EE 330
Lecture 28

Small-Signal Model Extension
Applications of the Small-signal Model
Nonlinear network characterized by 3 functions each functions of 3 variables

\[
\begin{align*}
I_1 &= f_1(V_1, V_2, V_3) \\
I_2 &= f_2(V_1, V_2, V_3) \\
I_3 &= f_3(V_1, V_2, V_3)
\end{align*}
\]
Consider now 3 functions, each a function of 3 variables

\[ i_1 = y_{11} v_1 + y_{12} v_2 + y_{13} v_3 \]

Extending this approach to the two nonlinear functions \( I_2 \) and \( I_3 \)

\[ i_2 = y_{21} v_1 + y_{22} v_2 + y_{23} v_3 \]

\[ i_3 = y_{31} v_1 + y_{32} v_2 + y_{33} v_3 \]

where

\[ y_{ij} = \left. \frac{\partial f_i(V_1, V_2, V_3)}{\partial V_j} \right|_{\tilde{V}=\bar{V}_Q} \]

This is a small-signal model of a 4-terminal network and it is linear. 9 small-signal parameters characterize the linear 4-terminal network. Small-signal model parameters dependent upon Q-point!
A small-signal equivalent circuit of a 4-terminal nonlinear network

\[ y_{ij} = \left. \frac{\partial f_i(V_1, V_2, V_3)}{\partial V_j} \right|_{\tilde{V} = \tilde{V}_Q} \]

Review from Last Lecture

Equivalent circuit is not unique
Equivalent circuit is a three-port network
Small-Signal Model

Consider 3-terminal network

\[
\begin{align*}
\mathbf{\dot{i}}_1 &= \mathbf{y}_{11} \mathbf{V}_1 + \mathbf{y}_{12} \mathbf{V}_2 + \mathbf{y}_{13} \mathbf{V}_3 \\
\mathbf{\dot{i}}_2 &= \mathbf{y}_{21} \mathbf{V}_1 + \mathbf{y}_{22} \mathbf{V}_2 + \mathbf{y}_{23} \mathbf{V}_3 \\
\mathbf{\dot{i}}_3 &= \mathbf{y}_{31} \mathbf{V}_1 + \mathbf{y}_{32} \mathbf{V}_2 + \mathbf{y}_{33} \mathbf{V}_3
\end{align*}
\]

\[
\begin{align*}
\mathbf{i}_1 &= \mathbf{g}_1(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3) \\
\mathbf{i}_2 &= \mathbf{g}_2(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3) \\
\mathbf{i}_3 &= \mathbf{g}_3(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3)
\end{align*}
\]

\[
\mathbf{y}_{ij} = \left. \frac{\partial \mathbf{f}_i(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3)}{\partial \mathbf{V}_j} \right|_{\mathbf{V} = \mathbf{V}_0}
\]
A Small Signal Equivalent Circuit

\[ i_1 = y_{11} V_1 + y_{12} V_2 \]
\[ i_2 = y_{21} V_1 + y_{22} V_2 \]

4 small-signal parameters characterize this 3-terminal (two-port) linear network
Small signal parameters dependent upon Q-point
Small-Signal Model

\[ i_1 = g_1(v_1, v_2, v_3) \]
\[ i_2 = g_2(v_1, v_2, v_3) \]
\[ i_3 = g_3(v_1, v_2, v_3) \]

\[ i_1 = y_{11} v_1 + y_{12} v_2 + y_{13} v_3 \]
\[ i_2 = y_{21} v_1 + y_{22} v_2 + y_{23} v_3 \]
\[ i_3 = y_{31} v_1 + y_{32} v_2 + y_{33} v_3 \]

**Review from Last Lecture**

Consider 2-terminal network
Small-Signal Model

\[ i_1 = y_{11} V_1 \]

\[ y_{11} = \left. \frac{\partial f_1(V_1)}{\partial V_1} \right|_{V=V_0} \]

\[ V = V_{1Q} \]

A Small Signal Equivalent Circuit

Review from Last Lecture
Consider 2-terminal network
Small Signal Model of MOSFET

Review from Last Lecture

**Large Signal Model**

\[
I_G = 0
\]

