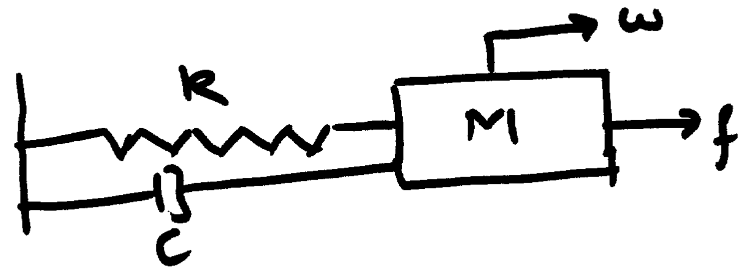
**Example1:**

# STATE-SPACE APPROACH:

## LECTURE: 11.




The dynamics are given by the equation

$$M\ddot{w} + Kw + C\dot{w} = f \quad - - - - (1)$$

- Suppose, a intial-time is $t=0$. Then we must specify $w(0)$, $\dot{w}(0)$ in order to solve (1) for a specified $f$. The internal variables $w$ and $\dot{w}$ are an example of state variables.

- Define $x_1 = w$

  $x_2 = \dot{w}$

Then we rewrite the $2^{nd}$ order differential equation (1) [ $2^{nd}$ order because the highest time-derivative is the $2^{nd}$ derivative] as an equivalent first order set of differential equation in terms of the new variables $x_1$ and $x_2$ as

$$\dot{x}_1 = \dot{\omega} = x_2$$
$$\dot{x}_2 = \ddot{\omega} = \frac{f}{m} - \frac{k}{m}x_1 - \frac{c}{m}x_2 .$$

So the state-space representation of the above system is:

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{f}{m}. \\ Y &= [1 \;\; 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}\right\}$$

Initial conditions $x_1(0)$, $x_2(0)$ are to be specified.

where the output ~~is~~ $y$ is $\omega(t)$.

Note that if we define the state to be

$$x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{then}$$

$$\frac{dx}{dt} := \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{f}{m} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} f$$

$$y = \underbrace{[1 \quad 0]}_{C} x.$$

Thus, the above state-space description is in the form

$$\dot{x} = Ax + Bf \qquad ; \; x(0) \text{ Specified.}$$
$$y = Cx + Du$$

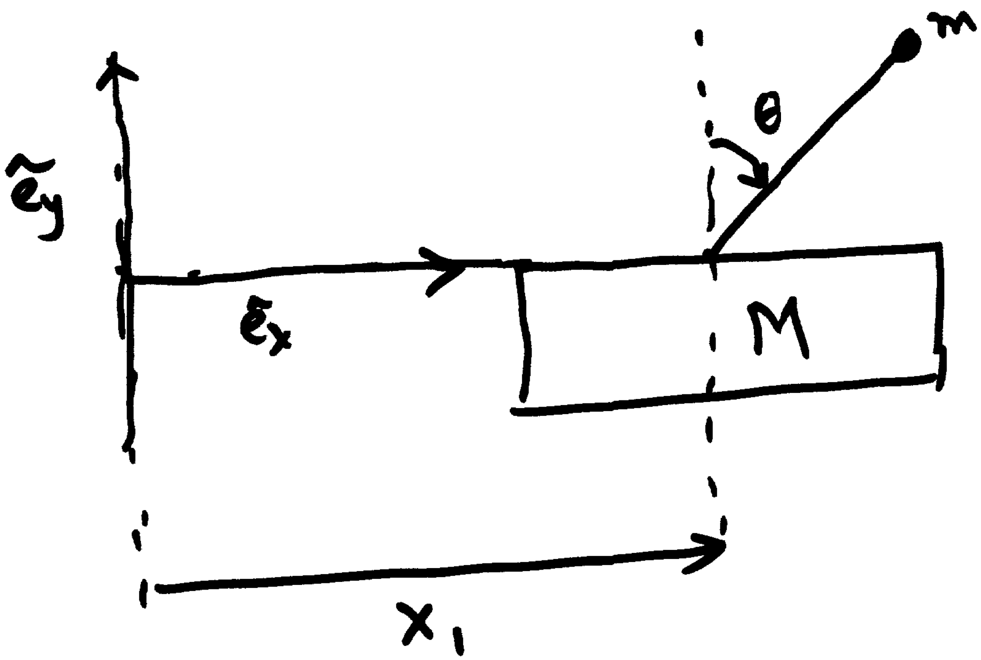
Any system which ~~to be~~ admits a state-space realization of the form:

$$\left.\begin{aligned} \dot{x} &= Ax + Bf \\ y &= Cx + Df \end{aligned}\right\} \quad x(0) \text{ specified.}$$

where A, B, C, D are constant-matrices is an LINEAR-TIME-INVARIANT-CAUSAL SYSTEM. (LTIC).

