternate equivalent expressions for \( g_m \):

\[ I_{dQ} = \mu C_{ox} \frac{W}{2L} (V_{GSQ} - V_T)^2 (1 + \lambda V_{dSO}) \approx \mu C_{ox} \frac{W}{2L} (V_{GSQ} - V_T)^2 \]

\[ g_m = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \]

\[ g_m = \sqrt{2\mu C_{ox}} \frac{W}{L} \cdot \sqrt{I_{dQ}} \]

\[ g_m = \frac{2I_{dQ}}{V_{GSQ} - V_T} \]
Small Signal Model of BJT

An equivalent circuit

\[ i_B = y_{11} V_{BE} + y_{12} V_{CE} \]
\[ i_C = y_{21} V_{BE} + y_{22} V_{CE} \]

\[ g_\pi = \frac{I_{CQ}}{\beta V_t} \]
\[ g_m = \frac{I_{CQ}}{V_t} \]
\[ g_o = \frac{I_{CQ}}{V_{AF}} \]

Review from Last Lecture
Small Signal BJT Model – alternate representation

\[ g_m = \frac{I_{CQ}}{V_t} \]
\[ g_\pi = \frac{I_{CQ}}{\beta V_t} \]
\[ g_o \approx \frac{I_{CQ}}{V_{AF}} \]

Alternate equivalent small signal model

\[ g_\pi = \frac{I_{CQ}}{\beta V_t} \]
\[ g_o \approx \frac{I_{CQ}}{V_{AF}} \]
Small-Signal Model Representations

The good, the bad, and the unnecessary !!

- Equivalent circuits often given for each representation
- All provide identical characterization
- Easy to move from any one to another

Review from Last Lecture
What are the simplified dc equivalent models?
Active Device Model Summary

What are the simplified dc equivalent models?

dc equivalent

- Simplified

- Simplified

- Simplified

- Simplified

\[ V_{GSQ} \]

\[ \mu C_{OX} W \frac{(V_{GSQ} - V_{Tn})^2}{2L} \]

\[ \mu C_{OX} W \frac{(V_{GSQ} - V_{Tp})^2}{2L} \]

\[ I_{BQ} \]

\[ \beta I_{BQ} \]
Small-signal Operation of Nonlinear Circuits

- Small-signal principles
- Example Circuit
- Small-Signal Models

Small-Signal Analysis of Nonlinear Circuits
Recall:

Alternative Approach to small-signal analysis of nonlinear networks

1. Linearize nonlinear devices
   *(have small-signal model for key devices!)*

2. Replace all devices with small-signal equivalent

3. Solve linear small-signal network

Remember that the small-signal model is operating point dependent!

Thus need Q-point to obtain values for small signal parameters
Example:

Determine the small signal voltage gain $A_V = \frac{v_{out}}{v_{in}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$. 
Example:  
*Determine the small signal voltage gain* \( A_V = \frac{V_{OUT}}{V_{IN}} \). *Assume* \( M_1 \) and \( M_2 \) *are operating in the saturation region and that* \( \lambda = 0 \)

---

**Small-signal circuit**
Example: \( A_V = \frac{v_{OUT}}{v_{IN}} \). Assume \( M_1 \) and \( M_2 \) are operating in the saturation region and that \( \lambda = 0 \).
Example: Determine the small signal voltage gain \( A_v = \frac{v_{OUT}}{v_{IN}} \). Assume \( M_1 \) and \( M_2 \) are operating in the saturation region and that \( \lambda = 0 \).
Example:

Small-signal circuit

Analysis:

By KCL

\[ g_{m1} \, v_{GS1} = g_{m2} \, v_{GS2} \]

but

\[ v_{GS1} = v_{IN} \]

\[ -v_{GS2} = v_{OUT} \]

thus:

\[ A_v = \frac{v_{OUT}}{v_{IN}} = -\frac{g_{m1}}{g_{m2}} \]
Example:

