EE 435

Lecture 29

Spectral Performance
– Tool Use and Validation
If \( f(t) \) is periodic

\[
f(t) = A_0 + \sum_{k=1}^{\infty} A_k \sin(k \omega t + \theta_k)
\]

alternately

\[
f(t) = A_0 + \sum_{k=1}^{\infty} a_k \sin(k \omega t) + \sum_{k=1}^{\infty} b_k \cos(k \omega t)
\]

\[
\omega = \frac{2\pi}{T}
\]

\[
A_k = \sqrt{a_k^2 + b_k^2}
\]

Termed the Fourier Series Representation of \( f(t) \)
Distortion Analysis

Total Harmonic Distortion, THD

\[
THD = \frac{\text{RMS voltage in harmonics}}{\text{RMS voltage of fundamental}}
\]

\[
THD = \sqrt{\frac{A_2^2}{\sqrt{2}} + \frac{A_3^2}{\sqrt{2}} + \frac{A_4^2}{\sqrt{2}} + \ldots}
\]

\[
THD = \frac{\sqrt{\sum_{k=2}^{\infty} A_k^2}}{A_1}
\]

Review from last lecture...
Distortion Analysis

- Often noise is present at other non-harmonic frequencies
- At higher frequencies the harmonics are often buried in the noise

$|A_k|$
Distortion Analysis

**Theorem:** In a fully differential symmetric circuit, all even harmonics are absent in the differential output for symmetric differential excitations!

Proof: Expanding in a Taylor’s series around \( V_{ID} = 0 \), we obtain

\[
V_{OD} = f(V_{ID}) = \sum_{k=0}^{\infty} h_k V_{ID}^k
\]

Assume \( V_{ID} = K \sin(\omega t) \)

W.L.O.G. assume \( K = 1 \)

\[
V_{O1} = \sum_{k=0}^{\infty} h_k [\sin(\omega t)]^k, \quad V_{O2} = \sum_{k=0}^{\infty} h_k [-\sin(\omega t)]^k
\]

\[
V_{OD} = V_{O1} - V_{O2} = \sum_{k=0}^{\infty} h_k \left( [\sin(\omega t)]^k - [-\sin(\omega t)]^k \right) = \sum_{k=0}^{\infty} h_k \left( [\sin(\omega t)]^k - (-1)^k [\sin(\omega t)]^k \right)
\]

Observe the even-ordered harmonics are absent in this last sum.
THEOREM: If \( N_P \) is an integer and \( x(t) \) is band limited to \( f_{\text{MAX}} \), then

\[
|A_m| = \frac{2}{N} |X(mN_P + 1)| \quad 0 \leq m \leq h - 1
\]

and

\[
X(k) = 0 \quad \text{for all } k \text{ not defined above}
\]

where \( \langle X(k) \rangle_{k=0}^{N-1} \) is the DFT of the sequence \( \langle x(kT_s) \rangle_{k=0}^{N-1} \)

\[
f = 1/T, \text{ and } f_{\text{MAX}} = \frac{f}{2} \cdot \left\lceil \frac{N}{N_P} \right\rceil
\]
Distortion Analysis

If the hypothesis of the theorem are satisfied, we thus have

\[ |X(k)| \]

FFT is a computationally efficient way of calculating the DFT, particularly when \( N \) is a power of 2.
FFT Examples

Recall the theorem that provided for the relationship between the DFT terms and the Fourier Series Coefficients required

1. The sampling window be an integral number of periods

2. \[ N > \frac{2 f_{\text{max}}}{f_{\text{SIGNAL}}} N_p \]
Considerations for Spectral Characterization

• Tool Validation
• FFT Length
• Importance of Satisfying Hypothesis
• Windowing
Considerations for Spectral Characterization

• Tool Validation
• FFT Length
• Importance of Satisfying Hypothesis
• Windowing
Recall the theorem that provided for the relationship between the DFT terms and the Fourier Series Coefficients required

1. The sampling window be an integral number of periods
2. \[ N > \frac{2 f_{\text{max}}}{f_{\text{SIGNAL}}} N_p \]
Example

