EE 330
Lecture 24

• Small Signal Analysis
• Small Signal Analysis of BJT Amplifier
Exam 2  Friday March  9

Exam 3  Friday April  13

Review Session for Exam 2:
6:00 p.m. on Thursday March 8 in Room Sweeney 1116
Review from Last Lecture

Comparison of Gains for MOSFET and BJT Circuits

**BJT**

\[ A_{VB} = -\frac{I_{CQ} R_1}{V_t} \]

If \( I_{DQ} R = I_{CQ} R_1 = 2V \), \( V_{SS} + V_T = -1V \), \( V_t = 25mV \)

\[ A_{VB} = -\frac{2V}{25mV} = -80 \]

**MOSFET**

\[ A_{VM} = \frac{2I_{DQ} R}{[V_{SS} + V_T]} \]

\[ A_{VM} = \frac{4V}{-1V} = -4 \]

Observe \( A_{VB} >> A_{VM} \)

Due to exponential-law rather than square-law model
Review from Last Lecture

Operation with Small-Signal Inputs

- Analysis procedure for these simple circuits was very tedious
- This approach will be unmanageable for even modestly more complicated circuits
- **Faster analysis method is needed!**
Review from Last Lecture

Small-Signal Analysis

- Will commit next several lectures to developing this approach
- Analysis will be MUCH simpler, faster, and provide significantly more insight
- Applicable to many fields of engineering
Small-Signal Principle

Relationship is nearly linear in a small enough region around Q-point
Region of linearity is often quite large
Linear relationship may be different for different Q-points
Device Behaves Linearly in Neighborhood of Q-Point
Can be characterized in terms of a small-signal coordinate system
Small-Signal Principle

\[ y = f(x_Q) + \left. \frac{\partial f}{\partial x} \right|_{x=x_Q} x_{ss} \]

Quiescent Output

ss Gain

How can a circuit be linearized at an operating point as an alternative to linearizing a nonlinear function at an operating point?

Consider arbitrary nonlinear one-port network
**Arbitrary Nonlinear One-Port**

The equation for an arbitrary nonlinear one-port is given by:

\[ I = f(V) \]

At the Q-point, the steady-state current and voltage are defined as:

\[ i_{SS} = \left. \frac{\partial I}{\partial V} \right|_{V=V_Q} V_{SS} \]

\[ i_{SS} = i \]

\[ V_{SS} = \mathcal{U} \]

Linear model of the nonlinear device at the Q-point:

\[ i = y \mathcal{U} \]
The small-signal model of this 2-terminal electrical network is a resistor of value $1/y$ or a conductor of value $y$.

One small-signal parameter characterizes this one-port but it is dependent on Q-point.

This applies to ANY nonlinear one-port that is differentiable at a Q-point (e.g. a diode).
Small-Signal Principle

Goal with small signal model is to predict performance of circuit or device in the vicinity of an operating point (Q-point)

Will be extended to functions of two and three variables (e.g. BJT$s$ and MOSFET$s$)
Solution for the example of the previous lecture was based upon solving the nonlinear circuit for \( V_{\text{OUT}} \) and then linearizing the solution by doing a Taylor’s series expansion.

- Solution of nonlinear equations very involved with two or more nonlinear devices
- Taylor’s series linearization can get very tedious if multiple nonlinear devices are present

**Natural approach to small-signal analysis of nonlinear networks**

1. Solve nonlinear network
2. Linearize solution

**Alternative Approach to small-signal analysis of nonlinear networks**

1. Linearize nonlinear devices (all)
2. Obtain small-signal model from linearized device models
3. Replace all devices with small-signal equivalent
4. Solve linear small-signal network
“Alternative” Approach to small-signal analysis of nonlinear networks

Nonlinear Network

dc Equivalent Network

Q-point

Values for small-signal parameters

Small-signal (linear) equivalent network

Small-signal output

Total output

(good approximation)
Linearized nonlinear devices

Nonlinear Device \[\rightarrow\] Linearized Small-signal Device

This terminology will be used in THIS course to emphasize difference between nonlinear model and linearized small signal model
Example:

It will be shown that the nonlinear circuit has the linearized small-signal network given
Linearized Small-Signal Circuit Elements

Must obtain the linearized small-signal circuit element for ALL linear and nonlinear circuit elements

(Will also give models that are usually used for Q-point calculations: Simplified dc models)
Small-signal and simplified dc equivalent elements