![Small-signal circuit diagram]

**Analysis:**

Small-signal circuit

\[ A_v = \frac{V_{OUT}}{V_{IN}} = -\frac{g_{m1}}{g_{m2}} \]

Recall:

\[ g_m = -\sqrt{2I_D \mu C_{ox} \frac{W_1}{L_1}} \]

\[ A_v = -\sqrt{\frac{2I_D \mu C_{ox} \frac{W_1}{L_1}}{\frac{W_2}{L_2}}} = -\sqrt{\frac{W_1}{W_2}} \sqrt{\frac{L_2}{L_1}} \]
Example:

Small-signal circuit

\[ A_v = \frac{V_{OUT}}{V_{IN}} = -\frac{g_{m1}}{g_{m2}} \]

Recall:

\[ A_v = -\sqrt{\frac{W_1}{W_2}} \sqrt{\frac{L_2}{L_1}} \]

If \( L_1 = L_2 \), obtain

\[ A_v = -\sqrt{\frac{W_1}{W_2}} \sqrt{\frac{L_2}{L_1}} = -\sqrt{\frac{W_1}{W_2}} \]

The width and length ratios can be accurately set when designed in a standard CMOS process.
Example: Obtain the small signal model of the following circuit. Assume MOSFET is operating in the saturation region.
Example

Obtain the small signal model of the following circuit. Assume MOSFET is operating in the saturation region

Solution:

\[ V \left( g_m + g_0 \right) = I \]

\[ R_{EQ} = \frac{1}{g_m + g_0} \]
Graphical Analysis and Interpretation

Consider Again

\[ V_{\text{OUT}} = V_{\text{DD}} - I_D R \]

\[ I_D = \frac{\mu C_{\text{ox}} W}{2L} \left( V_{\text{IN}} - V_{\text{SS}} - V_T \right)^2 \]
Graphical Analysis and Interpretation

Device Model (family of curves) \[ I_{DQ} = \frac{\mu C_{ox} W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS}) \]

Load Line

Device Model

Device Model at Operating Point

\[ V_{OUT} = V_{DD} - I_D R \]

\[ I_D = \frac{\mu C_{ox} W}{2L} (V_{IN} - V_{SS} - V_T)^2 \]
Graphical Analysis and Interpretation

Device Model (family of curves)

\[ I_{DQ} = \frac{\mu C_{ox} W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS}) \]

\[ V_{OUT} = V_{DD} - I_D R \]

\[ I_D = \frac{\mu C_{ox} W}{2L} (V_{IN} - V_{SS} - V_T)^2 \]

Must satisfy both equations all of the time!
Graphical Analysis and Interpretation

Device Model (family of curves) \( I_D = \frac{\mu C_{ox} W}{2L} \left( V_{IN} - V_{SS} - V_T \right)^2 \left( 1 + \lambda V_{DS} \right) \)

- As \( V_{IN} \) changes around Q-point, due to changes \( V_{IN} \) induces in \( V_{GS} \), the operating point must remain on the load line!
- Small sinusoidal changes of \( V_{IN} \) will be nearly symmetric around the \( V_{GSQ} \) line
- This will cause nearly symmetric changes in both \( I_D \) and \( V_{DS} \)!
- Since \( V_{SS} \) is constant, change in \( V_{DS} \) is equal to change in \( V_{OUT} \)

\[ \begin{align*}
V_{GSQ} &= -V_{SS} \\
= A_V
\end{align*} \]
Graphical Analysis and Interpretation

Device Model (family of curves) \( I_d = \frac{\mu C_{ox} W}{2L} (V_{IN} - V_{SS} - V_T)^2 (1 + \lambda V_{DS}) \)

As \( V_{IN} \) changes around Q-point, due to changes \( V_{IN} \) induces in \( V_{GS} \), the operating point must remain on the load line!
Graphical Analysis and Interpretation