WLOG assume $f_{SIG}=50\text{Hz}$

$$V_{IN} = \sin(\omega t) + 0.5 \sin(2\omega t)$$

$$\omega = 2\pi f_{SIG}$$

Consider $N_P=20 \quad N=512$

Recall $20\log_{10}(0.5)=-6.0205999$
Input Waveform
Input Waveform
Input Waveform

Location of First Point if Extended Into Periodic Function
Spectral Response

Rect. Window N=512  Np =20

Mag(dB)

Frequency

0 200 400 600 800 1000 1200
DFT Horizontal Axis Converter to Frequency: \[ f_{\text{AXIS}} = f_{\text{SIGNAL}} \frac{n - 1}{N_p} \]
Spectral Response

Rect. Window  N=512  Np =20
Fundamental will appear at position $1+N_p = 21$

Columns 1 through 5

-316.1458  -312.9517  -329.5203  -311.1473  -314.2615

Columns 6 through 10

-315.2584  -330.6258  -317.2896  -312.2316  -311.6335

Columns 11 through 15


Columns 16 through 20

-314.0088  -302.6391  -306.6650  -311.3733  -308.3689

Columns 21 through 25

-0.0000  -307.7012  -312.9902  -312.8737  -305.4320

Observe system noise floor due to both spectral limitations of signal generator and numerical limitations in FFT are below -300db
**Second Harmonic at 1+2Np = 41**

Columns 26 through 30

-307.8301  -309.0737  -305.8503  -312.2772  -315.7544

Columns 31 through 35

-311.9316  -316.0581  -318.3454  -306.4977  -308.6679

Columns 36 through 40

-309.9702  -305.9809  -322.1270  -310.6723  -310.3506

Columns 41 through 45


Columns 46 through 50

-313.0745  -304.2330  -310.8487  -317.7966  -316.3385
Third Harmonic at $1 + 3N_p = 61$

Columns 51 through 55

-307.0529 -312.7787 -312.9340 -323.2969 -314.9297

Columns 56 through 60

-318.7605 -303.5929 -305.2994 -310.6430 -306.7613

Columns 61 through 65

-304.8298 -301.4463 -301.1410 -303.1784 -317.8343

Columns 66 through 70

-308.6310 -307.0135 -321.6015 -316.6548 -309.8946

Columns 71 through 75

-306.3472 -323.0110 -319.3267 -314.7873 -310.4085
Fourth Harmonic at 1+4Np = 81

Columns 76 through 80


Columns 81 through 85


Columns 86 through 90

-313.4988 -303.4513 -310.4969 -317.9652 -312.5846

Columns 91 through 95

-309.8121 -311.6403 -312.8374 -310.5414 -308.7807

Columns 96 through 100

-316.7549 -316.3395 -308.4113 -307.3766 -311.0358
Question: How much noise is in the computational environment?

Is this due to quantization in the computational environment or to numerical rounding in the FFT?
Question: How much noise is in the computational environment?

Observation: This noise is nearly uniformly distributed. The level of this noise at each component is around -310dB.
Question: How much noise is in the computational environment?

Assume $A_k = -310 \text{ dB}$ for $0 \leq k \leq N$

$$A_{k DB} = 20 \log_{10} A_k$$

$$A_k \approx 10^{\frac{-310}{20}} = 10^{-15.5}$$

$$V_{\text{Noise, RMS}} \approx \sqrt{\sum_{k=1}^{N-1} A_k^2} \quad A_k = \bar{A} = \sqrt{N \bar{A}}$$

$$V_{\text{Noise, RMS}} \approx \sqrt{N} \ \bar{A} = \sqrt{512 \ 10^{-15.5}} = 1.8 \times 10^{-14} = 18 \text{ fV}$$

Note: This computational environment has a very low total computational noise and does not become significant until the 45-bit resolution level is reached!!
Considerations for Spectral Characterization

• Tool Validation
• FFT Length
• Importance of Satisfying Hypothesis
• Windowing
Example

WLOG assume $f_{\text{SIG}}=50\text{Hz}$

$$V_{\text{IN}} = \sin(\omega t) + 0.5 \sin(2\omega t)$$

$$\omega = 2\pi f_{\text{SIG}}$$

Consider $N_P=20$  $N=4096$
Spectral Response
Fundamental will appear at position $1+N_p = 21$