<table>
<thead>
<tr>
<th>Element</th>
<th>ss equivalent</th>
<th>Simplified dc equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>dc Voltage Source</td>
<td>$V_{DC}$</td>
<td>$V_{DC}$</td>
</tr>
<tr>
<td>ac Voltage Source</td>
<td>$V_{AC}$</td>
<td>$V_{AC}$</td>
</tr>
<tr>
<td>dc Current Source</td>
<td>$I_{DC}$</td>
<td>$I_{DC}$</td>
</tr>
<tr>
<td>ac Current Source</td>
<td>$I_{AC}$</td>
<td>$I_{AC}$</td>
</tr>
<tr>
<td>Resistor</td>
<td>$R$</td>
<td>$R$</td>
</tr>
</tbody>
</table>
Small-signal and simplified dc equivalent elements

- **Capacitors**
  - Large: \( \frac{C}{\text{Large}} \)
  - Small: \( \frac{C}{\text{Small}} \)

- **Inductors**
  - Large: \( \frac{L}{\text{Large}} \)
  - Small: \( \frac{L}{\text{Small}} \)

- **Diodes**

- **MOS transistors**
  - MOSFET (enhancement or depletion, JFET)
Small-signal and simplified dc equivalent elements

<table>
<thead>
<tr>
<th>Element</th>
<th>ss equivalent</th>
<th>Simplified dc equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bipolar Transistors</td>
<td></td>
<td></td>
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<td></td>
<td></td>
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<tr>
<td>Dependent Sources (Linear)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V_O = A_V V_{IN}$</td>
<td>$I_O = A_I I_{IN}$</td>
<td>$V_O = R_I I_{IN}$</td>
</tr>
</tbody>
</table>

Small-signal and simplified dc equivalent elements
Example: Obtain the small-signal equivalent circuit
Example: Obtain the small-signal equivalent circuit
Example: Obtain the small-signal equivalent circuit

\[ V_{INSS} \quad V_{DD} \quad V_{OUT} \quad V_{SS} \]

\[ R_1, R_2, R_4, R_5, R_6, R_L \]

\[ C_1, C_2, C_3 \text{ large} \]

\[ C_4 \text{ small} \]

\[ I_{DD} \]

\[ Q_1 \]

\[ M_1 \]
How is the small-signal equivalent circuit obtained from the nonlinear circuit?

What is the small-signal equivalent of the MOSFET, BJT, and diode?
A Small Signal Equivalent Circuit

Thus, for the diode

\[ R_d = \left( \frac{\partial I_D}{\partial V_D} \right)_Q^{-1} \]
Small-Signal Diode Model

For the diode

\[
R_d = \left( \frac{\partial I_D}{\partial V_D} \right)_Q^{-1}
\]

\[
I_D = I_{Se} \frac{V_D}{V_t}
\]

\[
\left. \frac{\partial I_D}{\partial V_D} \right|_Q = \left[ \left( I_{Se} \frac{V_D}{V_t} \right) \frac{1}{V_t} \right]_Q = \frac{I_{DQ}}{V_t}
\]

\[
R_d = \frac{V_t}{I_{DQ}}
\]
Example of diode circuit where small-signal diode model is useful

Voltage Reference

Small-signal model of Voltage Reference (useful for compensation when parasitic Cs included)
Small-Signal Model of BJT and MOSFET

Consider 4-terminal network
(3-terminal and 2-terminal and 1-terminal devices then become special cases)

\[
\begin{align*}
I_1 &= f_1(V_1, V_2, V_3) \\
I_2 &= f_2(V_1, V_2, V_3) \\
I_3 &= f_3(V_1, V_2, V_3)
\end{align*}
\]

4 different ways to choose reference terminal

Six port electrical variables \(\{I_1, I_2, I_3, V_1, V_2, V_3\}\)

Number of ways to choose independent variables
\[
\binom{6}{3} = \frac{6!}{(6-3)!3!} = 20
\]

Number of potentially different ways to represent same device 80

We will choose one of these 80 which uses port voltages as independent variables
Small-Signal Model of BJT and MOSFET

Consider 4-terminal network

Define

\[ i_1 = I_1 - I_{1Q} \]
\[ i_2 = I_2 - I_{2Q} \]
\[ i_3 = I_3 - I_{3Q} \]

\[ \nu_1 = V_1 - V_{1Q} \]
\[ \nu_2 = V_2 - V_{2Q} \]
\[ \nu_3 = V_3 - V_{3Q} \]