3-terminal device

\[
I_D = \begin{cases} 
0 & V_{GS} \leq V_T \\
\mu C_{OX} \frac{W}{L} \left( V_{GS} - V_T - \frac{V_{DS}}{2} \right) V_{DS} & V_{GS} \geq V_T, V_{DS} < V_{GS} - V_T \\
\mu C_{OX} \frac{W}{2L} (V_{GS} - V_T)^2 \left( 1 + \lambda V_{DS} \right) & V_{GS} \geq V_T, V_{DS} \geq V_{GS} - V_T
\end{cases}
\]

**MOSFET is usually operated in saturation region in linear applications where a small-signal model is needed so will develop the small-signal model in the saturation region**
Small Signal Model of MOSFET

\[ g_m = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \]

\[ g_o = \lambda I_{DQ} \]

Alternate equivalent expressions:

\[ I_{DQ} = \mu C_{ox} \frac{W}{2L} (V_{GSQ} - V_T)^2 (1 + \lambda V_{DSQ}) \approx \mu C_{ox} \frac{W}{2L} (V_{GSQ} - V_T)^2 \]

\[ g_m = \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \]

\[ g_m = \sqrt{2\mu C_{ox}} \frac{W}{L} \cdot \sqrt{I_{DQ}} \]

\[ g_m = \frac{2I_{DQ}}{V_{GSQ} - V_T} \]
Small Signal Model of BJT

3-terminal device

Usually operated in Forward Active Region when small-signal model is needed

Forward Active Model:

\[ I_C = J_S A_E e^{\frac{V_{BE}}{V_t}} \left( 1 + \frac{V_{CE}}{V_{AF}} \right) \]

\[ I_B = \frac{J_S A_E}{\beta} e^{\frac{V_{BE}}{V_t}} \]
Small Signal Model of BJT

\[ i_B = y_{11} V_{BE} + y_{12} V_{CE} \]
\[ i_C = y_{21} V_{BE} + y_{22} V_{CE} \]

\[ g_\pi = \frac{I_{CQ}}{\beta V_t} \quad g_m = \frac{I_{CQ}}{V_t} \quad g_o = \frac{I_{CQ}}{V_{AF}} \]

Review from Last Lecture
Review from Last Lecture

Active Device Model Summary

<table>
<thead>
<tr>
<th>Element</th>
<th>ss equivalent</th>
<th>dc equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diodes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MOS transistors</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bipolar Transistors</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What are the simplified dc equivalent models?
Active Device Model Summary

What are the simplified dc equivalent models?

dc equivalent

Review from Last Lecture
Recall:

**Alternative Approach to small-signal analysis of nonlinear networks**

1. **Linearize nonlinear devices**  
   *(have small-signal model for key devices!)*

2. **Replace all devices with small-signal equivalent**

3. **Solve linear small-signal network**

Remember that the small-signal model is operating point dependent!

Thus need Q-point to obtain values for small signal parameters
Comparison of Gains for MOSFET and BJT Circuits

**BJT**

\[ A_{VB} = -\frac{I_{CQ} R}{V_t} \]

**MOSFET**

\[ A_{VM} = \frac{2I_{DQ} R_{1}}{[V_{SS} + V_T]} \]

Verify the gain expression obtained for the BJT using a small signal analysis.
Neglect λ effects to be consistent with earlier analysis

\[
\begin{align*}
V_{OUT} &= -g_m R V_{BE} \\
V_{IN} &= V_{BE}
\end{align*}
\]

\[
A_v = \frac{V_{OUT}}{V_{IN}} = -g_m R
\]

\[
g_m = \frac{I_{CQ}}{V_t} \\
A_v = -\frac{I_{CQ} R}{V_t}
\]

Note this is identical to what was obtained with the direct nonlinear analysis.
Example:

Determine the small signal voltage gain $A_V = \frac{V_{OUT}}{V_{IN}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$. 

\[ V_{DD} \]

\[ V_{OUT} \]

\[ V_{IN} \]

\[ V_{SS} \]
Example: Determine the small signal voltage gain $A_V = \frac{V_{OUT}}{V_{IN}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$. 

Small-signal circuit
Example: Determine the small signal voltage gain $A_V = \frac{V_{OUT}}{V_{IN}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$. 

Small-signal circuit

Small-signal MOSFET model for $\lambda = 0$
Example: Determine the small signal voltage gain $A_V = \frac{V_{OUT}}{V_{IN}}$. Assume $M_1$ and $M_2$ are operating in the saturation region and that $\lambda = 0$. 

