EE 330
Lecture 26

Small-Signal Models
ss models of BJT
Quiz 20

Obtain the small signal model of the following circuit. Assume MOSFET is operating in the saturation region.
And the number is ....
And the number is ....
Quiz 20

Obtain the small signal model of the following circuit. Assume
MOSFET is operating in the saturation region.

Solution:

\[ V \left( g_m + g_0 \right) = I \]

\[ R_{EQ} = \frac{1}{g_m + g_0} \approx \frac{1}{g_m} \]
Small Signal Model of BJT

3-terminal device

Forward Active Model:

\[ I_C = J_S A E \frac{V_{BE}}{V_t} \left( 1 + \frac{V_{CE}}{V_{AF}} \right) \]

\[ I_B = \frac{J_S A E}{\beta} \frac{V_{BE}}{V_t} e^{V_{BE}/V_t} \]

Usually operated in Forward Active Region when small-signal model is needed
Small Signal Model of BJT

\[ i_B = y_{11} V_{BE} + y_{12} V_{CE} \]
\[ i_C = y_{21} V_{BE} + y_{22} V_{CE} \]

\[ i_B = g_{\pi} V_{BE} \]
\[ i_C = g_m V_{BE} + g_o V_{CE} \]

\[ g_{\pi} = \frac{I_{CO}}{\beta V_t} \]
\[ g_m = \frac{I_{CO}}{V_t} \]
\[ g_o = \frac{I_{CO}}{V_{AF}} \]
What are the simplified dc equivalent models?
What are the simplified dc equivalent models?

Active Device Model Summary

Review from Last Time
Example: Determine the small signal voltage gain $A_V = \frac{V_{OUT}}{V_{IN}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$. 

Small-signal circuit
Review from Last Time

Example:

![Small-signal circuit diagram]

Analysis:

$$A_v = \frac{V_{OUT}}{V_{IN}} = -\frac{g_{m1}}{g_{m2}}$$

Recall:

$$A_v = -\sqrt{\frac{W_1}{W_2}} \sqrt{\frac{L_2}{L_1}}$$

If $L_1 = L_2$, obtain

$$A_v = -\sqrt{\frac{W_1}{W_2}} \sqrt{\frac{L_2}{L_1}} = -\sqrt{\frac{W_1}{W_2}}$$

The width and length ratios can be accurately set when designed in a standard CMOS process.

But care must be taken in the layout to obtain this accuracy!
Graphical Analysis and Interpretation

Consider Again

Review from Last Time

\[ V_{OUT} = V_{DD} - I_D R \]

\[ I_{DQ} = \frac{\mu C_{ox} W}{2L} (V_{ss} + V_T)^2 \]

\[ I_d = \frac{\mu C_{ox} W}{2L} (V_{IN} - V_{ss} - V_T)^2 \]
Graphical Analysis and Interpretation

Device Model (family of curves)

\[
I_{DQ} = \frac{\mu C_{ox} W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS})
\]

Review from Last Time

\[V_{GSQ} = -V_{SS}\]

\[
V_{OUT} = V_{DD} - I_D R
\]

\[
I_{DQ} = \frac{\mu C_{ox} W}{2L} (V_{SS} + V_T)^2
\]

\[
I_D = \frac{\mu C_{ox} W}{2L} (V_{IN} - V_{SS} - V_T)^2
\]
Graphical Analysis and Interpretation

Device Model (family of curves) \[ I_{DQ} = \frac{\mu C_{ox} W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS}) \]

Saturation region

\[ V_{GSQ} = -V_{SS} \]

Q-Point

Load Line
Graphical Analysis and Interpretation

Device Model (family of curves) \( I_{DQ} = \frac{\mu C_{ox} W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS}) \)

**Review from Last Time**

- Signal swing can be maximized by judicious location of Q-point
- Often selected to be at middle of load line in saturation region

Saturation region

Q-Point

Load Line

\( V_{GSQ} = -V_{SS} \)
Small-Signal MOSFET Model Extension

Existing model does not depend upon the bulk voltage!