Small signal model is that which represents the relationship between the small signal voltages and the small signal currents.
Small-Signal Model of 4-Terminal Network

Small signal model is that which represents the relationship between the small signal voltages and the small signal currents. For small signals, this relationship should be linear. Can be thought of as a change in coordinate systems from the large signal coordinate system to the small-signal coordinate system.
Small-Signal Model of 4-Terminal Network

\[
\begin{align*}
I_1 &= f_1(V_1, V_2, V_3) \\
I_2 &= f_2(V_1, V_2, V_3) \\
I_3 &= f_3(V_1, V_2, V_3)
\end{align*}
\]

\[
\begin{align*}
i_1 &= g_1(V_1, V_2, V_3) \\
i_2 &= g_2(V_1, V_2, V_3) \\
i_3 &= g_3(V_1, V_2, V_3)
\end{align*}
\]

Mapping is unique (with same models)
Small-Signal Model of 4-Terminal Network

\[ I_1 = f_1(V_1, V_2, V_3) \]
\[ I_2 = f_2(V_1, V_2, V_3) \]
\[ I_3 = f_3(V_1, V_2, V_3) \]

\[ i_1 = g_1(V_1, V_2, V_3) \]
\[ i_2 = g_2(V_1, V_2, V_3) \]
\[ i_3 = g_3(V_1, V_2, V_3) \]

Does inverse mapping exist? Yes

Is it unique (with same models)? No

Multiple nonlinear circuits can have same small-signal circuit
Recall for a function of one variable

\[ y = f(x) \]

Taylor's Series Expansion about the point \( x_0 \)

\[
y = f(x) = f(x)\bigg|_{x=x_0} + \frac{\partial f}{\partial x}\bigg|_{x=x_0} (x-x_0) + \frac{\partial^2 f}{\partial x^2}\bigg|_{x=x_0} \frac{1}{2!} (x-x_0)^2 + \ldots
\]

If \( x-x_0 \) is small

\[
y \approx f(x)\bigg|_{x=x_0} + \frac{\partial f}{\partial x}\bigg|_{x=x_0} (x-x_0)
\]

\[
y \approx y_0 + \frac{\partial f}{\partial x}\bigg|_{x=x_0} (x-x_0)
\]

\( x_0 \) is termed the expansion point or the Q-point.
Recall for a function of one variable
\[ y = f(x) \]

If \( x-x_0 \) is small

\[
y \approx y_0 + \left. \frac{\partial f}{\partial x} \right|_{x=x_0} (x-x_0)
\]

\[
y - y_0 = \left. \frac{\partial f}{\partial x} \right|_{x=x_0} (x-x_0)
\]

If we define the small signal variables as

\[
y = y - y_0
\]

\[
x = x - x_0
\]
Recall for a function of one variable

\[ y = f(x) \]

If \( x - x_0 \) is small

\[ y \approx y_0 + \left. \frac{\partial f}{\partial x} \right|_{x=x_0} (x - x_0) \]

\[ y - y_0 = \left. \frac{\partial f}{\partial x} \right|_{x=x_0} (x - x_0) \]

If we define the small signal variables as

\[ y = y - y_0 \]

\[ x = x - x_0 \]

Then

\[ y = \left. \frac{\partial f}{\partial x} \right|_{x=x_0} \cdot x \]

This relationship is linear!
Consider now a function of $n$ variables

$$y = f(x_1, \ldots x_n) = f(\bar{x})$$

If we consider an arbitrary expansion point $\bar{X}_0 = \{x_{10}, x_{20}, \ldots x_{n0}\}$

The multivariate Taylor’s series expansion around the point $\bar{X}_0$ is given by

$$y = f(\bar{x}) = f(x)|_{\bar{x} = x_0} + \sum_{k=1}^{n} \left( \frac{\partial f}{\partial x_k} \right) \bigg|_{\bar{x} = x_0} (x_k - x_{k0})$$

$$+ \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k} \bigg|_{\bar{x} = x_0} \frac{1}{2!} (x_j - x_{j0})(x_k - x_{k0}) + \ldots \text{(H.O.T.)}$$