Small-signal circuit
Example:

Small-signal circuit

Analysis:

By KCL

\[ g_{m1} V_{GS1} = g_{m2} V_{GS2} \]

but

\[ V_{GS1} = V_{IN} \]

\[ -V_{GS2} = V_{OUT} \]

thus:

\[ A_v = \frac{V_{OUT}}{V_{IN}} = -\frac{g_{m1}}{g_{m2}} \]
Example:

Small-signal circuit

Analysis:

\[ A_v = \frac{v_{OUT}}{v_{IN}} = -\frac{g_{m1}}{g_{m2}} \]

Recall:

\[ g_m = -\sqrt{2I_D \mu C_{ox}} \sqrt{\frac{W_1}{L_1}} \]

\[ A_v = -\sqrt{2I_D \mu C_{ox} \frac{W_1}{L_1}} = -\sqrt{\frac{W_1}{W_2}} \sqrt{\frac{L_2}{L_1}} \]
Example:

![Small-signal circuit diagram]

**Analysis:**

Small-signal circuit

\[ A_v = \frac{V_{OUT}}{V_{IN}} = -\frac{g_{m1}}{g_{m2}} \]

**Recall:**

\[ A_v = -\sqrt{\frac{W_1}{W_2}} \sqrt{\frac{L_2}{L_1}} \]

*If \( L_1 = L_2 \), obtain*

\[ A_v = -\sqrt{\frac{W_1}{W_2}} \sqrt{\frac{L_2}{L_1}} = -\sqrt{\frac{W_1}{W_2}} \]

*The width and length ratios can be accurately set when designed in a standard CMOS process*
Graphical Analysis and Interpretation

Consider Again

\[ V_{OUT} = V_{DD} - I_D R \]

\[ I_{DQ} = \frac{\mu C_{ox} W}{2L} (V_{SS} + V_T)^2 \]

\[ I_D = \frac{\mu C_{ox} W}{2L} (V_{IN} - V_{SS} - V_T)^2 \]
Graphical Analysis and Interpretation

Device Model (family of curves)

\[
I_{DQ} = \frac{\mu C_{ox} W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS})
\]

Load Line

Device Model at Operating Point

\[
V_{OUT} = V_{DD} - I_D R
\]

\[
I_{DQ} = \frac{\mu C_{ox} W}{2L} (V_{SS} + V_T)^2
\]
Graphical Analysis and Interpretation

Device Model (family of curves)

\[ I_{DQ} = \frac{\mu C_{ox} W}{2L} \left( V_{GS} - V_T \right)^2 \left( 1 + \lambda V_{DS} \right) \]

\[ V_{OUT} = V_{DD} - I_D R \]

\[ I_{DQ} = \frac{\mu C_{ox} W}{2L} \left( V_{SS} + V_T \right)^2 \]

\[ I_D = \frac{\mu C_{ox} W}{2L} \left( V_{IN} - V_{SS} - V_T \right)^2 \]
Graphical Analysis and Interpretation

Device Model (family of curves)  

\[ I_{DQ} = \frac{\mu C_{ox} W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS}) \]

Saturation region
Graphical Analysis and Interpretation

Device Model (family of curves)

\[ I_{DQ} = \frac{\mu C_{ox} W}{2L} \left( V_{GS} - V_T \right)^2 \left( 1 + \lambda V_{DS} \right) \]

- Linear signal swing region smaller than saturation region
- Modest nonlinear distortion provided saturation region operation maintained
- Symmetric swing about Q-point
- Signal swing can be maximized by judicious location of Q-point
Graphical Analysis and Interpretation

Device Model (family of curves)

\[ I_{DQ} = \frac{\mu C_{ox} W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS}) \]

- **Q-Point**
- **Load Line**
- **Saturation region**

Very limited signal swing with non-optimal Q-point location
Graphical Analysis and Interpretation

Device Model (family of curves)  \[ I_{DQ} = \frac{\mu C_{ox} W}{2L} (V_{GS} - V_T)^2 (1 + \lambda V_{DS}) \]

- Signal swing can be maximized by judicious location of Q-point
- Often selected to be at middle of load line in saturation region
Small-Signal MOSFET Model Extension

Existing model does not depend upon the bulk voltage!

Observe that changing the bulk voltage will change the electric field in the channel region!
Further Model Extensions

Existing model does not depend upon the bulk voltage!

Observe that changing the bulk voltage will change the electric field in the channel region!

