EE 435

Lecture 30

Spectral Performance
– Tool Use and Validation
If \( f(t) \) is periodic

\[
f(t) = A_0 + \sum_{k=1}^{\infty} A_k \sin\left( k\omega t + \theta_k \right)
\]

alternately

\[
f(t) = A_0 + \sum_{k=1}^{\infty} a_k \sin( k\omega t ) + \sum_{k=1}^{\infty} b_k \cos( k\omega t )
\]

\[
\omega = \frac{2\pi}{T}
\]

\[
A_k = \sqrt{a_k^2 + b_k^2}
\]

Termed the Fourier Series Representation of \( f(t) \)
Distortion Analysis

Total Harmonic Distortion, THD

\[
THD = \frac{\text{RMS voltage in harmonics}}{\text{RMS voltage of fundamental}}
\]

\[
THD = \sqrt{\left(\frac{A_2}{\sqrt{2}}\right)^2 + \left(\frac{A_3}{\sqrt{2}}\right)^2 + \left(\frac{A_4}{\sqrt{2}}\right)^2 + \ldots}
\]

\[
THD = \frac{\sqrt{\sum_{k=2}^{\infty} A_k^2}}{A_1}
\]
Distortion Analysis

- Often noise is present at other non-harmonic frequencies
- At higher frequencies the harmonics are often buried in the noise

Graph showing the magnitude of the harmonics $A_k$ against frequency $k$. The SFDR (Spurious Free Dynamic Range) is indicated by a vertical red line extending from the harmonic at $k=3$.
Distortion Analysis

**Theorem:** In a fully differential symmetric circuit, all even harmonics are absent in the differential output for symmetric differential excitations!

\[ V_{OD} = f(V_{ID}) = \sum_{k=0}^{\infty} h_k V_{ID}^k \]

Assume \( V_{ID} = K \sin(\omega t) \)

\[ V_{O1} = \sum_{k=0}^{\infty} h_k [\sin(\omega t)]^k \]
\[ V_{O2} = \sum_{k=0}^{\infty} h_k [-\sin(\omega t)]^k \]

\[ V_{OD} = V_{O1} - V_{O2} = \sum_{k=0}^{\infty} h_k \left( [\sin(\omega t)]^k - [-\sin(\omega t)]^k \right) = \sum_{k=0}^{\infty} h_k \left( [\sin(\omega t)]^k - (-1)^k [\sin(\omega t)]^k \right) \]

Observe the even-ordered harmonics are absent in this last sum.
Distortion Analysis

NOTATION:

\( T \): Period of Excitation
\( T_S \): Sampling Period
\( N_P \): Number of periods over which samples are taken
\( N \): Total number of samples

\[
N_P = \frac{NT_S}{T}
\]

Note: \( N_P \) is not an integer unless a specific relationship exists between \( N \), \( T_S \) and \( T \)

\[
h = \text{Int}\left(\left\lfloor \frac{N}{2} - 1 \right\rfloor \frac{1}{N_P}\right)
\]

Note: The function \( \text{Int}(x) \) is the integer part of \( x \)
THEOREM: If \( N_P \) is an integer and \( x(t) \) is band limited to \( f_{\text{MAX}} \), then

\[
|A_m| = \frac{2}{N} |X(mN_P + 1)| \quad 0 \leq m \leq h - 1
\]

and

\[
X(k) = 0 \quad \text{for all } k \text{ not defined above}
\]

where \( \langle X(k) \rangle_{k=0}^{N-1} \) is the DFT of the sequence \( \langle x(kT_s) \rangle_{k=0}^{N-1} \)

\[
f = \frac{1}{T}, \quad f_{\text{MAX}} = \frac{f}{2} \cdot \left\lfloor \frac{N}{N_P} \right\rfloor, \text{ and } \quad h = \text{Int} \left( \left\lfloor \frac{N}{2} \right\rfloor - 1 \cdot \frac{1}{N_P} \right)
\]
THEOREM?: If $N_P$ is an integer and $x(t)$ is band limited to $f_{\text{MAX}}$, then

$$|A_m| = \frac{2}{N} |X(mN_P + 1)|$$

for all $0 \leq m \leq h$

and

$$X(k) = 0$$

for all $k$ not defined above

where $\langle X(k) \rangle_{k=0}^{N-1}$ is the DFT of the sequence $\langle x(kT_s) \rangle_{k=0}^{N-1}$

$$f = \frac{1}{T}, \quad f_{\text{MAX}} = \frac{f}{2} \cdot \left[ \frac{N}{N_P} \right], \quad \text{and} \quad h = \text{Int}\left( \frac{f_{\text{MAX}}}{f} \right)$$
Distortion Analysis