Device Model (family of curves)\[ I_{\text{DQ}} = \frac{\mu C_{\text{ox}} W}{2L} (V_{\text{GS}} - V_{\text{T}})^2 (1 + \lambda V_{\text{DS}}) \]

- Linear signal swing region smaller than saturation region
- Modest nonlinear distortion provided saturation region operation maintained
- Symmetric swing about Q-point
- Signal swing can be maximized by judicious location of Q-point

\[ V_{\text{GSQ}} = -V_{\text{SS}} \]

Saturation region
Graphical Analysis and Interpretation

Device Model (family of curves) \[ I_{DQ} = \frac{\mu C_{ox} W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS}) \]

Saturation region

Very limited signal swing with non-optimal Q-point location
Graphical Analysis and Interpretation

Device Model (family of curves) \( I_{DQ} = \frac{\mu C_{ox} W}{2L} (V_{GS} - V_{T})^2 (1 + \lambda V_{DS}) \)

- Signal swing can be maximized by judicious location of Q-point
- Often selected to be at middle of load line in saturation region
Small-Signal MOSFET Model Extension

Existing model does not depend upon the bulk voltage!

Observe that changing the bulk voltage will change the electric field in the channel region!
Further Model Extensions

Existing model does not depend upon the bulk voltage!

Observe that changing the bulk voltage will change the electric field in the channel region!

Changing the bulk voltage will change the thickness of the inversion layer.
Changing the bulk voltage will change the threshold voltage of the device.
Typical Effects of Bulk on Threshold Voltage for n-channel Device

\[ \gamma \approx 0.4V^{\frac{1}{2}} \quad \phi \approx 0.6V \]

Bulk-Diffusion Generally Reverse Biased \((V_{BS} < 0\) or at least less than 0.3V\) for n-channel
Shift in threshold voltage with bulk voltage can be substantial
Often \(V_{BS} = 0\)
Typical Effects of Bulk on Threshold Voltage for p-channel Device

Bulk-Diffusion Generally Reverse Biased ($V_{BS} > 0$ or at least greater than -0.3V) for n-channel

Same functional form as for n-channel devices but $V_{T0}$ is now negative and the magnitude of $V_T$ still increases with the magnitude of the reverse bias.
Model Extension Summary

\( I_G = 0 \)
\( I_B = 0 \)

\[
I_d = \begin{cases} 
0 & V_{GS} \leq V_T \\
\mu C_{OX} \frac{W}{L} \left( V_{GS} - V_T - \frac{V_{DS}}{2} \right) V_{DS} & V_{GS} \geq V_T \quad V_{DS} < V_{GS} - V_T \\
\mu C_{OX} \frac{W}{2L} \left( V_{GS} - V_T \right)^2 \cdot (1 + \lambda V_{DS}) & V_{GS} \geq V_T \quad V_{DS} \geq V_{GS} - V_T 
\end{cases}
\]

\[ V_T = V_{T0} + \gamma \left( \sqrt{\phi - V_{BS}} - \sqrt{\phi} \right) \]

Model Parameters : \( \{ \mu, C_{OX}, V_{T0}, \phi, \gamma, \lambda \} \)

Design Parameters : \( \{ W, L \} \) but only one degree of freedom \( W/L \)
Small-Signal Model Extension

\[ I_G = 0 \]
\[ I_B = 0 \]

\[ I_D = \begin{cases} 0 & \text{if } V_{GS} \leq V_T \\ \mu C_{ox} \frac{W}{L} \left( V_{GS} - V_T - \frac{V_{DS}}{2} \right) V_{DS} & \text{if } V_{GS} \geq V_T, V_{DS} < V_{GS} - V_T \\ \mu C_{ox} \frac{W}{2L} \left( V_{GS} - V_T \right)^2 \cdot (1 + \lambda V_{DS}) & \text{if } V_{GS} \geq V_T, V_{DS} \geq V_{GS} - V_T \end{cases} \]

