Small-signal model of BJT
Small-Signal Model Extension
Applications of the Small-signal Model
Review from Last Lecture

4-terminal small-signal network summary

\[
\begin{align*}
I_1 &= f_1(V_1, V_2, V_3) \\
I_2 &= f_2(V_1, V_2, V_3) \\
I_3 &= f_3(V_1, V_2, V_3)
\end{align*}
\]

Small signal model:

\[
\begin{align*}
\dot{i}_1 &= y_{11} \dot{v}_1 + y_{12} \dot{v}_2 + y_{13} \dot{v}_3 \\
\dot{i}_2 &= y_{21} \dot{v}_1 + y_{22} \dot{v}_2 + y_{23} \dot{v}_3 \\
\dot{i}_3 &= y_{31} \dot{v}_1 + y_{32} \dot{v}_2 + y_{33} \dot{v}_3
\end{align*}
\]

\[
y_{ij} = \left. \frac{\partial f_i(V_1, V_2, V_3)}{\partial V_j} \right|_{\bar{V}=\bar{V}_0}
\]
3-terminal small-signal network summary

\[ \begin{align*}
    I_1 &= f_1(V_1, V_2) \\
    I_2 &= f_2(V_1, V_2)
\end{align*} \]

Small signal model:

\[ \begin{align*}
    I_1 &= y_{11}V_1 + y_{12}V_2 \\
    I_2 &= y_{21}V_1 + y_{22}V_2
\end{align*} \]
Small-Signal Model

\[ i_1 = g_1(v_1, v_2, v_3) \]
\[ i_2 = g_2(v_1, v_2, v_3) \]
\[ i_3 = g_3(v_1, v_2, v_3) \]

\[ i_1 = y_{11}v_1 + y_{12}v_2 + y_{13}v_3 \]
\[ i_2 = y_{21}v_1 + y_{22}v_2 + y_{23}v_3 \]
\[ i_3 = y_{31}v_1 + y_{32}v_2 + y_{33}v_3 \]

\[ y_{ij} = \left. \frac{\partial f_i(v_1, v_2, v_3)}{\partial v_j} \right|_{\tilde{v} = \tilde{v}_0} \]
Small Signal Model of MOSFET

\[ g_m = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \]

\[ g_o \equiv \lambda I_{DQ} \]

Alternate equivalent expressions for \( g_m \):

\[ I_{DQ} = \mu C_{ox} \frac{W}{2L} (V_{GSQ} - V_T)^2 (1 + \lambda V_{DSQ}) \approx \mu C_{ox} \frac{W}{2L} (V_{GSQ} - V_T)^2 \]

\[ g_m = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \]

\[ g_m = \sqrt{2} \mu C_{ox} \frac{W}{L} \cdot \sqrt{I_{DQ}} \]

\[ g_m = \frac{2I_{DQ}}{V_{GSQ} - V_T} \]
Small signal analysis example

For $\lambda=0$, $g_0 = \lambda I_{DQ} = 0$

More accurate gain can be obtained if $\lambda$ effects are included and does not significantly increase complexity of small signal analysis
Small Signal Model of BJT

3-terminal device

Usually operated in Forward Active Region when small-signal model is needed

Forward Active Model:

\[ I_C = J_S A_E e^{v_{BE}/v_t} \left( 1 + \frac{v_{CE}}{v_{AF}} \right) \]

\[ I_B = \frac{J_S A_E}{\beta} e^{v_{BE}/v_t} \]

\[ y_{21} = g_m = \left. \frac{\partial I_C}{\partial V_{BE}} \right|_{V = V_Q} \]
Small Signal Model of BJT

Nonlinear model:

\[ I_1 = f_1 (V_1, V_2) \quad \iff \quad I_B = \frac{J_S A_E}{\beta} e^{\frac{V_{BE}}{V_t}} \]

\[ I_2 = f_2 (V_1, V_2) \quad \iff \quad I_C = J_S A_E e^{\frac{V_{BE}}{V_t}} \left(1 + \frac{V_{CE}}{V_{AF}}\right) \]

Small-signal model:

\[ i_B = y_{11} V_{BE} + y_{12} V_{CE} \]

\[ i_C = y_{21} V_{BE} + y_{22} V_{CE} \]

\[ y_{ij} = \left. \frac{\partial f_i (V_1, V_2)}{\partial V_j} \right|_{V=V_Q} \]

\[ y_{11} = g_\pi = \left. \frac{\partial I_B}{\partial V_{BE}} \right|_{V=V_Q} \]

\[ y_{12} = \left. \frac{\partial I_B}{\partial V_{CE}} \right|_{V=V_Q} \]

\[ y_{21} = g_m = \left. \frac{\partial I_C}{\partial V_{BE}} \right|_{V=V_Q} \]

\[ y_{22} = g_o = \left. \frac{\partial I_C}{\partial V_{CE}} \right|_{V=V_Q} \]

Note: \( g_m, g_\pi \) and \( g_o \) used for notational consistency with legacy terminology
Small Signal Model of BJT