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<td>-319.7032</td>
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<td>-315.5166</td>
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The \( k^{th} \) harmonic will appear at position \( 1+k\cdot N_p \).
Example

WLOG assume $f_{\text{SIG}}=50\text{Hz}$

$$V_{\text{IN}} = \sin(\omega t) + 0.5 \sin(2\omega t)$$

$$\omega = 2\pi f_{\text{SIG}}$$

Consider $N_p=50 \quad N=4096$
Spectral Response
Fundamental will appear at position $1+N_p = 51$

Columns 1 through 7

-322.4309 -325.5445 -322.2645 -321.6226 -319.5894 -323.4895 -327.3216

Columns 8 through 14

-321.2981 -316.1855 -312.3071 -310.4889 -309.6790 -309.9436 -309.3734

Columns 15 through 21


Columns 22 through 28

-310.1735 -311.1633 -308.9079 -312.0709 -310.6683 -310.6908 -307.6761

Columns 29 through 35

-312.9440 -310.5706 -316.2098 -318.9565 -327.6885 -326.4021 -322.3135
**Fundamental will appear at position 1+Np = 51**

Columns 36 through 42


Columns 43 through 49


Columns 50 through 56

-309.5231  \[0\]  -308.8842  -316.1343  -314.5406  -333.4024  -313.7342

Columns 57 through 63

-319.6023  -314.9029  -316.6932  -314.7123  -311.9567  -312.0200  -309.8825

Columns 64 through 70

-308.7103  -309.8064  -314.9393  -312.4610  -322.7229  -328.0350  -326.6767
The $k^{th}$ harmonic will appear at position $1+k\cdot N_p$.

Columns 71 through 77


Columns 78 through 84


Columns 85 through 91


Columns 92 through 98

-313.6855 -313.3882 -330.4962 -324.4762 -333.2237 -325.8694 -313.9127

Columns 99 through 105

-315.4869 -308.6364 -6.0206 -309.2723 -314.4098 -316.3311 -328.2626
$k^{th}$ harmonic will appear at position $1+k\cdot N_p$

Columns 106 through 112


Columns 113 through 119

-319.9292 -325.4840 -318.0998 -328.0000 -321.7632 -326.5097 -328.5867

Columns 120 through 126


Columns 127 through 133

-315.0684 -308.6315 -312.9640 -309.5056 -311.6251 -316.1369 -316.1064

Columns 134 through 140

-320.4989 -331.2686 -314.3479 -310.0891 -308.0023 -308.1556 -309.0616
$k^{\text{th}}$ harmonic will appear at position $1+k\cdot N_p$

Columns 141 through 147

-311.2372 -312.6180 -319.0565 -325.6750 -323.7759 -320.7444 -318.0752

Columns 148 through 154


Columns 155 through 161


Columns 162 through 168


Columns 169 through 175

Considerations for Spectral Characterization

**FFT Length**

- FFT Length does not affect the computational noise floor

- Although not shown here yet, FFT length does reduce the quantization noise floor coefficients

\[
E_{\text{QUANT}} \approx \sqrt{\sum_{k=2}^{2^{n_{\text{DFT}}}} A_k^2}
\]

If we assume \(E_{\text{QUANT}}\) is fixed

If the \(A_k\)'s are constant and equal

\[
E_{\text{QUANT}} \approx A_k \frac{2^{n_{\text{DFT}}}}{2}
\]

Solving for \(A_k\), obtain

\[
A_k \approx \frac{E_{\text{QUANT}}}{2^{n_{\text{DFT}}}/2}
\]

If input is full-scale sinusoid with only amplitude quantization with \(n\)-bit res,

\[
E_{\text{QUANT}} \approx \frac{X_{\text{LSB}}}{\sqrt{12}} = \frac{X_{\text{REF}}}{\sqrt{3} \cdot 2^{n+1}}
\]
Considerations for Spectral Characterization

FFT Length

\[ E_{QUANT} \approx \frac{X_{LSB}}{\sqrt{12}} = \frac{X_{REF}}{\sqrt{3 \cdot 2^{n+1}}} \]