\[ V_T = V_{T0} + \gamma \left( \sqrt{\phi} - V_{BS} - \sqrt{\phi} \right) \]

\[ y_{11} = \left. \frac{\partial I_G}{\partial V_{GS}} \right|_{V=V_Q} = 0 \quad y_{12} = \left. \frac{\partial I_G}{\partial V_{DS}} \right|_{V=V_Q} = 0 \quad y_{13} = \left. \frac{\partial I_G}{\partial V_{GS}} \right|_{V=V_Q} = 0 \]

\[ y_{31} = \left. \frac{\partial I_B}{\partial V_{GS}} \right|_{V=V_Q} = 0 \quad y_{32} = \left. \frac{\partial I_B}{\partial V_{DS}} \right|_{V=V_Q} = 0 \quad y_{33} = \left. \frac{\partial I_B}{\partial V_{GS}} \right|_{V=V_Q} = 0 \]

\[ y_{21} = \left. \frac{\partial I_D}{\partial V_{GS}} \right|_{V=V_Q} = g_m \quad y_{12} = \left. \frac{\partial I_D}{\partial V_{DS}} \right|_{V=V_Q} = g \quad y_{13} = \left. \frac{\partial I_D}{\partial V_{BS}} \right|_{V=V_Q} = g_{nb} \]
\[ I_D = \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 \cdot (1 + \lambda V_{DS}) \]

\[ V_T = V_{T0} + \gamma \left( \sqrt{\phi - V_{BS}} - \sqrt{\phi} \right) \]

\[ g_m = \frac{\partial I_D}{\partial V_{GS}} \bigg|_{V = V_Q} = \mu C_{ox} \frac{W}{2L} 2 (V_{GS} - V_T)^1 \cdot (1 + \lambda V_{DS}) \bigg|_{V = V_Q} \approx \mu C_{ox} \frac{W}{L} V_{EBQ} \]

\[ g_o = \frac{\partial I_D}{\partial V_{DS}} \bigg|_{V = V_Q} = \mu C_{ox} \frac{W}{2L} 2 (V_{GS} - V_T)^2 \cdot \lambda \bigg|_{V = V_Q} \approx \lambda I_{DQ} \]

\[ g_{mb} = \frac{\partial I_D}{\partial V_{BS}} \bigg|_{V = V_Q} = \mu C_{ox} \frac{W}{2L} 2 (V_{GS} - V_T)^1 \cdot \left( -\frac{\partial V_T}{\partial V_{BS}} \right) \cdot (1 + \lambda V_{DS}) \bigg|_{V = V_Q} \]

\[ g_{mb} = \frac{\partial I_D}{\partial V_{BS}} \bigg|_{V = V_Q} \approx \mu C_{ox} \frac{W}{L} V_{EBQ} \cdot \frac{\partial V_T}{\partial V_{BS}} \bigg|_{V = V_Q} = \left( \mu C_{ox} \frac{W}{L} V_{EBQ} \right) (-\lambda) \gamma \frac{1}{2} (\phi - V_{BS})^{-\frac{1}{2}} \bigg|_{V = V_Q} (-1) \]

\[ g_{mb} \approx g_m \frac{\gamma}{2\sqrt{\phi - V_{BSQ}}} \]
Small Signal Model Summary

\[ i_g = 0 \]
\[ i_b = 0 \]
\[ i_d = g_m v_{gs} + g_{mb} v_{bs} + g_o v_{ds} \]

\[ g_m = \frac{\mu C_{OX} W}{L} V_{EBQ} \]

\[ g_o = \lambda I_{DQ} \]

\[ g_{mb} = g_m \left( \frac{\gamma}{2\sqrt{\phi - V_{BSQ}}} \right) \]
Relative Magnitude of Small Signal MOS Parameters

Consider:

\[ i_d = g_m v_{gs} + g_{mb} v_{bs} + g_o v_{ds} \]