Observe that changing the bulk voltage will change the electric field in the channel region!

\[ V_{BS} \] \[ V_{GS} \] \[ V_{DS} \] \[ I_D \] \[ I_G \] \[ I_B \]
Further Model Extensions

Existing model does not depend upon the bulk voltage!

Observe that changing the bulk voltage will change the electric field in the channel region!

Changing the bulk voltage will change the thickness of the inversion layer

Changing the bulk voltage will change the threshold voltage of the device

\[ V_T = V_{T0} + \gamma \left( \sqrt{\phi - V_{BS}} - \sqrt{\phi} \right) \]
Typical Effects of Bulk on Threshold Voltage for n-channel Device

\[ V_T = V_{T0} + \gamma \left( \sqrt{\phi - V_{BS}} - \sqrt{\phi} \right) \]

\[ \gamma \approx 0.4V^{\frac{1}{2}} \quad \phi \approx 0.6V \]

Bulk-Diffusion Generally Reverse Biased (\( V_{BS} < 0 \) or at least less than 0.3V) for n-channel
Shift in threshold voltage with bulk voltage can be substantial
Often \( V_{BS} = 0 \)
Typical Effects of Bulk on Threshold Voltage for p-channel Device

\[ V_T = V_{T0} - \gamma \left( \sqrt{\phi + V_{BS}} - \sqrt{\phi} \right) \]

\[ \gamma \approx 0.4V^{\frac{1}{2}} \quad \phi \approx 0.6V \]

Bulk-Diffusion Generally Reverse Biased (\(V_{BS} > 0\) or at least greater than -0.3V) for n-channel

Same functional form as for n-channel devices but \(V_{T0}\) is now negative and the magnitude of \(V_T\) still increases with the magnitude of the reverse bias
Model Extension Summary

$I_G = 0$
$I_B = 0$

$I_D = \begin{cases} 
0 & \text{if } V_{GS} \leq V_T \\
\mu C_{ox} \frac{W}{L} \left( V_{GS} - V_T - \frac{V_{DS}}{2} \right) V_{DS} & \text{if } V_{GS} \geq V_T \text{ and } V_{DS} < V_{GS} - V_T \\
\mu C_{ox} \frac{W}{2L} \left( V_{GS} - V_T \right)^2 \cdot (1 + \lambda V_{DS}) & \text{if } V_{GS} \geq V_T \text{ and } V_{DS} \geq V_{GS} - V_T 
\end{cases}
$

$V_T = V_{T0} + \gamma \left( \sqrt{\phi} - V_{BS} - \sqrt{\phi} \right)$

Model Parameters: \{\mu, C_{OX}, V_{T0}, \phi, \gamma, \lambda\}

Design Parameters: \{W, L\} but only one degree of freedom W/L
Small-Signal Model Extension

\[ I_G = 0 \]
\[ I_B = 0 \]

\[ I_D = \begin{cases} 
0 & \text{if } V_{GS} \leq V_T \\
\mu C_{ox} \frac{W}{L} \left( V_{GS} - V_T - \frac{V_{DS}}{2} \right) V_{DS} & \text{if } V_{GS} \geq V_T \quad V_{DS} < V_{GS} - V_T \\
\mu C_{ox} \frac{W}{2L} \left( V_{GS} - V_T \right)^2 \cdot (1 + \lambda V_{DS}) & \text{if } V_{GS} \geq V_T \quad V_{DS} \geq V_{GS} - V_T
\end{cases} \]

\[ V_T = V_{T0} + \gamma \left( \sqrt{\phi} - V_{BS} - \sqrt{\phi} \right) \]

\[ y_{11} = \left. \frac{\partial I_G}{\partial V_{GS}} \right|_{V = V_a} = 0 \quad y_{12} = \left. \frac{\partial I_G}{\partial V_{DS}} \right|_{V = V_a} = 0 \quad y_{13} = \left. \frac{\partial I_G}{\partial V_{GS}} \right|_{V = V_a} = 0 \]

