EE 476
Homework 4

Exercise 1 (20 points)**.** Assume as part of prototyping some design, you order some springs from a certain manufacturer. They provide you with the following model of the force as a function of displacement from equilibrium that the model line you have chosen should produce:

$$F(x) = \operatorname{sgn}\left(\max\{0, L - l - |x|\}\right)\left[kx + \left(-\log\left(\frac{L-x}{L}\right)\right)^p + \left(\log\left(\frac{-L-x}{-L}\right)\right)^p\right]$$

For the manufacturers model:

F	force
x	displacement from equilibrium
k	small displacement parameter ($k > 0$)
L	coil length parameter ($L > 0$)
p	deformation parameter ($p > 1$)
l	break parameter ($l < L$)

(a) Investigate the meaning of the parameters by plotting $F(x)$ as a function of x for various sensible k, L, p, and l. Choose one nice looking set of parameters and the corresponding plot along with your interpretation of each parameters role. Make sure to clearly label all axes. Comment on what is happening to the spring in various regions of displacement.

(b) Linearize the model at equilibrium. (*Hint: you should obtain a nice, simple, familiar looking result.*)

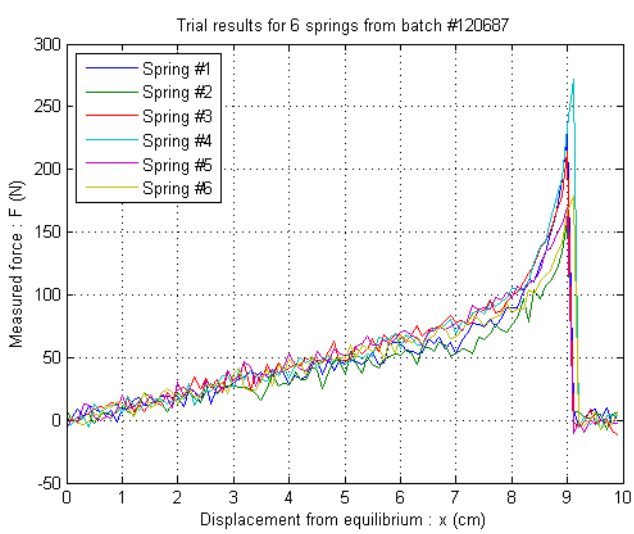


Figure 1: Example Spring Trials Results Plot

Exercise 2 (40 points)**.** You have asked a reliable technician to perform some test for you to investigate the quality of the manufacturers product before you use it in a prototype. The parts you have ordered specify

the following nominal parameter values: You asked the technician to take 6 random springs from a set you

k	1000 N/m
L	10 cm
p	5
l	8 mm

Table 1: Nominal Parameters

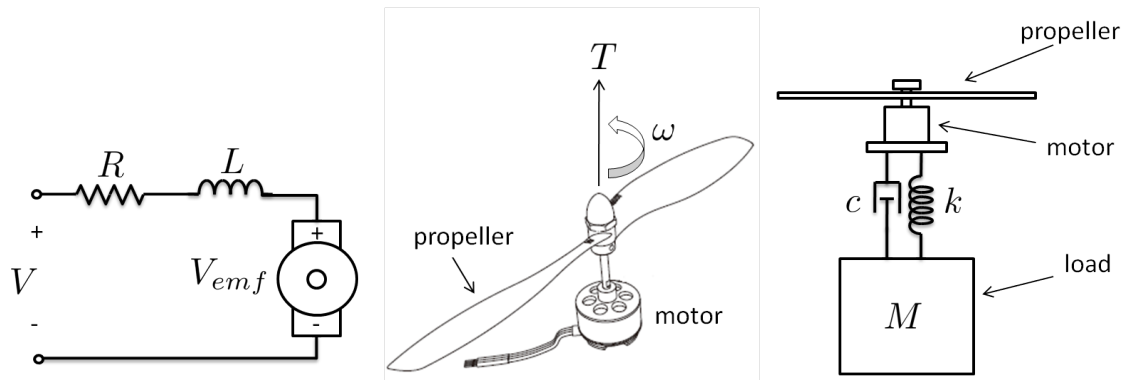
ordered and measure the spring force at 1mm increments between $x = 0$ and $x = 10$cm. An *example* plot of the results is shown in Figure 1.

(a) Use the provided function 'csm' to obtain *your* trial data. The input to the function is your birth date in mm/dd/yy format, e.g. for January 1, 1990 you would enter 010190 into the function.

(b) Modify the code to include on the plot a black line representing the nominal $F(x)$ over the same range of x with no noise. Ensure that this is indicated in the legend as well. Give some qualitative commentary on what conclusions you can reach about the manufacturing consistency and accuracy keeping your own measurement noise in mind.