If the hypothesis of the theorem are satisfied, we thus have

|X(k)|

FFT is a computationally efficient way of calculating the DFT, particularly when N is a power of 2.
FFT Examples

Recall the theorem that provided for the relationship between the DFT terms and the Fourier Series Coefficients required

1. The sampling window be an integral number of periods

2. \[ N > \frac{2 \, f_{\text{max}}}{f_{\text{SIGNAL}}} \, N_p \] (from \( f_{\text{MAX}} \leq \frac{f}{2 \cdot \left[ \frac{N}{N_p} \right]} \))
Considerations for Spectral Characterization

• Tool Validation

• FFT Length

• Importance of Satisfying Hypothesis

• Windowing
Considerations for Spectral Characterization

- Tool Validation (MATLAB)
- FFT Length
- Importance of Satisfying Hypothesis
- Windowing
Recall the theorem that provided for the relationship between the DFT terms and the Fourier Series Coefficients required

1. The sampling window must be an integral number of periods

2. \[ N > \frac{2 f_{\text{max}}}{f_{\text{SIGNAL}}} N_p \]
Example

WLOG assume $f_{SIG}=50\text{Hz}$

$$V_{IN} = \sin(\omega t) + 0.5 \sin(2\omega t)$$

$$\omega = 2\pi f_{SIG}$$

Consider $N_P=20$ $N=512$

Recall $20\log_{10}(1.0)=0.0000000$

Recall $20\log_{10}(0.5)=-6.0205999$
Input Waveform
Input Waveform
Input Waveform

Location of First Point if Extended Into Periodic Function
Spectral Response (expressed in dB)

(Actually Stem plots but points connected in plotting program)

(Horizontal axis is the “Index” axis but converted to frequency) \[ f_{\text{AXIS}} = f_{\text{SIGNAL}} \frac{n - 1}{N_p} \]
DFT Horizontal Axis Converter to Frequency:

\[ f_{\text{AXIS}} = f_{\text{SIGNAL}} \frac{n-1}{N_p} \]
Spectral Response

Rect. Window N=512  Np =20
Fundamental will appear at position 1+Np = 21

Columns 1 through 5

-316.1458 -312.9517 -329.5203 -311.1473 -314.2615

Columns 6 through 10

-315.2584 -330.6258 -317.2896 -312.2316 -311.6335

Columns 11 through 15


Columns 16 through 20

-314.0088 -302.6391 -306.6650 -311.3733 -308.3689

Columns 21 through 25

-0.0000 -307.7012 -312.9902 -312.8737 -305.4320

Observe system noise floor due to both spectral limitations of signal generator and numerical limitations in FFT are below -300db
Second Harmonic at $1+2Np = 41$

Columns 26 through 30

-307.8301 -309.0737 -305.8503 -312.2772 -315.7544

Columns 31 through 35

-311.9316 -316.0581 -318.3454 -306.4977 -308.6679

Columns 36 through 40

-309.9702 -305.9809 -322.1270 -310.6723 -310.3506

Columns 41 through 45

-6.0206  -309.6071 -314.1026 -307.6405 -302.9277

Columns 46 through 50

-313.0745 -304.2330 -310.8487 -317.7966 -316.3385
Third Harmonic at $1+3Np = 61$

Columns 51 through 55

-307.0529 -312.7787 -312.9340 -323.2969 -314.9297

Columns 56 through 60

-318.7605 -303.5929 -305.2994 -310.6430 -306.7613

Columns 61 through 65

-304.8298 -301.4463 -301.1410 -303.1784 -317.8343

Columns 66 through 70

-308.6310 -307.0135 -321.6015 -316.6548 -309.8946

Columns 71 through 75

-306.3472 -323.0110 -319.3267 -314.7873 -310.4085
Fourth Harmonic at $1+4Np = 81$

Columns 76 through 80


Columns 81 through 85


Columns 86 through 90

-313.4988 -303.4513 -310.4969 -317.9652 -312.5846

Columns 91 through 95

-309.8121 -311.6403 -312.8374 -310.5414 -308.7807

Columns 96 through 100

-316.7549 -316.3395 -308.4113 -307.3766 -311.0358
Question: How much noise is in the computational environment?

Is this due to quantization in the computational environment or to numerical rounding in the FFT?
Question: How much noise is in the computational environment?

Observation: This noise is nearly uniformly distributed. The level of this noise at each component is around -310dB.
Question: How much noise is in the computational environment?

