EE 476
Homework 1

Exercise 1 (20 points)**.** Explain which of the following form a vector space over real scalars (**Hint:** Apply definition of vector spaces over Real scalars).

1. Set of solutions $x(t)$ of the ODE
$$\frac{dx(t)}{dt} + 3x(t) = 0$$
2. Set of points (x, y, z) satisfying $x + y + z = 2$
3. Set of points (x, y, z) satisfying $x + y + z = 0$
4. Space of all complex numbers $\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R},\ i^2 = -1\}$

Exercise 2 (20 points)**.** Let V be a vector space and S_1 and S_2 be subspaces of vector space V. Show the following

1. Intersection of S_1 and S_2 is a subspace.
2. Union of S_1 and S_2 is in general not a subspace.
3. **Bonus 10 points:** Space formed by sum of vectors from S_1 and S_2 is a subspace. This space is known as the direct sum of S_1 and S_2 and denoted by $S_1 \oplus S_2$.

Exercise 3 (20 points)**.** Let P_n be the set of polynomials of maximum degree n given by,

$$P_n = \left\{ p(t) = \sum_{i=0}^{n} \alpha_i t^i \mid \alpha_i \in \mathbb{R} \text{ for all } i = \{0, 1, \ldots, n\},\ t \neq 0 \right\}$$

In class we have seen that P_n is a vector space. Show that the set $B = \{1, t, t^2, \ldots, t^n\}$ forms a basis for this space.

Bonus 10 points: Does the set $\hat{B} = \{1, (1+t), (1+t)^2, \ldots, (1+t)^n\}$ also form a basis? Why?

Exercise 4 (20 points)**.** Let P_n be the set of polynomials of maximum degree n given by,

$$P_n = \left\{ p(t) = \sum_{i=0}^{n} \alpha_i t^i \mid \alpha_i \in \mathbb{R} \text{ for all } i = \{0, 1, \ldots, n\},\ t \neq 0 \right\}$$

Let $S = \{s(t)\}$ denote the space of polynomials obtained by differentiating polynomials in P_n with respect to t

$$s(t) = \frac{d}{dt}p(t) = \sum_{i=i}^{n} i\alpha_i t^{i-1} = \alpha_1 + 2\alpha_2 t + 3\alpha_3 t^2 + \cdots + n\alpha_n t^{n-1}$$

1. Show that $\frac{d}{dt} : P_n \to S$ is a linear transformation. (**Hint:** Apply summation and product rule for derivatives)
2. Find a matrix representation A for the linear transformation $\frac{d}{dt}$. What is the rank of A.
3. What is the null space $\mathcal{N}(A)$ of the transformation? What is the dimension of $\mathcal{N}(A)$?
4. What is the range space $\mathcal{R}(A)$ of the transformation? What is the dimension of $\mathcal{R}(A)$?

Exercise 5 (20 points)**.** Consider the following matrices

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$$

1. Find characteristic polynomial for matrices A_1 and A_2.
2. Find eigenvalues of matrices A_1 and A_2. Verify your answer using Matlab command **eig**. ("help eig" in Matlab will give you the syntax for the command)
3. Find eigenvectors corresponding to eigenvalues in part (b) for matrix A_2 without the use of Matlab. Show all your steps. Verify your answer using Matlab command **eig**.
4. Find eigenvalues and eigenvectors of A_2^2 and A_2^{-1}
5. **Bonus 10 points:** If matrix $A \in \mathbb{R}^{n\times n}$ is invertible with eigenvalues and eigenvectors given by λ_i, and v_i, respectively for all $i = \{1, 2, \dots, n\}$. Then for any integer k, positive or negative, show that eigenvalues of A^k are given by λ_i^k and corresponding eigenvectors are given by v_i.