## Example 2 : PENDULUM ON A CART






We used the Lagrangian to obtain the following equations of motion:

$$\ddot{\theta} = \frac{g}{l}\sin\theta - \frac{\ddot{x}_1}{l}\cos\theta$$

$$\ddot{x}_1 = \frac{F}{M+m} - \frac{m}{M+m}\, l\cos\theta\,\ddot{\theta} + \frac{m\,\dot{\theta}^2 l\sin\theta}{m+M}.$$

$$\therefore \ddot{x}_1 = \frac{F}{M+m} - \frac{m}{M+m} l\cos\theta \left[ \frac{g}{l}\sin\theta - \frac{\ddot{x}_1}{l}\cos\theta \right] + \frac{m\dot{\theta}^2}{M+M} l \sin\theta .$$

$$= \frac{F}{M+m} - \frac{m}{M+m} g \sin\theta\cos\theta + \frac{m}{M+m}\ddot{x}_1 \cos^2\theta + \frac{m\dot{\theta}^2}{m+M} l\sin\theta .$$

$$\therefore \left[ 1 - \frac{m}{M+m}\cos^2\theta \right]\ddot{x}_1 = \frac{F}{M+m} - \frac{mg}{M+m}\sin\theta\cos\theta + \frac{m\dot{\theta}^2}{M+m} l\sin\theta .$$

$$\Rightarrow \ddot{x}_1 = \frac{M+m}{M+m-m\cos^2\theta}\left[ \frac{F}{M+m} - \frac{mg}{M+m}\sin\theta\cos\theta + \frac{m\dot{\theta}^2}{M+M} l\sin\theta \right].$$

<u>Similarly</u>

$$\ddot{\theta} = \frac{g}{l}\sin\theta - \frac{\ddot{x}_1}{l}\cos\theta$$

$$= \frac{g}{l}\sin\theta - \left[\frac{F}{M+m} - \frac{m}{M+m}\, l\cos\theta\,\ddot{\theta} + \frac{m\dot{\theta}^2}{m+M}\, l\sin\theta\right]\frac{1}{l}\cos\theta.$$

$$\therefore \left[1 - \frac{1}{l}\cos\theta\,\frac{m}{M+m}\, l\cos\theta\right]\ddot{\theta} = \frac{g}{l}\sin\theta - \frac{F}{M+m}\cdot\frac{1}{l}\cos\theta - \frac{m\dot{\theta}^2}{m+M}\sin\theta\cos\theta.$$

$$\therefore \quad \ddot{\theta} = \frac{M+m}{M+m-m\cos^2\theta}\left[\frac{g}{l}\sin\theta - \frac{F}{(M+m)l}\cos\theta - \frac{m\dot{\theta}^2}{m+M}\sin\theta\cos\theta\right].$$

Let

$x_2 = \dot{x}_1$ ,

$\cancel{x_3}$

$x_3 = \theta$

$x_4 = \dot{\theta}$

$$\therefore \quad \dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{x}_1 = \frac{M+m}{M+m-m\cos^2 x_3}\left[-\frac{mg}{2(M+m)}\sin 2x_3 + \frac{ml}{M+M}x_4^2 \sin x_3 + \frac{F}{M+m}\right].$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \ddot{\theta} = \frac{M+m}{M+m-m\cos^2 x_3}\left[\frac{g}{l}\sin x_3 - \frac{F}{(M+m)l}\cos x_3 - \frac{m x_4^2}{2(M+M)}\sin 2x_3\right].$$

$$\therefore$$

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{M+m}{M+m-m\cos^2 x_3}\left(-\dfrac{mg}{2(M+m)}\sin 2x_3 + \dfrac{ml}{M+m}x_4^2 \sin x_3 + \dfrac{F}{M+m}\right). \\ x_4 \\ \dfrac{M+m}{M+m-m\cos^2 x_3}\left[\dfrac{g}{l}\sin x_3 - \dfrac{m x_4^2}{2(M+m)}\sin 2x_3 - \dfrac{F}{(M+m)l}\cos x_3\right] \end{bmatrix}$$

$$=: f(x, F).$$

$$y = \begin{bmatrix} x_1 \\ \theta \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} X.$$

In this case

$\dot{x}$ cannot be written ~~to~~ as

$$\dot{x} = Ax + BF$$
$$y = Cx + Du.$$

$\therefore$ The above system is not linear.

lets assume that $M \gg m$. Then, the State-Space Equations can be simplified as

$$\dot{X} = \begin{bmatrix} x_2 \\ -\frac{m}{2M} \sin 2x_3 + \frac{m}{M} \ell x_4^2 \sin x_3 + \frac{F}{M} \\ x_4 \\ \frac{g}{\ell} \sin x_3 - \frac{m}{2M} x_4^2 \sin 2x_3 - \frac{F}{M\ell} \cos x_3 \end{bmatrix} \approx \begin{bmatrix} x_2 \\ F/M \\ x_4 \\ \frac{g}{\ell} \sin x_3 - \frac{F}{M\ell} \cos x_3 \end{bmatrix}$$

$$Y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} X.$$