Changing the bulk voltage will change the thickness of the inversion layer
Changing the bulk voltage will change the threshold voltage of the device

\[ V_T = V_{T0} + \gamma \left( \sqrt{\phi} - V_{BS} - \sqrt{\phi} \right) \]
Typical Effects of Bulk on Threshold Voltage for n-channel Device

\[ V_T = V_{T0} + \gamma \left( \sqrt{\phi} - V_{BS} - \sqrt{\phi} \right) \]

\[ \gamma \approx 0.4V^{\frac{1}{2}} \quad \phi \approx 0.6V \]

Bulk-Diffusion Generally Reverse Biased \((V_{BS} < 0 \text{ or at least less than } 0.3V)\) for n-channel

Shift in threshold voltage with bulk voltage can be substantial

Often \(V_{BS}=0\)
Typical Effects of Bulk on Threshold Voltage for p-channel Device

\[ V_T = V_{T0} - \gamma \left( \sqrt{\phi + V_{BS}} - \sqrt{\phi} \right) \]

\[ \gamma \approx 0.4V^{\frac{1}{2}} \quad \phi \approx 0.6V \]

Bulk-Diffusion Generally Reverse Biased \((V_{BS} > 0 \text{ or at least greater than } -0.3V)\) for n-channel

Same functional form as for n-channel devices but \(V_{T0}\) is now negative and the magnitude of \(V_T\) still increases with the magnitude of the reverse bias.
Model Extension Summary

\( I_G = 0 \)
\( I_B = 0 \)

\[
I_d = \begin{cases} 
0 & V_{GS} \leq V_T \\
\mu C_{OX} \frac{W}{L} \left( V_{GS} - V_T - \frac{V_{DS}}{2} \right) V_{DS} & V_{GS} \geq V_T \quad V_{DS} < V_{GS} - V_T \\
\mu C_{OX} \frac{W}{2L} \left( V_{GS} - V_T \right)^2 \cdot (1 + \lambda V_{DS}) & V_{GS} \geq V_T \quad V_{DS} \geq V_{GS} - V_T 
\end{cases}
\]

\( V_T = V_{T0} + \gamma \left( \sqrt{\phi} - V_{BS} - \sqrt{\phi} \right) \)

Model Parameters: \{\mu, C_{OX}, V_{T0}, \phi, \gamma, \lambda\}

Design Parameters: \{W, L\} but only one degree of freedom W/L
Small-Signal Model Extension

\[ I_G = 0 \]
\[ I_B = 0 \]

\[ I_D = \begin{cases} 
0 & \text{if } V_{GS} \leq V_T \\
\mu C_{OX} \frac{W}{L} \left( V_{GS} - V_T - \frac{V_{DS}}{2} \right) V_{DS} & \text{if } V_{GS} \geq V_T \\
\mu C_{OX} \frac{W}{2L} \left( V_{GS} - V_T \right)^2 \cdot (1 + \lambda V_{DS}) & \text{if } V_{GS} \geq V_T \text{ and } V_{DS} \geq V_{GS} - V_T 
\end{cases} \]

\[ V_T = V_{T0} + \gamma \left( \sqrt{\phi - V_{BS}} - \sqrt{\phi} \right) \]

\[ y_{11} = \frac{\partial I_G}{\partial V_{GS}} \bigg|_{v=v_a} = 0 \quad y_{12} = \frac{\partial I_G}{\partial V_{DS}} \bigg|_{v=v_a} = 0 \quad y_{13} = \frac{\partial I_G}{\partial V_{GS}} \bigg|_{v=v_a} = 0 \]

\[ y_{31} = \frac{\partial I_B}{\partial V_{GS}} \bigg|_{v=v_a} = 0 \quad y_{32} = \frac{\partial I_B}{\partial V_{DS}} \bigg|_{v=v_a} = 0 \quad y_{33} = \frac{\partial I_B}{\partial V_{GS}} \bigg|_{v=v_a} = 0 \]

\[ y_{21} = \frac{\partial I_d}{\partial V_{GS}} \bigg|_{v=v_o} = g_m \quad y_{12} = \frac{\partial I_d}{\partial V_{DS}} \bigg|_{v=v_o} = g_n \quad y_{13} = \frac{\partial I_d}{\partial V_{GS}} \bigg|_{v=v_o} = g_{mb} \]
\[ I_D = \mu C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2 \cdot (1 + \lambda V_{DS}) \]

\[ V_T = V_{T0} + \gamma \left( \sqrt{\phi} - V_{BS} - \sqrt{\phi} \right) \]