Truncating after first-order terms, we obtain the approximation

$$y - y_0 \approx \sum_{k=1}^{n} \left( \frac{\partial f}{\partial x_k} \right) \bigg|_{\bar{x} = x_0} (x_k - x_{k0})$$

where $y_0 = f(x)|_{\bar{x} = x_0}$
Multivariate Taylors Series Expansion

\[ y = f(x_1, \ldots, x_n) = f(\bar{x}) \]

Linearized approximation

\[ y - y_0 \approx \sum_{k=1}^{n} \left( \frac{\partial f}{\partial x_k} \bigg|_{\bar{x} = \bar{x}_0} (x_k - x_{k0}) \right) \]

This can be expressed as

\[ y_{ss} \approx \sum_{k=1}^{n} a_k x_{ssk} \]

where

\[ y_{ss} = y - y_0 \]
\[ x_{ssk} = x_k - x_{k0} \]
\[ a_k = \frac{\partial f}{\partial x_k} \bigg|_{\bar{x} = \bar{x}_0} \]
In the more general form\(^1\), the multivariate Taylor's series expansion can be expressed as

\[
f(x_1, \ldots, x_n) = \alpha_o + \sum_{m=1}^{\infty} \sum_{k_1, \ldots, k_n}^{\sum k_j = m} \alpha_{k_1, \ldots, k_n; m} (x_1 - x_{1,o})^{k_1} \cdots (x_n - x_{n,o})^{k_n}
\]

(7)

\[
\alpha_o = f(x_{1o}, \ldots, x_{no})
\]

\[
\alpha_{k_1, \ldots, k_n; m} = \frac{1}{k_1! \cdots k_n!} \frac{\partial^m f}{\partial^{k_1} x_1 \cdots \partial^{k_n} x_n} \bigg|_{x_{1o}, \ldots, x_{no}}
\]

(8)

\(^1\) http://www.chem.mtu.edu/~tbco/cm416/taylor.html
Consider 4-terminal network

\[ I_1 = f_1(V_1, V_2, V_3) \]
\[ I_2 = f_2(V_1, V_2, V_3) \]
\[ I_3 = f_3(V_1, V_2, V_3) \]

Nonlinear network characterized by 3 functions each functions of 3 variables
Consider now 3 functions each functions of 3 variables

\[
\begin{align*}
I_1 &= f_1(V_1, V_2, V_3) \\
I_2 &= f_2(V_1, V_2, V_3) \\
I_3 &= f_3(V_1, V_2, V_3)
\end{align*}
\]

Define

\[
\tilde{V}_Q = \begin{bmatrix}
V_{1Q} \\
V_{2Q} \\
V_{3Q}
\end{bmatrix}
\]

In what follows, we will use \( \tilde{V}_Q \) as an expansion point in a Taylor’s series expansion.
Consider now 3 functions each functions of 3 variables

\[ I_1 = f_1(V_1, V_2, V_3) \]
\[ I_2 = f_2(V_1, V_2, V_3) \]
\[ I_3 = f_3(V_1, V_2, V_3) \]

Define \[ \bar{V}_Q = \begin{bmatrix} V_{1Q} \\ V_{2Q} \\ V_{3Q} \end{bmatrix} \]

Consider first the function \( I_1 \)

The multivariate Taylors Series expansion of \( I_1 \), around the operating point \( \bar{V}_Q \) , when truncated after first-order terms, can be expressed as:

\[
I_1 = f_1(V_1, V_2, V_3) \approx f_1(V_{1Q}, V_{2Q}, V_{3Q}) + \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_1} \bigg|_{V=V_0} (V_1 - V_{1Q}) + \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_2} \bigg|_{V=V_0} (V_2 - V_{2Q}) + \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_3} \bigg|_{V=V_0} (V_3 - V_{3Q})
\]

or equivalently as:

\[
I_1 - I_{1Q} = \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_1} \bigg|_{V=V_0} (V_1 - V_{1Q}) + \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_2} \bigg|_{V=V_0} (V_2 - V_{2Q}) + \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_3} \bigg|_{V=V_0} (V_3 - V_{3Q})
\]
repeating from previous slide:

\[ I_1 - I_{1Q} = \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_1} \bigg|_{V=V_Q} (V_1 - V_{1Q}) + \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_2} \bigg|_{V=V_Q} (V_2 - V_{2Q}) + \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_3} \bigg|_{V=V_Q} (V_3 - V_{3Q}) \]