(c) Assume that the spring is to be used as part of a mechanism where it will be displaced no more than 5cm from equilibrium. Fit a linear model to the data appropriately with this in mind. Use a least-squares minimization.[1] Comment on your result in relation to the nominal parameters.

(d) Modify the code to include a plot of your linear fit as a dot-dash black line, and again indicate this in the legend.



(a) Motor Circuit (b) Motor & propeller (c) Motor & Propeller & Load

Figure 2: Motor Circuit and Physical Setup

Exercise 3 (40 points)**.** Assume that we attach a propeller to a DC motor. See Figure 2(a) and 2(b). The motor is fixed to a small platform, and this platform is then attached to a larger load mass via a semi-flexible link. See Figure 2c. Assume that the load mass only moves in the up/down (z) direction for simplicity (i.e. only the motor+propeller rotate). Also assume the following:

- The motor inductance is negligable, i.e. $L = 0$.
- The motor torque is $Q_m = k_v(i - i_0 \operatorname{sgn}(\omega))$ where i_0 is what is often times called zero-load current.
- The motor back-emf voltage is given by $V_{emf} = k_v\omega$.
- The motor and propeller have combined moment of inertia J.

[1]Note that the data is returned in rows of a matrix. You may want to alter this form before working with it. Be careful how you format/shape your data. If you use 'reshape' make sure to consult the documentation to avoid issues as it operates along specific dimensions. Consult your TA if you have issues/questions.

- The motor and propeller and small platform have mass m, while the load mass is M.
- The semi-flexible link can be represented as a typical spring and damping using k and c respectively.
- The thrust produced by the propeller is given by $T = k_t\omega^2$.
- The torque generated by the propeller drag is $Q_p = k_d\omega^2$.
- The acceleration of gravity is g and pointing 'down' in Figure 2c

In a previous homework you were asked to model this system. Defining the displacement of the top mass m as q_1, the displacement of the load mass M as q_2 and the angular displacement of the motor and propeller inertia J as q_3, the total kinetic energy $\mathcal{T}$ and total potential energy $\mathcal{V}$ in the system are:

$$\mathcal{T} = \frac{1}{2}m\dot{q}_1^2 + \frac{1}{2}M\dot{q}_2^2 + \frac{1}{2}J\dot{q}_3^2$$
$$\mathcal{V} = \frac{1}{2}k(q_1 - q_2)^2 + mgq_1 + Mgq_2$$

So the Lagrangian $\mathcal{L} = \mathcal{T} - \mathcal{V}$ is:

$$\mathcal{L} = \frac{1}{2}m\dot{q}_1^2 + \frac{1}{2}M\dot{q}_2^2 + \frac{1}{2}J\dot{q}_3^2 - \frac{1}{2}k(q_1 - q_2)^2 - mgq_1 - Mgq_2$$

and with generalized external forces $\mathcal{Q}$ we have $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = \mathcal{Q}$ which is:

$$\begin{bmatrix} m\ddot{q}_1 \\ M\ddot{q}_2 \\ J\ddot{q}_3 \end{bmatrix} - \begin{bmatrix} -k(q_1 - q_2) - mg \\ k(q_1 - q_2) - Mg \\ 0 \end{bmatrix} = \begin{bmatrix} -c(\dot{q}_1 - \dot{q}_2) + T \\ c(\dot{q}_1 - \dot{q}_2) \\ Q_m - Q_p \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} m\ddot{q}_1 \\ M\ddot{q}_2 \\ J\ddot{q}_3 \end{bmatrix} = \begin{bmatrix} k_t\dot{q}_3^2 - c(\dot{q}_1 - \dot{q}_2) - k(q_1 - q_2) - mg \\ c(\dot{q}_1 - \dot{q}_2) + k(q_1 - q_2) - Mg \\ \frac{k_v}{R}V - \frac{k_v^2}{R}\dot{q}_3 - k_v i_0 \operatorname{sgn}(\dot{q}_3) - k_d\dot{q}_3^2 \end{bmatrix}$$

(a) Define the necessary state vector x and write the above model as a nonlinear state space $\dot{x} = f(x, V)$.

(b) Find the equilibrium state $\bar{x}$ and input voltage $\bar{V}$ for zero velocity and acceleration–i.e.'hovering'.

(c) Linearize the system around hovering equilibrium. Assume full state measurement and provide the A, B, C, D matrices. *Hint: In the region our operating point exists in, what is sgn($\dot{q}_3$)?*

(d) Draw a block diagram showing how you would have to implement a linear state feedback control gain on the nonlinear system at the previously specified equilibrium.