\[ y_{31} = \left. \frac{\partial I_B}{\partial V_{GS}} \right|_{V = V_a} = 0 \quad y_{32} = \left. \frac{\partial I_B}{\partial V_{DS}} \right|_{V = V_a} = 0 \quad y_{33} = \left. \frac{\partial I_B}{\partial V_{GS}} \right|_{V = V_a} = 0 \]

\[ y_{21} = \left. \frac{\partial I_D}{\partial V_{GS}} \right|_{V = V_a} = g_w \quad y_{22} = \left. \frac{\partial I_D}{\partial V_{DS}} \right|_{V = V_a} = g_v \quad y_{23} = \left. \frac{\partial I_D}{\partial V_{GS}} \right|_{V = V_a} = g_{wb} \]
\[ I_D = \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 \cdot (1 + \lambda V_{DS}) \]

\[ V_T = V_{T0} + \gamma \left( \sqrt{\phi - V_{BS}} - \sqrt{\phi} \right) \]

\[ g_m = \frac{\partial I_D}{\partial V_{GS}} \bigg|_{\phi = \phi_Q} = \mu C_{ox} \frac{W}{2L} 2 \left( V_{GS} - V_T \right)^1 \cdot (1 + \lambda V_{DS}) \bigg|_{\phi = \phi_Q} \cong \mu C_{ox} \frac{W}{L} V_{EB} \]

\[ g_o = \frac{\partial I_D}{\partial V_{DS}} \bigg|_{\phi = \phi_Q} = \mu C_{ox} \frac{W}{2L} 2 \left( V_{GS} - V_T \right)^2 \cdot \lambda \bigg|_{\phi = \phi_Q} \cong \lambda I_{DQ} \]

\[ g_{mb} = \frac{\partial I_D}{\partial V_{BS}} \bigg|_{\phi = \phi_Q} = \mu C_{ox} \frac{W}{2L} 2 \left( V_{GS} - V_T \right)^1 \cdot \left( - \frac{\partial V_T}{\partial V_{BS}} \right) \cdot (1 + \lambda V_{DS}) \bigg|_{\phi = \phi_Q} \]

\[ g_{mb} \approx \mu C_{ox} \frac{W}{L} V_{EB} \cdot \frac{\partial V_T}{\partial V_{BS}} \bigg|_{\phi = \phi_Q} = \left( \mu C_{ox} \frac{W}{L} V_{EB} \right) (-\lambda) \gamma \frac{1}{2} \left( \phi - V_{BS} \right)^{\frac{1}{2}} \bigg|_{\phi = \phi_Q} \left( -\lambda \right) \]

\[ g_{mb} \approx g_m \frac{\gamma}{2\sqrt{\phi - V_{BS}}} \]
Small Signal Model Summary

\[ i_g = 0 \]

\[ i_b = 0 \]

\[ i_d = g_m v_{gs} + g_{mb} v_{bs} + g_o v_{ds} \]

\[ g_m = \mu C_{ox} W L \]

\[ g_o = \lambda I_{DQ} \]

\[ g_{mb} = g_m \left( \frac{\gamma}{2\sqrt{\phi - V_{BSQ}}} \right) \]
Relative Magnitude of Small Signal MOS Parameters

Consider:

\[ i_d = g_m v_{gs} + g_{mb} v_{bs} + g_o v_{ds} \]

3 alternate equivalent expressions for \( g_m \)

\[ g_m = \frac{\mu C_{ox} W}{L} V_{EBQ} \quad \quad g_m = \sqrt{\frac{2\mu C_{ox} W}{L}} \sqrt{I_{DQ}} \quad \quad g_m = \frac{2I_{DQ}}{V_{EBQ}} \]