Substituting for \( E_{QUANT} \), obtain

\[ A_k = \frac{X_{REF}}{\sqrt{3 \cdot 2^{n+1} 2^{n_{DFT}/2}}} \]

This value for \( A_k \) thus decreases with the length of the DFT window.
Considerations for Spectral Characterization

- Tool Validation
- FFT Length
- Importance of Satisfying Hypothesis
  - NP is an integer
  - Band-limited excitation
- Windowing
Example

WLOG assume $f_{\text{SIG}}=50\text{Hz}$

$$V_{\text{IN}} = \sin(\omega t) + 0.5 \sin(2\omega t)$$

$$\omega = 2\pi f_{\text{SIG}}$$

Consider $N_{P}=20.2$  $N=4096$

Recall $20\log_{10}(0.5)=-6.0205999$
Input Waveform
Input Waveform
Input Waveform
Spectral Response

Rect. Window  N=4096  Np =20.2

Freqency

Mag(dB)
Fundamental will appear at position 1+Np = 21

Columns 1 through 7

-35.0366  -35.0125  -34.9400  -34.8182  -34.6458  -34.4208  -34.1403

Columns 8 through 14


Columns 15 through 21


Columns 22 through 28


Columns 29 through 35

-34.1902  -35.2163  -35.9043  -36.1838  -35.9965  -35.3255  -34.1946

Note there is a dramatic increase in the noise floor and a significant change in and spreading of the fundamental!!
\[ k^{th} \text{ harmonic will appear at position } 1+k\cdot Np \]

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<th>Columns 50 through 56</th>
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<tbody>
<tr>
<td>-33.0833  -33.8720  -34.5759  -35.2113  -35.7902</td>
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\( k^{th} \) harmonic will appear at position \( 1 + k \cdot N_p \)

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<tbody>
<tr>
<td>-33.0833  -33.8720  -34.5759  -35.2113  -35.7902  -36.3218  -36.8133</td>
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Observations

• Modest change in sampling window of 0.2 out of 20 periods (1%) results in a big error in both fundamental and harmonic

• More importantly, dramatic raise in the noise floor !!! (from over -300dB to only -12dB)
Example

WLOG assume $f_{SIG}=50$Hz

$$V_{IN} = \sin(\omega t) + 0.5 \sin(2\omega t)$$

$$\omega = 2\pi f_{SIG}$$

Consider $N_P=20.01 \quad N=4096$

Deviation from hypothesis is .05% of the sampling window
Input Waveform
Input Waveform
Input Waveform
Input Waveform
Spectral Response

Rect. Window  N=4096  Np =20.01

Mag(dB)

Frequency

0  50  100  150  200

-90  -80  -70  -60  -50  -40  -30  -20  -10
**Fundamental will appear at position 1+Np = 21**

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<tr>
<td>-89.8679</td>
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<td>-83.0583</td>
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<td>-77.7239</td>
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<td>-74.2607</td>
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<td>-71.6830</td>
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<td>-69.5948</td>
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<td>-67.8044</td>
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<td>-66.2037</td>
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<td>-63.3167</td>
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<td>-60.5707</td>
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<td>-56.0866</td>
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<td>-54.2966</td>
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<td>-52.2035</td>
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<td>-49.6015</td>
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<td>-46.0326</td>
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<td>-40.0441</td>
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<td>-0.0007</td>
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<td>-40.0162</td>
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<td>-46.2516</td>
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<td>-50.0399</td>
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<td>-52.8973</td>
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<td>-55.3185</td>
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<td>-57.5543</td>
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<td>-59.7864</td>
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<td>-62.2078</td>
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<td>-65.1175</td>
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<td>-69.1845</td>
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<td>-76.9560</td>
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<td>-81.1539</td>
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<td>-69.6230</td>
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<td>-64.0636</td>
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$k^{th}$ harmonic will appear at position $1 + k \cdot N_p$

Columns 36 through 42

Observations

• Modest change in sampling window of 0.01 out of 20 periods (.05%) still results in a modest error in both fundamental and harmonic

• More importantly, substantial raise in the computational noise floor !!! (from over -300dB to only -40dB)