Make the following definitions

\[ y_{11} = \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_1} \bigg|_{V=V_Q} \]

\[ y_{12} = \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_2} \bigg|_{V=V_Q} \]

\[ y_{13} = \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_3} \bigg|_{V=V_Q} \]

It thus follows that

\[ \mathbf{i}_1 = y_{11} \mathbf{v}_1 + y_{12} \mathbf{v}_2 + y_{13} \mathbf{v}_3 \]

This is a linear relationship between the small signal electrical variables!
Small Signal Model Development

Nonlinear Model

\[ I_1 = f_1(V_1, V_2, V_3) \]

\[ I_2 = f_2(V_1, V_2, V_3) \]

\[ I_3 = f_3(V_1, V_2, V_3) \]

Linear Model at \( \vec{V}_Q \) (alt. small signal model)

\[ i_1 = y_{11} u_1 + y_{12} u_2 + y_{13} u_3 \]

Extending this approach to the two nonlinear functions \( I_2 \) and \( I_3 \)

\[ i_2 = y_{21} u_1 + y_{22} u_2 + y_{23} u_3 \]

\[ i_3 = y_{31} u_1 + y_{32} u_2 + y_{33} u_3 \]

where

\[ y_{ij} = \left. \frac{\partial f_i(V_1, V_2, V_3)}{\partial V_j} \right|_{V=\vec{V}_Q} \]
Small Signal Model Development

Nonlinear Model

\[ I_1 = f_1(V_1, V_2, V_3) \]
\[ I_2 = f_2(V_1, V_2, V_3) \]
\[ I_3 = f_3(V_1, V_2, V_3) \]

Linear Model at \( \tilde{V}_q \)
(alternative small signal model)

\[ i_1 = y_{11} v_1 + y_{12} v_2 + y_{13} v_3 \]
\[ i_2 = y_{21} v_1 + y_{22} v_2 + y_{23} v_3 \]
\[ i_3 = y_{31} v_1 + y_{32} v_2 + y_{33} v_3 \]

where

\[ y_{ij} = \frac{\partial f_i(V_1, V_2, V_3)}{\partial V_j} \bigg|_{V=\tilde{V}_q} \]
Small Signal Model

\[ i_1 = y_{11} v_1 + y_{12} v_2 + y_{13} v_3 \]
\[ i_2 = y_{21} v_1 + y_{22} v_2 + y_{23} v_3 \]
\[ i_3 = y_{31} v_1 + y_{32} v_2 + y_{33} v_3 \]

where

\[ y_{ij} = \left. \frac{\partial f_i (V_1, V_2, V_3)}{\partial V_j} \right|_{v=q} \]

- This is a small-signal model of a 4-terminal network and it is linear
- 9 small-signal parameters characterize the linear 4-terminal network
- Small-signal model parameters dependent upon Q-point!
- Termed the y-parameter model or “admittance” –parameter model
A small-signal equivalent circuit of a 4-terminal nonlinear network
(equivalent circuit because has exactly the same port equations)

\[ y_{ij} = \left. \frac{\partial f_i(V_1, V_2, V_3)}{\partial V_j} \right|_{\tilde{V} = \tilde{V}_Q} \]

Equivalent circuit is not unique
Equivalent circuit is a three-port network
4-terminal small-signal network summary

Small signal model:

\[
\begin{align*}
\mathbf{I}_1 &= f_1(V_1, V_2, V_3) \\
\mathbf{I}_2 &= f_2(V_1, V_2, V_3) \\
\mathbf{I}_3 &= f_3(V_1, V_2, V_3)
\end{align*}
\]

\[
\begin{align*}
\mathbf{i}_1 &= y_{11} \mathbf{v}_1 + y_{12} \mathbf{v}_2 + y_{13} \mathbf{v}_3 \\
\mathbf{i}_2 &= y_{21} \mathbf{v}_1 + y_{22} \mathbf{v}_2 + y_{23} \mathbf{v}_3 \\
\mathbf{i}_3 &= y_{31} \mathbf{v}_1 + y_{32} \mathbf{v}_2 + y_{33} \mathbf{v}_3
\end{align*}
\]

\[
\mathbf{y}_{ij} = \left. \frac{\partial f_i(V_1, V_2, V_3)}{\partial V_j} \right|_{\bar{V} = \bar{V}_Q}
\]
Consider 3-terminal network