Assume $A_k = -310$ dB for $0 \leq k \leq N$

$$A_{kdB} = 20 \log_{10} A_k$$

$$A_k \approx 10^{\frac{-310}{20}} = 10^{-15.5} \quad \text{defn} \quad \bar{A}$$

$$V_{\text{Noise,RMS}} \approx \sqrt{\sum_{k=1}^{N-1} \left( \frac{A_k}{\sqrt{2}} \right)^2}$$

$$V_{\text{Noise,RMS}} \approx \bar{A} \sqrt{\frac{N}{2}} = 10^{-15.5} \sqrt{\frac{512}{2}} = 5.1 \cdot 10^{-15} \approx 5fV$$

Note: This computational environment has a very low total computational noise and does not become significant until the 46-bit resolution level is reached!!
Considerations for Spectral Characterization

• Tool Validation
• FFT Length
• Importance of Satisfying Hypothesis
• Windowing
Example

- Increase length from 512 to 4096

WLOG assume $f_{SIG} = 50$Hz

$V_{IN} = \sin(\omega t) + 0.5 \sin(2\omega t)$

$\omega = 2\pi f_{SIG}$

Consider $N_p = 20 \quad N = 4096$
Spectral Response

Rect. Window N=4096  Np =20
**Fundamental will appear at position 1+Np = 21**

Columns 1 through 7


Columns 8 through 14

-319.7032 -317.4419 -327.4933 -321.1968 -318.2241 -312.7300 -316.8359

Columns 15 through 21

-315.5166 -316.1801 -307.8072 -304.3414 -301.3326 -301.7993 0

Columns 22 through 28


Columns 29 through 35

$k^{\text{th}}$ harmonic will appear at position $1+k \cdot N_p$

| Columns 36 through 42 |  
|-----------------------|---

| Columns 43 through 49 |  
|-----------------------|---
| -300.8222  -301.6722  -304.8150  -313.0288  -313.5963  -312.1136  -310.7740 |

| Columns 50 through 56 |  
|-----------------------|---

| Columns 57 through 63 |  
|-----------------------|---
| -320.2843  -320.9910  -316.8320  -318.3531  -318.4341  -322.1619  -321.6183 |

| Columns 64 through 70 |  
|-----------------------|---
Example

WLOG assume \( f_{\text{SIG}} = 50 \text{Hz} \)

\[
V_{\text{IN}} = \sin(\omega t) + 0.5 \sin(2\omega t)
\]

\[
\omega = 2\pi f_{\text{SIG}}
\]

Consider \( N_P = 50 \) \( N = 4096 \)
Spectral Response
**Fundamental will appear at position 1+Np = 51**

**Columns 1 through 7**

-322.4309  -325.5445  -322.2645  -321.6226  -319.5894  -323.4895  -327.3216

**Columns 8 through 14**

-321.2981  -316.1855  -312.3071  -310.4889  -309.6790  -309.9436  -309.3734

**Columns 15 through 21**


**Columns 22 through 28**

-310.1735  -311.1633  -308.9079  -312.0709  -310.6683  -310.6908  -307.6761

**Columns 29 through 35**

-312.9440  -310.5706  -316.2098  -318.9565  -327.6885  -326.4021  -322.3135
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\( k^{\text{th}} \) harmonic will appear at position 1+k\( \cdot N_p \)

Columns 71 through 77

\(-329.1687 \ -321.1102 \ -328.3790 \ -326.9774 \ -323.4227 \ -323.3388 \ -325.1652\)

Columns 78 through 84

\(-325.3417 \ -332.1905 \ -320.4431 \ -322.1461 \ -323.8993 \ -325.4370 \ -329.8160\)

Columns 85 through 91

\(-319.1702 \ -317.1792 \ -312.4734 \ -310.2585 \ -309.5426 \ -310.8963 \ -310.6955\)

Columns 92 through 98

\(-313.6855 \ -313.3882 \ -330.4962 \ -324.4762 \ -333.2237 \ -325.8694 \ -313.9127\)

Columns 99 through 105

\(-315.4869 \ -308.6364 \ -6.0206 \ -309.2723 \ -314.4098 \ -316.3311 \ -328.2626\)
$k^{th}$ harmonic will appear at position $1+k \cdot N_p$

Columns 106 through 112


Columns 113 through 119

-319.9292 -325.4840 -318.0998 -328.0000 -321.7632 -326.5097 -328.5867

Columns 120 through 126


Columns 127 through 133

-315.0684 -308.6315 -312.9640 -309.5056 -311.6251 -316.1369 -316.1064

Columns 134 through 140

-320.4989 -331.2686 -314.3479 -310.0891 -308.0023 -308.1556 -309.0616
The $k^{th}$ harmonic will appear at position $1+k\cdot N_p$.