Nonlinear model:

$$I_B = \frac{J_S A_E}{\beta} e^{\frac{V_{BE}}{V_t}}$$

$$I_C = J_S A_E e^{\frac{V_{BE}}{V_t}} \left(1 + \frac{V_{CE}}{V_{AF}}\right)$$

$$i_B = y_{11} v_{BE} + y_{12} v_{CE}$$

$$i_C = y_{21} v_{BE} + y_{22} v_{CE}$$

$$y_{ij} = \frac{\partial f_i (V_1, V_2)}{\partial V_j} \Bigg|_{V=V_Q}$$

Small-signal model:

$$y_{11} = g_{\pi} = \left. \frac{\partial I_B}{\partial V_{BE}} \right|_{V=V_Q} = ?$$

$$y_{12} = \left. \frac{\partial I_B}{\partial V_{CE}} \right|_{V=V_Q} = ?$$

$$y_{21} = g_m = \left. \frac{\partial I_C}{\partial V_{BE}} \right|_{V=V_Q} = ?$$

$$y_{22} = g_o = \left. \frac{\partial I_C}{\partial V_{CE}} \right|_{V=V_Q} = ?$$
Small Signal Model of BJT

**Nonlinear model:**

\[
I_B = \frac{J S A_E}{\beta} e^{\frac{V_{BE}}{V_t}}
\]

\[
I_C = J S A_E e^{\frac{V_{BE}}{V_t}} \left(1 + \frac{V_{CE}}{V_{AF}}\right)
\]

\[
i_B = y_{11} V_{BE} + y_{12} V_{CE}
\]

\[
i_C = y_{21} V_{BE} + y_{22} V_{CE}
\]

\[
y_{11} = g_{\pi} = \left. \frac{\partial I_B}{\partial V_{BE}} \right|_{V_{CE}=V_Q} = \frac{1}{I_t} \frac{J S A_E}{\beta} e^{\frac{V_{BE}}{V_t}} \bigg|_{V_{CE}=V_Q} \approx \frac{I_C}{\beta V_t}
\]

\[
y_{21} = g_m = \left. \frac{\partial I_C}{\partial V_{BE}} \right|_{V_{CE}=V_Q} = \frac{1}{V_t} J S A_E e^{\frac{V_{BE}}{V_t}} \left(1 + \frac{V_{CE}}{V_{AF}}\right) \bigg|_{V_{CE}=V_Q} = \frac{I_C}{V_t}
\]

\[
y_{12} = \left. \frac{\partial I_B}{\partial V_{CE}} \right|_{V_{BE}=V_Q} = 0
\]

\[
y_{22} = g_o = \left. \frac{\partial I_C}{\partial V_{CE}} \right|_{V_{BE}=V_Q} = J S A E \frac{V_{BE}}{V_{AF}} \bigg|_{V_{BE}=V_Q} \approx \frac{I_C}{V_{AF}}
\]

**Note:** usually prefer to express in terms of \( I_{CQ} \)
Small Signal Model of BJT

\[ \begin{align*}
    i_B &= y_{11} V_{BE} + y_{12} V_{CE} \\
    i_C &= y_{21} V_{BE} + y_{22} V_{CE}
\end{align*} \]

\[ i_B = g_{\pi} V_{BE} \]
\[ i_C = g_m V_{BE} + g_o V_{CE} \]

\[ g_{\pi} = \frac{I_C}{\beta V_t} \]
\[ g_m = \frac{I_C}{V_t} \]
\[ g_o = \frac{I_C}{V_{AF}} \]

An equivalent circuit
Small Signal BJT Model

Observe:

\[ g_{\pi} v_{be} = i_b \]

\[ g_m v_{be} = i_b \frac{g_m}{g_{\pi}} \]

\[ \frac{g_m}{g_{\pi}} = \begin{bmatrix} \frac{I_Q}{V_t} \\ \frac{I_Q}{\beta V_t} \end{bmatrix} = \beta \]

\[ g_m v_{be} = \beta i_b \]

Can replace the voltage dependent current source with a current dependent current source.
Small Signal BJT Model

![Small Signal BJT Model Diagram]

\[ g_m = \frac{I_{CQ}}{V_t} \quad g_\pi = \frac{I_{CQ}}{\beta V_t} \quad g_o \approx \frac{I_{CQ}}{V_{AF}} \]

Alternate equivalent small signal model

![Alternate Equivalent Diagram]

\[ g_\pi = \frac{I_{CQ}}{\beta V_t} \quad g_o \approx \frac{I_{CQ}}{V_{AF}} \]
Small-Signal Model Representations

(3-terminal network – also relevant with 4-terminal networks)

Have developed small-signal models for the MOSFET and BJT.

Models have been based upon arbitrary assumption that $v_1, v_2$ are independent variables.

Have already seen some alternatives for parameter definitions in these models.

Alternative representations are sometimes used.
Small-Signal Model Representations

The good, the bad, and the unnecessary!!

what we have developed:

The hybrid parameters:
Small-Signal Model Representations

The z-parameters

\[ v_1 = z_{11}i_1 + z_{12}i_2 \]
\[ v_2 = z_{21}i_1 + z_{22}i_2 \]

The ABCD parameters:

\[ v_1 = Av_2 - Bi_2 \]
\[ i_1 = Cv_2 - Di_2 \]
Small-Signal Model Representations

The S-parameters

(embedded with source and load impedances)

The T parameters:

(embedded with source and load impedances)
Small-Signal Model Representations

The good, the bad, and the unnecessary!!