\[ g_m = \mu C_{ox} \frac{W}{2L} 2 \left( V_{GS} - V_T \right)^1 \cdot (1 + \lambda V_{DS}) \]

\[ g_o = \mu C_{ox} \frac{W}{2L} 2 \left( V_{GS} - V_T \right)^2 \cdot \lambda \]

\[ g_{mb} = \mu C_{ox} \frac{W}{2L} 2 \left( V_{GS} - V_T \right)^1 \cdot \left( -\frac{\partial V_T}{\partial V_{BS}} \right) \cdot (1 + \lambda V_{DS}) \]

\[ g_{mb} = \mu C_{ox} \frac{W}{L} V_{EBQ} \cdot \frac{\partial V_T}{\partial V_{BS}} \bigg|_{V=V_Q} = \left( \mu C_{ox} \frac{W}{L} V_{EBQ} \right) (-\lambda) \gamma \frac{1}{2} \left( \phi - V_{BS} \right)^{-\frac{1}{2}} \bigg|_{V=V_Q} \]

\[ g_{mb} \approx g_m \frac{\gamma}{2\sqrt{\phi - V_{BSQ}}} \]
Small Signal Model Summary

\[ i_g = 0 \]

\[ i_b = 0 \]

\[ i_d = g_m v_{gs} + g_{mb} v_{bs} + g_o v_{ds} \]

\[ g_m = \frac{\mu C_{OX} W}{L} V_{EBQ} \]

\[ g_o = \lambda I_{DQ} \]

\[ g_{mb} = g_m \left( \frac{\gamma}{2\sqrt{\phi - V_{BSQ}}} \right) \]
Relative Magnitude of Small Signal MOS Parameters

Consider:

\[ i_d = g_m v_{gs} + g_{mb} v_{bs} + g_o v_{ds} \]

3 alternate equivalent expressions for \( g_m \)

\[ g_m = \frac{\mu C_{OX} W}{L} v_{EBQ} \quad g_m = \sqrt{\frac{2\mu C_{OX} W}{L}} \sqrt{I_{DQ}} \quad g_m = \frac{2I_{DQ}}{V_{EBQ}} \]

If \( \mu C_{OX} = 100 \mu A/V^2 \), \( \lambda = 0.01 V^{-1} \), \( \gamma = 0.4 V^{0.5} \), \( V_{EBQ} = 1V \), \( W/L = 1 \), \( V_{BSQ} = 0V \)

\[ I_{DQ} \approx \frac{\mu C_{OX} W}{2L} v_{EBQ}^2 = \frac{10^{-4} W}{2L} (1V)^2 = 5E-5 \]

\[ g_m = \frac{\mu C_{OX} W}{L} v_{EBQ} = 1E-4 \]

\[ g_o = \lambda I_{DQ} = 5E-7 \]

\[ g_{mb} = g_m \left( \frac{\gamma}{2\sqrt{\phi - V_{BSQ}}} \right) = 0.26g_m \]

In this example

\( g_0 << g_m, g_{mb} \)

\( g_{mb} < g_m \)

This relationship is common

In many circuits, \( V_{BSQ} = 0V \) as well
Large and Small Signal Model Summary

Large Signal Model

\[ I_D = \begin{cases} 
0 & V_{GS} \leq V_T \\
\mu C_{ox} \frac{W}{L} \left( V_{GS} - V_T - \frac{V_{DS}}{2} \right) V_{DS} & V_{GS} \geq V_T, V_{DS} < V_{GS} - V_T \\
\mu C_{ox} \frac{W}{2L} \left( V_{GS} - V_T \right)^2 \cdot (1 + \lambda V_{DS}) & V_{GS} \geq V_T, V_{DS} \geq V_{GS} - V_T 
\end{cases} \]

\[ V_T = V_{T0} + \gamma \left( \sqrt{\phi - V_{BS}} - \sqrt{\phi} \right) \]

Small Signal Model

\[ i_g = 0 \]
\[ i_b = 0 \]
\[ i_d = g_m v_{gs} + g_{mb} v_{bs} + g_o v_{ds} \]

where

\[ g_m = \frac{\mu C_{ox} W}{L} v_{EBQ} \]
\[ g_{mb} = g_m \left( \frac{\gamma}{2\sqrt{\phi - V_{BSQ}}} \right) \]
\[ g_o = \lambda I_{DQ} \]
End of Lecture 28