If \( \mu C_{ox}=100\mu A/V^2 \), \( \lambda=.01V^{-1} \), \( \gamma = 0.4V^{0.5} \), \( V_{EBQ}=1V \), \( W/L=1 \), \( V_{BSQ}=0V \)

\[ I_{DQ} \approx \frac{\mu C_{ox} W}{2L} V_{EBQ}^2 = \frac{10^4 W}{2L} (1V)^2 = 5E-5 \]

\[ g_m = \frac{\mu C_{ox} W}{L} V_{EBQ} = 1E-4 \]

\[ g_o = \lambda I_{DQ} = 5E-7 \]

\[ g_{mb} = g_m \left( \frac{\gamma}{2\sqrt{\phi - V_{BSQ}}} \right) = .26g_m \]

In this example

\( g_o \ll g_m, g_{mb} \)

\( g_{mb} \ll g_m \)

This relationship is common

In many circuits, \( V_{BS} = 0 \) as well
Small Signal Model Summary

Large Signal Model

\[ I_D = \begin{cases} 
0 & V_{GS} \leq V_T \\
\mu C_{ox} \frac{W}{L} \left( V_{GS} - V_T - \frac{V_{DS}}{2} \right) V_{DS} & V_{GS} > V_T \quad V_{DS} < V_{GS} - V_T \\
\mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 \cdot (1 + \lambda V_{DS}) & V_{GS} \geq V_T \quad V_{DS} \geq V_{GS} - V_T 
\end{cases} \]

\[ V_T = V_{T0} + \gamma \left( \sqrt{\phi - V_{BS}} - \sqrt{\phi} \right) \]

Small Signal Model

\[ i_g = 0 \]
\[ i_b = 0 \]
\[ i_d = g_m v_{gs} + g_{mb} v_{bs} + g_o v_{ds} \]

where

\[ g_m = \frac{\mu C_{ox} W}{L} V_{EBQ} \]
\[ g_{mb} = g_m \left( \frac{\gamma}{2 \sqrt{\phi - V_{BSQ}}} \right) \]
\[ g_o = \lambda I_{DQ} \]
How does $g_m$ vary with $I_{DQ}$?

$$g_m = \sqrt{\frac{2\mu C_{ox} W}{L}} \sqrt{I_{DQ}}$$

Varies with the square root of $I_{DQ}$

$$g_m = \frac{2I_{DQ}}{V_{GSQ} - V_T} = \frac{2I_{DQ}}{V_{EBQ}}$$

Varies linearly with $I_{DQ}$

$$g_m = \frac{\mu C_{ox} W}{L} (V_{GSQ} - V_T)$$

Doesn’t vary with $I_{DQ}$
How does $g_m$ vary with $I_{DQ}$?

All of the above are true – but with qualification

$g_m$ is a function of more than one variable ($I_{DQ}$) and how it varies depends upon how the remaining variables are constrained.
Small Signal Model Summary

An equivalent circuit

![Diagram with symbols and equations]

\[ g_m = \frac{\mu C_{OX} W}{L} (V_{GSQ} - V_T) \]

\[ g_o = \lambda I_{DQ} \]

\[ g_{mb} = g_m \left( \frac{\gamma}{2\sqrt{\phi} - V_{BSQ}} \right) \]

This contains absolutely no more information than the previous model
Small Signal Model Summary

More convenient representation
Small Signal Model Summary

Simplification that is often adequate
Small Signal Model Summary

Even further simplification that is often adequate
Small Signal Model Summary

Alternate equivalent representations for $g_m$

$$g_m = \frac{\mu C_{ox} W}{L} (V_{GSQ} - V_T)$$

from

$$I_D \approx \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2$$

$$g_m = \sqrt{\frac{2\mu C_{ox} W}{L}} \sqrt{I_{DQ}}$$

$$g_m = \frac{2I_{DQ}}{V_{GSQ} - V_T} = \frac{2I_{DQ}}{V_{EBQ}}$$
Alternate Equivalent Small Signal Model of the BJT
Small Signal BJT Model

\[ g_m = \frac{I_{CQ}}{V_t} \]

\[ g_\pi = \frac{I_{CQ}}{\beta V_t} \]

\[ g_o \approx \frac{I_{CQ}}{V_{AF}} \]