- Equivalent circuits often given for each representation
- All provide identical characterization
- Easy to move from any one to another
Small-Signal Model Representations

The good, the bad, and the unnecessary !!

Conversions Between $S$, $Z$, $Y$, $h$, $ABCD$, and $T$ Parameters which are Valid for Complex Source and Load Impedances

Dean A. Frickey, Member, IEEE
What are the simplified dc equivalent models?
Active Device Model Summary

What are the simplified dc equivalent models?

dc equivalent

0.6V

G

\( V_{GSQ} \)

\( \frac{\mu_{COX} W}{2L} (V_{GSQ} - V_{Th})^2 \)

S

D

G

\( V_{GSQ} \)

\( \frac{\mu_{COX} W}{2L} (V_{GSQ} - V_{Tp})^2 \)

S

B

\( I_{BQ} \)

0.6V

C

E

\( \beta I_{BQ} \)

B

\( I_{BQ} \)

0.6V

C

E

\( \beta I_{BQ} \)
Small-signal Operation of Nonlinear Circuits

- Small-signal principles
- Example Circuit
- Small-Signal Models

Small-Signal Analysis of Nonlinear Circuits
Recall:  

**Alternative Approach to small-signal analysis of nonlinear networks**

1. **Linearize nonlinear devices**  
   *(have small-signal model for key devices!)*

2. **Replace all devices with small-signal equivalent**

3. **Solve linear small-signal network**

Remember that the small-signal model is operating point dependent!

Thus need Q-point to obtain values for small signal parameters
Comparison of Gains for MOSFET and BJT Circuits

Recall results from nonlinear analysis of these two circuits:

For BJT:

\[ A_{V_B} = -\frac{I_{CQ} R}{V_t} \]

For MOSFET:

\[ A_{V_M} = \frac{2I_{DQ} R \cdot R_1}{V_{SS} + V_T} \]

Verify the gain expression obtained for the BJT using a small signal analysis.
\[ V_{\text{OUT}} = -g_m R V_{\text{BE}} \]
\[ V_{\text{IN}} = V_{\text{BE}} \]

\[ A_V = \frac{V_{\text{OUT}}}{V_{\text{IN}}} = -g_m R \]

\[ g_m = \frac{I_{\text{CQ}}}{V_t} \]

\[ A_V = -\frac{I_{\text{CQ}} R}{V_t} \]

Note this is identical to what was obtained with the direct nonlinear analysis.
Example:

Determine the small signal voltage gain $A_v = \frac{v_{out}}{v_{in}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$. 

\[ v_{DD} \quad M_2 \quad v_{out} \quad v_{IN} \quad M_1 \quad v_{SS} \]
Example: Determine the small signal voltage gain $A_V = \frac{V_{OUT}}{V_{IN}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$. 

Small-signal circuit
Example: *Determine the small signal voltage gain $A_v = \frac{v_{OUT}}{v_{IN}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$.*

![Small-signal circuit diagram](image)

**Small-signal circuit**

![Small-signal MOSFET model](image)

**Small-signal MOSFET model for $\lambda = 0$**
Example: Determine the small signal voltage gain $A_v = \frac{v_{OUT}}{v_{IN}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$. 

Small-signal circuit
Example:

![Small-signal circuit diagram]

**Analysis:**

By KCL

\[ g_{m1} V_{GS1} = g_{m2} V_{GS2} \]

but

\[ V_{GS1} = V_{IN} \]
\[ -V_{GS2} = V_{OUT} \]

thus:

\[ A_{v} = \frac{V_{OUT}}{V_{IN}} = -\frac{g_{m1}}{g_{m2}} \]
Example:

Small-signal circuit

Analysis:

\[ A_v = \frac{V_{OUT}}{V_{IN}} = -\frac{g_{m1}}{g_{m2}} \]

Recall:

\[ g_m = -\sqrt{2I_D\mu C_{ox}} \frac{W_1}{L_1} \]

\[ A_v = -\sqrt{2I_D\mu C_{ox}} \frac{W_1}{L_1} \frac{W}{L} = -\sqrt{\frac{W_1}{W_2}} \sqrt{\frac{L_2}{L_1}} \]
Example:

Small-signal circuit

\[ A_v = \frac{V_{\text{OUT}}}{V_{\text{IN}}} = -\frac{g_{m1}}{g_{m2}} \]

Recall:

\[ A_v = -\sqrt{\frac{W_1}{W_2}} \sqrt{\frac{L_2}{L_1}} \]

If \( L_1 = L_2 \), obtain

\[ A_v = -\sqrt{\frac{W_1}{W_2}} \sqrt{\frac{L_2}{L_1}} = -\sqrt{\frac{W_1}{W_2}} \]

The width and length ratios can be accurately set when designed in a standard CMOS process.
End of Lecture 27