Small-Signal Model

\[ I_1 = f_1(V_1, V_2) \]
\[ I_2 = f_2(V_1, V_2) \]

Define

\[ i_1 = I_1 - I_{1Q} \]
\[ i_2 = I_2 - I_{2Q} \]
\[ v_1 = V_1 - V_{1Q} \]
\[ v_2 = V_2 - V_{2Q} \]

Small signal model is that which represents the relationship between the small signal voltages and the small signal currents.
Consider 3-terminal network

Small-Signal Model

\[ i_1 = g_1(v_1,v_2,v_3) \]
\[ i_2 = g_2(v_1,v_2,v_3) \]
\[ i_3 = g_3(v_1,v_2,v_3) \]

\[ i_1 = y_{11}v_1 + y_{12}v_2 + y_{13}v_3 \]
\[ i_2 = y_{21}v_1 + y_{22}v_2 + y_{23}v_3 \]
\[ i_3 = y_{31}v_1 + y_{32}v_2 + y_{33}v_3 \]

\[ y_{ij} = \frac{\partial f_i(v_1,v_2,v_3)}{\partial v_j} \bigg|_{\tilde{v} = \tilde{v}_q} \]
Consider 3-terminal network

**Small-Signal Model**

\[
\begin{align*}
    i_1 &= y_{11}v_1 + y_{12}v_2 \\
    i_2 &= y_{21}v_1 + y_{22}v_2
\end{align*}
\]

An equivalent small signal circuit can be shown.

\[
y_{ij} = \left. \frac{\partial f_i(v_1, v_2)}{\partial v_j} \right|_{\bar{v}=v_Q}
\]

\[
\bar{v} = \begin{pmatrix} v_{1Q} \\ v_{2Q} \end{pmatrix}
\]

4 small-signal parameters characterize this 3-terminal (two-port) linear network. Small signal parameters dependent upon Q-point.
3-terminal small-signal network summary

3-Terminal Device

\[ \begin{align*}
I_1 &= f_1(V_1, V_2) \\
I_2 &= f_2(V_1, V_2)
\end{align*} \]

Small signal model:

\[ \begin{align*}
i_1 &= y_{11}V_1 + y_{12}V_2 \\
i_2 &= y_{21}V_1 + y_{22}V_2
\end{align*} \]

\[ y_{ij} = \left. \frac{\partial f_i(V_1, V_2)}{\partial V_j} \right|_{V=V_Q} \]
Consider 2-terminal network

Small-Signal Model

\[ I_1 = f_1(V_1) \]

Define

\[ i_1 = I_1 - I_{1Q} \]
\[ v_1 = V_1 - V_{1Q} \]

Small signal model is that which represents the relationship between the small signal voltages and the small signal currents.
Consider 2-terminal network

Small-Signal Model

\[
\begin{align*}
\mathbf{i}_1 &= y_{11} \mathbf{V}_1 + y_{12} \mathbf{V}_2 + y_{13} \mathbf{V}_3 \\
\mathbf{i}_2 &= y_{21} \mathbf{V}_1 + y_{22} \mathbf{V}_2 + y_{23} \mathbf{V}_3 \\
\mathbf{i}_3 &= y_{31} \mathbf{V}_1 + y_{32} \mathbf{V}_2 + y_{33} \mathbf{V}_3
\end{align*}
\]

\[
\begin{align*}
\mathbf{i}_1 &= g_1(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3) \\
\mathbf{i}_2 &= g_2(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3) \\
\mathbf{i}_3 &= g_3(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3)
\end{align*}
\]

\[
y_{ij} = \left. \frac{\partial f_i(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3)}{\partial \mathbf{V}_j} \right|_{\mathbf{V} = \bar{\mathbf{V}}}
\]
Consider 2-terminal network

**Small-Signal Model**

\[ i_1 = y_{11} V_1 \]

\[ y_{11} = \frac{\partial f_1(V_1)}{\partial V_1} \bigg|_{V=V_Q} \]

\[ \tilde{V} = V_{1Q} \]

A Small Signal Equivalent Circuit

This was actually developed earlier!
Linearized nonlinear devices
How is the small-signal equivalent circuit obtained from the nonlinear circuit?

What is the small-signal equivalent of the MOSFET, BJT, and diode?
MOSFET is actually a 4-terminal device but for many applications acceptable predictions of performance can be obtained by treating it as a 3-terminal device by neglecting the bulk terminal.