3 alternate equivalent expressions for \( g_m \)

\[ g_m = \frac{\mu C_{OX} W}{L} v_{EBQ} \quad g_m = \sqrt{\frac{2\mu C_{OX} W}{L}} \sqrt{I_{DQ}} \quad g_m = \frac{2I_{DQ}}{V_{EBQ}} \]

If \( \mu C_{OX}=100\mu A/V^2 \), \( \lambda=0.01V^{-1} \), \( \gamma = 0.4V^{0.5} \), \( V_{EBQ}=1V \), \( W/L=1 \), \( V_{BSQ}=0V \)

\[ I_{DQ} \approx \frac{\mu C_{OX} W}{2L} v_{EBQ}^2 = \frac{10^{-4} W}{2L} (1V)^2 = 5E-5 \]

\[ g_m = \frac{\mu C_{OX} W}{L} v_{EBQ} = 1E-4 \]

\[ g_o = \lambda I_{DQ} = 5E-7 \]

\[ g_{mb} = g_m \left( \frac{\gamma}{2\sqrt{\phi - V_{BSQ}}} \right) = 0.26g_m \]

- Often the \( g_o \) term can be neglected in the small signal model because it is so small
- Be careful about neglecting \( g_o \) prior to obtaining a final expression

In this example

\[ g_0 << g_m, g_{mb} \]

\[ g_{mb} < g_m \]

This relationship is common

In many circuits, \( V_{BSQ}=0V \) as well
Large and Small Signal Model Summary

**Large Signal Model**

\[ I_D = \begin{cases} 
0 & V_{GS} \leq V_T \\
\mu C_{OX} \frac{W}{L} \left( V_{GS} - V_T - \frac{V_{DS}}{2} \right) V_{DS} & V_{GS} \geq V_T, V_{DS} < V_{GS} - V_T \\
\mu C_{OX} \frac{W}{2L} \left( V_{GS} - V_T \right)^2 \cdot \left( 1 + \lambda V_{DS} \right) & V_{GS} \geq V_T, V_{DS} \geq V_{GS} - V_T
\end{cases} \]

\[ V_T = V_{T0} + \gamma \left( \sqrt{\phi - V_{BS}} - \sqrt{\phi} \right) \]

**Small Signal Model**

\[ i_d = g_m v_{gs} + g_{mb} v_{bs} + g_o v_{ds} \]

where

\[ g_m = \frac{\mu C_{OX} W}{L} v_{EBQ} \]

\[ g_{mb} = g_m \left( \frac{\gamma}{2 \sqrt{\phi - V_{BSQ}}} \right) \]

\[ g_o = \lambda I_{DQ} \]
Large and Small Signal Model Summary

**Large Signal Model**

\[ I_C = \beta I_B \left( 1 + \frac{V_{CE}}{V_{AF}} \right) \]
\[ I_B = \frac{J_S A_E e}{\beta} \]

- \( V_{BE} > 0.4V \)
- \( V_{BC} < 0 \)

\[ V_{BE} = 0.7V \]
\[ V_{CE} = 0.2V \]

**Small Signal Model**

\[ i_b = g_{\pi} v_{be} \]
\[ i_c = g_m v_{be} + g_0 v_{ce} \]

where

\[ g_m = \frac{I_{CQ}}{V_t} \]
\[ g_{\pi} = \frac{I_{CQ}}{\beta V_t} \]
\[ g_o \approx \frac{I_{CQ}}{V_{AF}} \]
Relative Magnitude of Small Signal BJT Parameters

\[ g_m = \frac{I_{CQ}}{V_t} \quad g_\pi = \frac{I_{CQ}}{\beta V_t} \quad g_o \approx \frac{I_{CQ}}{V_{AF}} \]

\[ g_m = \begin{bmatrix} \frac{I_Q}{V_t} \\ \frac{I_Q}{\beta V_t} \end{bmatrix} \]

\[ g_\pi = \begin{bmatrix} \frac{I_Q}{\beta V_t} \end{bmatrix} \]

\[ g_o = \begin{bmatrix} \frac{I_Q}{V_{AF}} \end{bmatrix} \]

\[ g_m \gg g_\pi \gg g_o \]