Observe:

\[ g_{\pi} V_{be} = i_b \]

\[ g_m V_{be} = i_b \frac{g_m}{g_{\pi}} \]

\[ g_m = \begin{bmatrix} \frac{I_Q}{V_t} \\ \frac{I_Q}{\beta V_t} \end{bmatrix} = \beta \]

\[ g_m V_{be} = \beta i_b \]
Small Signal BJT Model

\[ g_m = \frac{l_{CQ}}{V_t} \]
\[ g_\pi = \frac{l_{CQ}}{\beta V_t} \]
\[ g_o \approx \frac{l_{CQ}}{V_{AF}} \]

Alternate equivalent small signal model

\[ g_\pi = \frac{l_{CQ}}{\beta V_t} \]
\[ g_o \approx \frac{l_{CQ}}{V_{AF}} \]
Relative Magnitude of Small Signal BJT Parameters

\[ g_m = \frac{I_{CQ}}{V_t} \]
\[ g_\pi = \frac{I_{CQ}}{\beta V_t} \]
\[ g_o \approx \frac{I_{CQ}}{V_{AF}} \]

\[ g_m = \begin{bmatrix} \frac{I_Q}{V_t} \\ \frac{I_Q}{\beta V_t} \end{bmatrix} \]
\[ g_\pi = \begin{bmatrix} \frac{I_Q}{\beta V_t} \end{bmatrix} \]
\[ g_o = \begin{bmatrix} \frac{I_Q}{V_{AF}} \end{bmatrix} \]

\[ g_m >> g_\pi >> g_o \]

Often the go term can be neglected in the small signal model because it is so small.
Relative Magnitude of Small Signal Parameters

\[
g_m = \frac{I_{CQ}}{V_t} \quad g_{\pi} = \frac{I_{CQ}}{\beta V_t} \quad g_o \approx \frac{I_{CQ}}{V_{AF}}
\]

\[
g_m = \begin{bmatrix} \frac{I_Q}{V_t} \end{bmatrix} = \beta
\]

\[
g_{\pi} = \begin{bmatrix} \frac{I_Q}{\beta V_t} \end{bmatrix} = \frac{V_{AF}}{100 \cdot 26mV} \approx \frac{200V}{100 \cdot 26mV} = 77
\]

\[
g_m \gg g_{\pi} \gg g_o
\]

Often the go term can be neglected in the small signal model because it is so small
Simplified small signal model
Comparison of BJT and MOSFET
The transconductance of the BJT is typically much larger than that of the MOSFET (and larger is better!)
This is due to the exponential rather than quadratic output/input relationship
Comparison of MOSFET and BJT

**BJT**

\[ g_m = \frac{I_{CQ}}{V_t} \]

**MOSFET**

\[ g_m = \frac{\mu C_{OX} W}{L} V_{EB} \]
\[ g_m = \sqrt{\frac{2\mu C_{OX} W}{L}} \sqrt{I_{DQ}} \]
\[ g_m = \frac{2I_{DQ}}{V_{EBQ}} \]

\[ g_{m_{BJT}} = \frac{I_{CQ}}{2I_{DQ}} \frac{V_t}{V_{EB}} = \frac{V_{EB}}{50mV} \]
\[ g_{m_{MOS}} = \frac{V_{EB}}{100mV} = 2 \text{ if } V_{EB} = 100mV \]
\[ g_{m_{MOS}} = \frac{V_{EB}}{50mV} > 2V \text{ if } V_{EB} = 2V \]

The transconductance of the BJT is typically much larger than that of the MOSFET (and larger is better)
This is due to the exponential rather than quadratic output/input relationship
Comparison of MOSFET and BJT

BJT

\[ g_o = \lambda I_{DQ} \]