• Errors at about the 6-bit level !
Example

WLOG assume $f_{\text{SIG}}=50\text{Hz}$

\[ V_{\text{IN}} = \sin(\omega t) + 0.5 \sin(2\omega t) \]

\[ \omega = 2\pi f_{\text{SIG}} \]

Consider $N_{\text{P}}=20.001$ $N=4096$

Deviation from hypothesis is .005% of the sampling window
Spectral Response

Rect. Window  N=4096  Np =20.001

Magnitude (dB) vs Frequency
**Fundamental will appear at position 1+Np = 21**

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<td>-112.2531</td>
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<td>-86.2014</td>
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<td>-76.0714</td>
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<td>-60.0947</td>
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<td>-82.2405</td>
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The $k^{th}$ harmonic will appear at position $1+k\cdot N_p$.

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<th>Columns 36 through 42</th>
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<tbody>
<tr>
<td>-79.8472  -76.1160  -72.2601  -67.6621  -60.7642</td>
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<tr>
<td>-64.8177  -67.8520  -69.9156  -71.4625  -72.6918</td>
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<td>-75.3225  -75.9857  -76.5796  -77.1173</td>
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<tr>
<td>-78.8721  -79.2387  -79.5837</td>
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Observations

• Modest change in sampling window of 0.01 out of 20 periods (.005%) results in a small error in both fundamental and harmonic

• More importantly, substantial raise in the computational noise floor !!! (from over -300dB to only -60dB)

• Errors at about the 10-bit level !
Spectral Response

Rect. Window N=4096  Np =20.0001

![Graph showing spectral response with peaks at specific frequencies and a maximum magnitude of -20 dB.](image)
Fundamental will appear at position $1+N_p = 21$

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<tbody>
<tr>
<td>-130.4427  -123.1634  -117.7467</td>
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<td>-114.2649  -111.6804  -109.5888</td>
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<td>-107.7965  -106.1944  -104.7137</td>
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<td>-103.3055  -101.9314  -100.5575</td>
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<td>-99.1499   -97.6702   -96.0691</td>
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<tr>
<td>-94.2764   -92.1793   -89.5706</td>
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<td>-85.9878   -79.9571   $0.0000$</td>
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<td>-96.0691  -94.2764  -92.1793</td>
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<td>-89.5706  -85.9878  -79.9571</td>
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<td>$0.0000$</td>
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$k^{th}$ harmonic will appear at position $1+k\cdot N_p$

Columns 36 through 42


Columns 43 through 49

-84.8247  -87.8566  -89.9190  -91.4652  -92.6940  -93.7098  -94.5736

Columns 50 through 56

-95.3241  -95.9872  -96.5810  -97.1187  -97.6100  -98.0625  -98.4821

Columns 57 through 63

-98.8732  -99.2398  -99.5847  -99.9107  -100.2197  -100.5135  -100.7937

Columns 64 through 70
Observations

• Modest change in sampling window of 0.001 out of 20 periods (.0005%) results in a small error in both fundamental and harmonic

• More importantly, substantial raise in the computational noise floor !!! (from over -300dB to only -80dB)

• Errors at about the 13-bit level !
Lecture 33

Spectral Characterization

Distortion Analysis

• Time Quantization Effects
• Spectral Characteristic of DAC
  – Time and Amplitude Quantization
THEOREM: If $N_p$ is an integer and $x(t)$ is band limited to $hf$, then

$$|A_m| = \frac{2}{N} |X(mN_p + 1)| \quad 0 \leq m \leq h - 1$$

and

$$X(k) = 0 \quad \text{for all } k \text{ not defined above}$$

where $\langle X(k) \rangle_{k=0}^{N-1}$ is the DFT of the sequence $\langle x(kT_s) \rangle_{k=0}^{N-1}$

and $f = 1/T$
Review

Spectral Response

![Spectral Response Graph](chart.png)
Recall the theorem that provided for the relationship between the DFT terms and the Fourier Series Coefficients required

1. The sampling window be an integral number of periods

2. \[ N > \frac{2f_{\text{max}}}{f_{\text{SIGNAL}}} N_p \]
Spectral Response

Rect. Window N=4096  Np =20.01
Observations

• Modest change in sampling window of 0.001 out of 20 periods (.0005%) results in a small error in both fundamental and harmonic