Columns 141 through 147

-311.2372 -312.6180 -319.0565 -325.6750 -323.7759 -320.7444 -318.0752

Columns 148 through 154


Columns 155 through 161


Columns 162 through 168


Columns 169 through 175

Considerations for Spectral Characterization

FFT Length

• FFT Length does not affect the computational noise floor

• Although not shown here yet, FFT length does reduce the quantization noise floor coefficients

\[ E_{\text{QUANT}} = \sqrt{\sum_{k=2}^{2^{n_{\text{DFT}}}} A_k^2} \]

If we assume \( E_{\text{QUANT}} \) is fixed

\[ E_{\text{QUANT}} \approx A_k 2^{n_{\text{DFT}}/2} \]

If the \( A_k \)'s are constant and equal

Solving for \( A_k \), obtain

\[ A_k \approx \frac{E_{\text{QUANT}}}{2^{n_{\text{DFT}}/2}} \]

If input is full-scale sinusoid with only amplitude quantization with \( n \)-bit res,

\[ E_{\text{QUANT}} \approx \frac{X_{\text{LSB}}}{\sqrt{12}} = \frac{X_{\text{REF}}}{\sqrt{3} \cdot 2^{n+1}} \]
Considerations for Spectral Characterization

FFT Length

\[ E_{\text{QUANT}} \approx \frac{X_{\text{LSB}}}{\sqrt{12}} = \frac{X_{\text{REF}}}{\sqrt{3 \cdot 2^{n+1}}} \]

Substituting for \( E_{\text{QUANT}} \), obtain

\[ A_k \approx \frac{X_{\text{REF}}}{\sqrt{3 \cdot 2^{n+1}} 2^{n_{\text{DFT}}/2}} \]

This value for \( A_k \) thus decreases with the length of the DFT window.

Example: if \( n=16 \), \( n_{\text{DFT}}=12 \) (4096 pt transform), and \( X_{\text{REF}}=1\text{V} \), then \( A_k=6.9\text{E-8V} \ (-143\text{dB}) \),

(Note \( A_k \gg \) computational noise for all practical \( n, n_{\text{DFT}} \))
Considerations for Spectral Characterization

- Tool Validation
- FFT Length
- Importance of Satisfying Hypothesis
  - NP is an integer
  - Band-limited excitation
- Windowing
Example

WLOG assume $f_{SIG}=50\text{Hz}$

$$V_{IN} = \sin(\omega t) + 0.5 \sin(2\omega t)$$

$$\omega = 2\pi f_{SIG}$$

Consider $N_P=20.2$, $N=4096$

Recall $20\log_{10}(0.5)=-6.0205999$
Input Waveform
Input Waveform
Input Waveform
Input Waveform
Spectral Response

Rect. Window  N=4096  Np =20.2
Fundamental will appear at position $1+N_p = 21$

Columns 1 through 7

\[-35.0366 \ -35.0125 \ -34.9400 \ -34.8182 \ -34.6458 \ -34.4208 \ -34.1403\]

Columns 8 through 14

\[-33.8005 \ -33.3963 \ -32.9206 \ -32.3642 \ -31.7144 \ -30.9535 \ -30.0563\]

Columns 15 through 21

\[-28.9855 \ -27.6830 \ -26.0523 \ -23.9155 \ -20.8888 \ -15.8561 \ -0.5309\]

Columns 22 through 28

\[-12.8167 \ -20.1124 \ -24.2085 \ -27.1229 \ -29.4104 \ -31.2957 \ -32.8782\]

Columns 29 through 35

\[-34.1902 \ -35.2163 \ -35.9043 \ -36.1838 \ -35.9965 \ -35.3255 \ -34.1946\]

Note there is a dramatic increase in the noise floor and a significant change in and spreading of the fundamental!!
$k^{th}$ harmonic will appear at position $1+k \cdot N_p$

Columns 36 through 42


Columns 43 through 49


Columns 50 through 56

-33.0833  -33.8720  -34.5759  -35.2113  -35.7902  -36.3218  -36.8133

Columns 57 through 63


Columns 64 through 70

$k^{th}$ harmonic will appear at position $1 + k \cdot N_p$

Columns 36 through 42

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Observations

• Modest change in sampling window of 0.2 out of 20 periods (1%) results in a big error in both fundamental and harmonic

• More importantly, dramatic raise in the noise floor !!! (from over -300dB to only -12dB)
End of Lecture 30