In this course, we have been treating it as a 3-terminal device and in this lecture will develop the small-signal model by treating it as a 3-terminal device.

When treated as a 4-terminal device, the bulk voltage introduces one additional term to the small signal model which is often either negligibly small or has no effect on circuit performance (will develop 4-terminal ss model later).
Small Signal Model of MOSFET

Large Signal Model

\[ I_G = 0 \]

3-terminal device

\[ I_D = \begin{cases} 
0 & V_{GS} \leq V_T \\
\mu C_{OX} \frac{W}{L} \left( V_{GS} - V_T - \frac{V_{DS}}{2} \right) V_{DS} & V_{GS} \geq V_T, V_{DS} < V_{GS} - V_T \\
\mu C_{OX} \frac{W}{2L} \left( V_{GS} - V_T \right)^2 \left(1 + \lambda V_{DS} \right) & V_{GS} \geq V_T, V_{DS} \geq V_{GS} - V_T 
\end{cases} \]

MOSFET is usually operated in saturation region in linear applications where a small-signal model is needed so will develop the small-signal model in the saturation region.
Small Signal Model of MOSFET

\[ I_1 = f_1(V_1, V_2) \quad \leftrightarrow \quad I_G = 0 \]

\[ I_2 = f_2(V_1, V_2) \quad \leftrightarrow \quad I_D = \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 (1 + a V_{DS}) \]

\[ I_G = f_1(V_{GS}, V_{DS}) \]

\[ I_D = f_2(V_{GS}, V_{DS}) \]

Small-signal model:

\[ y_{ij} = \frac{\partial f_i(V_1, V_2)}{\partial V_j} \bigg|_{V=V_Q} \]

\[ y_{11} = \frac{\partial I_G}{\partial V_{GS}} \bigg|_{V=V_Q} \]

\[ y_{12} = \frac{\partial I_G}{\partial V_{DS}} \bigg|_{V=V_Q} \]

\[ y_{21} = \frac{\partial I_D}{\partial V_{GS}} \bigg|_{V=V_Q} \]

\[ y_{22} = \frac{\partial I_D}{\partial V_{DS}} \bigg|_{V=V_Q} \]
Small Signal Model of MOSFET

\[ I_G = 0 \]

\[ I_D = \mu C_{OX} \frac{W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS}) \]

\[ y_{11} = \frac{\partial I_G}{\partial V_{GS}} \bigg|_{V=V_0} = ? \]

\[ y_{12} = \frac{\partial I_G}{\partial V_{DS}} \bigg|_{V=V_0} = ? \]

\[ y_{21} = \frac{\partial I_D}{\partial V_{GS}} \bigg|_{V=V_0} = ? \]

\[ y_{22} = \frac{\partial I_D}{\partial V_{DS}} \bigg|_{V=V_0} = ? \]

Recall: termed the \( y \)-parameter model
Small Signal Model of MOSFET

\[ I_1 = f_1(V_1, V_2) \quad \rightarrow \quad I_G = 0 \]

\[ I_2 = f_2(V_1, V_2) \quad \rightarrow \quad I_D = \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS}) \]

**Small-signal model:**

\[ y_{11} = \frac{\partial I_G}{\partial V_{GS}} \bigg|_{V=V_q} = 0 \]

\[ y_{12} = \frac{\partial I_G}{\partial V_{DS}} \bigg|_{V=V_q} = 0 \]

\[ y_{21} = \frac{\partial I_D}{\partial V_{GS}} \bigg|_{V=V_q} = 2\mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS}) \bigg|_{V=V_q} = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T)(1 + \lambda V_{DSQ}) \]

\[ y_{21} \approx \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \]

\[ y_{22} = \frac{\partial I_D}{\partial V_{DS}} \bigg|_{V=V_q} = \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 \lambda \bigg|_{V=V_q} \approx \lambda I_{DQ} \]
Small Signal Model of MOSFET

\[ I_G = 0 \]
\[ I_D = \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS}) \]

\[ y_{12} = 0 \]
\[ y_{11} = 0 \]
\[ y_{21} \approx \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \]
\[ y_{22} \approx \lambda I_{DQ} \]

\[ i_G = y_{11} V_{GS} + y_{12} V_{DS} \]
\[ i_D = y_{21} V_{GS} + y_{22} V_{DS} \]

An equivalent circuit (y-parameter model)
End of Lecture 24