Often the \( g_o \) term can be neglected in the small signal model because it is so small.
Relative Magnitude of Small Signal Parameters

\[ g_m = \frac{I_{CQ}}{V_t} \]

\[ g_\pi = \frac{I_{CQ}}{\beta V_t} \]

\[ g_o \approx \frac{I_{CQ}}{V_{AF}} \]

\[ \frac{g_m}{g_\pi} = \frac{\left[ \frac{I_Q}{V_t} \right]}{\left[ \frac{I_Q}{\beta V_t} \right]} = \beta \]

\[ \frac{g_\pi}{g_o} = \frac{\left[ \frac{I_Q}{\beta V_t} \right]}{\left[ \frac{I_Q}{V_{AF}} \right]} = \frac{V_{AF}}{\beta V_t} \approx \frac{200V}{100 \cdot 26mV} = 77 \]

\[ g_m \gg g_\pi \gg g_o \]

- Often the \( g_o \) term can be neglected in the small signal model because it is so small
- Be careful about neglecting \( g_o \) prior to obtaining a final expression
Small Signal Model Simplifications for the MOSFET and BJT

Often simplifications of the small signal model are adequate for a given application

These simplifications will be discussed next
Small Signal MOSFET Model Summary

An equivalent Circuit:

![Diagram of MOSFET equivalent circuit]

\[
g_m = \frac{\mu C_{ox} W}{L} (V_{GSQ} - V_T)
\]

\[
g_o = \lambda I_{DQ}
\]

\[
g_{mb} = g_m \left( \frac{\gamma}{2\sqrt{\phi - V_{BSQ}}} \right)
\]

Alternate equivalent representations for \(g_m\)

from

\[
I_D \approx \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2
\]

\[
g_m = \sqrt{\frac{2\mu C_{ox} W}{L}} \sqrt{I_{DQ}}
\]

\[
g_m = \frac{2I_{DQ}}{V_{GSQ} - V_T} = \frac{2I_{DQ}}{V_{EBQ}}
\]

\[
g_{mb} < g_m
\]

\[
g_0 << g_m, g_{mb}
\]
Simplification that is often adequate
Small Signal Model Simplifications

Even further simplification that is often adequate
Small Signal BJT Model Summary

An equivalent circuit

$$g_m = \frac{I_{CQ}}{V_t}$$

$$g_\pi = \frac{I_{CQ}}{\beta V_t}$$

$$g_o \approx \frac{I_{CQ}}{V_{AF}}$$

$$g_m \gg g_\pi \gg g_o$$

This contains absolutely no more information than the set of small-signal model equations.
Small Signal BJT Model Simplifications

Simplification that is often adequate
Gains for MOSFET and BJT Circuits

For both circuits
\[ A_v = -g_m R \]

Gains vary linearly with small signal parameter \( g_m \)

Power is often a key resource in the design of an integrated circuit
In both circuits, power is proportional to \( I_{CQ}, I_{DQ} \)
How does $g_m$ vary with $I_{DQ}$?

$$g_m = \sqrt{\frac{2\mu C_{OX} W}{L}} \sqrt{I_{DQ}}$$

Varies with the square root of $I_{DQ}$

$$g_m = \frac{2I_{DQ}}{V_{GSQ} - V_T} = \frac{2I_{DQ}}{V_{EBQ}}$$

Varies linearly with $I_{DQ}$

$$g_m = \frac{\mu C_{OX} W}{L} (V_{GSQ} - V_T)$$

 Doesn’t vary with $I_{DQ}$
How does $g_m$ vary with $I_{DQ}$?

All of the above are true – but with qualification

g_m is a function of more than one variable ($I_{DQ}$) and how it varies depends upon how the remaining variables are constrained
End of Lecture 24