• More importantly, substantial raise in the computational noise floor !!! (from over -300dB to only -80dB)

• Errors at about the 13-bit level !
Considerations for Spectral Characterization

- Tool Validation
- FFT Length
- Importance of Satisfying Hypothesis
  - NP is an integer
  - Band-limited excitation
- Windowing
FFT Examples

Recall the theorem that provided for the relationship between the DFT terms and the Fourier Series Coefficients required

1. The sampling window be an integral number of periods

2. \[ N > \frac{2f_{\text{max}}}{f_{\text{SIGNAL}}} N_p \]
Example

If $f_{\text{SIG}} = 50\text{Hz}$

and $N_p = 20$  $N = 512$

$$N > \frac{2f_{\text{max}}}{f_{\text{SIGNAL}}} N_p \quad \Rightarrow \quad f_{\text{max}} < 640\text{Hz}$$
Example

Consider $N_P=20$ $N=512$

If $f_{SIG}=50\text{Hz}$

$V_{IN} = \sin(\omega t) + 0.5 \sin(2\omega t) + 0.5 \sin(14\omega t)$

$\omega = 2\pi f_{SIG}$

(i.e. a component at 700 Hz which violates the band limit requirement)

Recall $20\log_{10}(0.5)=-6.0205999$
Effects of High-Frequency Spectral Components
Effects of High-Frequency Spectral Components
Effects of High-Frequency Spectral Components

Rect. Window  N=512  Np =20

fhigh=14fo

Mag(dB)

0  200  400  600  800  1000  1200
Frequency
Effects of High-Frequency Spectral Components

\[ f_{\text{high}} = 14f_0 \]

<table>
<thead>
<tr>
<th>Columns 1 through 7</th>
<th>Columns 8 through 14</th>
<th>Columns 15 through 21</th>
<th>Columns 22 through 28</th>
<th>Columns 29 through 35</th>
</tr>
</thead>
</table>
Effects of High-Frequency Spectral Components

\[ f_{\text{high}} = 14f_0 \]

Columns 36 through 42


Columns 43 through 49

-298.9215 -309.4829 -306.7363 -293.0808 -300.0882 -306.5530 -302.9962

Columns 50 through 56

-318.4706 -294.8956 -304.4663 -300.8919 -298.7732 -301.2474 -293.3188
Effects of High-Frequency Spectral Components

Aliased components at

\[ f_{\text{alias}} = 2f_{\text{sample}} - f \]

\[ f_{\text{alias}} = 2 \cdot 12.8 f_{\text{sig}} - 14 f_{\text{sig}} = 11.6 f_{\text{sig}} \]

thus position in sequence = \( 1 + N_p \frac{f_{\text{alias}}}{f_{\text{sig}}} = 1 + 20 \cdot 11.6 = 233 \)

Columns 225 through 231

-296.8883 -292.8175 -295.8882 -286.7494 -300.3477 -284.4253 -282.7639

Columns 232 through 238


Columns 239 through 245

-299.1299 -305.8361 -295.1772 -295.1670 -300.2698 -293.6406 -304.2886

Columns 246 through 252

-302.0233 -306.6100 -297.7242 -305.4513 -300.4242 -298.1795 -299.0956
Effects of High-Frequency Spectral Components
Effects of High-Frequency Spectral Components

Rect. Window N=512 Np =20

f_{high}=25 \, f_0
Effects of High-Frequency Spectral Components

![Graph showing effects of high-frequency spectral components with specific parameters and frequencies marked.](image-url)
Effects of High-Frequency Spectral Components
Effects of High-Frequency Spectral Components

Rect. Window  N=512  Np =20

f_{high}=24.5fo
Observations

• Aliasing will occur if the band-limited part of the hypothesis for using the DFT is not satisfied
• Modest aliasing will cause high frequency components that may or may not appear at a harmonic frequency
• More egregious aliasing can introduce components near or on top of fundamental and lower-order harmonics
• Important to avoid aliasing if the DFT is used for spectral characterization
End of Lecture 29