MOSFET

\[ g_o \approx \frac{I_{CQ}}{V_{AF}} \]

\[
\frac{g_{oBJT}}{g_{oMOS}} = \frac{I_{CQ}}{V_{AF} \lambda I_{DQ}} = \frac{1}{\lambda V_{AF}} \approx \frac{1}{0.01 V^{-1} 200 V} = 0.5
\]

The output conductances are comparable but that of the BJT is usually modestly smaller (and smaller is better!)
Comparison of MOSFET and BJT

**BJT**

$$g_\pi = \frac{I_{CQ}}{\beta V_t}$$

**MOSFET**

$$g_\pi = 0$$

$g_\pi$ is the reciprocal of the input impedance.

$g_\pi$ of a MOSFET is much smaller than that of a BJT (and smaller is better!)
Standard Approach to small-signal analysis of nonlinear networks

Nonlinear Network

dc Equivalent Network

Q-point

Values for small-signal parameters

Small-signal equivalent network

Small-signal output

Total output

(good approximation)
Systematic Approach to Small-Signal Circuit Analysis

- Obtain dc equivalent circuit by replacing all elements with large-signal (dc) equivalent circuits
- Obtain dc operating points (Q-point)
- Obtain ac equivalent circuit by replacing all elements with small-signal equivalent circuits
- Analyze linear small-signal equivalent circuit
## Dc and small-signal equivalent elements

<table>
<thead>
<tr>
<th>Element</th>
<th>ss equivalent</th>
<th>dc equivalent</th>
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<tbody>
<tr>
<td>MOS Transistors</td>
<td></td>
<td>Simplified</td>
</tr>
<tr>
<td>Bipolar Transistors</td>
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<td></td>
<td>Simplified</td>
</tr>
</tbody>
</table>
Square-Law Model

\[ I_G = 0 \]
\[ I_B = 0 \]

\[ I_D = \begin{cases} 
0 & \text{if } V_{GS} \leq V_T \\
\mu C_{ox} \frac{W}{L} \left( V_{GS} - V_T - \frac{V_{DS}}{2} \right) V_{DS} & \text{if } V_{GS} \geq V_T \text{ and } V_{DS} < V_{GS} - V_T \\
\mu C_{ox} \frac{W}{2L} \left( V_{GS} - V_T \right)^2 \left( 1 + \lambda V_{DS} \right) & \text{if } V_{GS} \geq V_T \text{ and } V_{DS} \geq V_{GS} - V_T 
\end{cases} \]

\[ V_T = V_{T0} + \gamma \left( \sqrt{\phi - V_{BS}} - \sqrt{\phi} \right) \]
Simplified MOS Model for Q-point Analysis

\[ I_G = 0 \]
\[ I_B = 0 \]
\[ I_D = \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 \]

Simplified dc equivalent circuit

\[ G \quad + \quad \frac{\mu C_{ox} W}{2L} (V_{GS} - V_T)^2 \quad \downarrow \quad \quad \downarrow \quad \quad D \quad S \]

Simplified
dc BJT model

\[ I_C = \beta I_B \left(1 + \frac{V_{CE}}{V_{AF}}\right) \]

\[ I_B = \frac{J_S A_E}{\beta} e^{\frac{V_{BE}}{V_t}} \]

\[ V_t = \frac{kT}{q} \]

- \( V_{BE} > 0.4V \)
- \( V_{BC} < 0 \)

Forward Active

- \( V_{BE} = 0.7V \)
- \( V_{CE} = 0.2V \)

Saturation

- \( I_C < \beta I_B \)

Cutoff

- \( I_C = I_B = 0 \)
- \( V_{BE} < 0 \)
- \( V_{BC} < 0 \)

A small portion of the operating region is missed with this model but seldom operate in the missing region.
Simplified dc BJT model for Q-point Analysis

\[ I_C = \beta I_B \]
\[ I_B = \frac{J_S A_E}{\beta} e^{\frac{V_{BE}}{V_t}} \]

\[ V_{BE} = 0.6V \]

Simplified dc equivalent circuit
End of Lecture 26