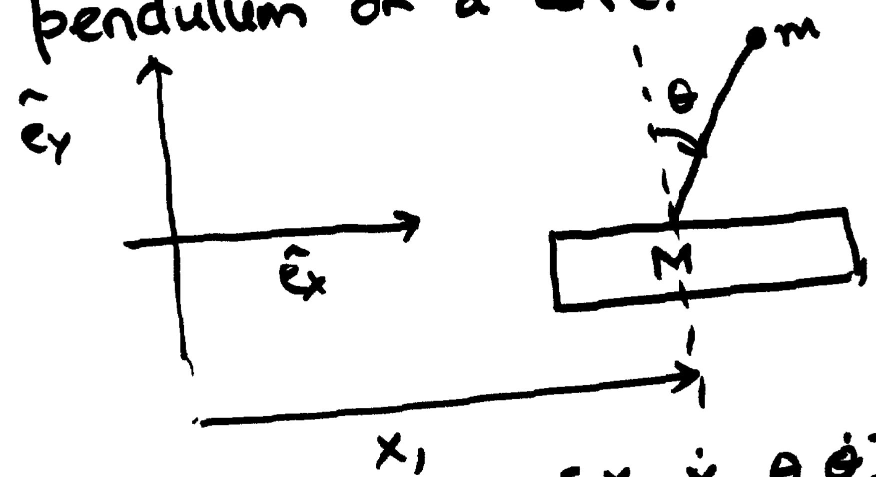
LECTURE 12

Last class we obtained the state-space representation of a pendulum on a cart:



Let the state be $x = [x_1 \ \dot{x}_1 \ \theta \ \dot{\theta}]^T =: [x_1 \ x_2 \ x_3 \ x_4]^T$.

After making a approximation that $m \ll M$ we have

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ F/M \\ x_4 \\ \frac{g}{\ell}\sin x_3 - \frac{F}{ml}\cos x_3 + \frac{\tau}{ml^2} \end{bmatrix} =: f(x,u) \text{ with}$$

$$u = \begin{bmatrix} F \\ \tau \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Thus,

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ u_1/M \\ x_4 \\ \frac{g}{\ell}\sin x_3 - \frac{u_1}{m\ell}\cos x_3 + \frac{u_2}{m\ell^2} \end{bmatrix} =: f(x,u),$$

Note that

$$f(x,u) \neq Ax + Bu$$ for any matrices A, B.

The output equation is given by

$$Y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x =: h(x).$$

Suppose our interest is in the state

$$X(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix}.$$

Note that the state is completely determined by specifying position of the cart x_1, velocity of the cart $x_2 = \dot{x}_1$, angular position of the pendulum $\theta =: x_3$ and the angular velocity $\dot{\theta} =: x_4$. ~~at the~~

Just specifying the pendulum angle and the cart position is not sufficient; one needs to also specify their velocities.

Suppose, we are interested in the nominal state $\bar{x}$, and the nominal control $\bar{u}$.

We want to analyze how the system behaves with the state x near $\bar{x}$ and the control effort u near $\bar{u}$, ie if $|x(t) - \bar{x}|$ and $|u - \bar{u}|$ are small.

The answer is provided by linearization about $\bar{x}$ and $\bar{u}$, which is obtained by ignoring the higher order terms in the Taylor series expansion of $f(x,u)$ about $\bar{x}$ and $\bar{u}$.

LINEARIZATION

Given the nonlinear system

$$\frac{dx}{dt} = f(x,u)$$

where $x \in R^n$, $u \in R^m$ and $f(x,u)$ is

$$f(x,u) = \begin{bmatrix} f_1(x_1, x_2 \ldots x_n, u_1, u_2 \ldots u_m) \\ f_2(x_1, x_2 \ldots x_n, u_1, u_2 \ldots u_m) \\ \vdots \\ f_n(x_1, x_2 \ldots x_n, u_1, u_2, \ldots u_m) \end{bmatrix}$$

~~The~~ Define the $n \times n$ matrix

$$\frac{\partial f}{\partial x} := \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

and the $n \times m$ matrix:

$$\frac{\partial f}{\partial u} := \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ \vdots & & & \vdots \\ \partial f_n / \partial u_1 & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}$$

Then the linearized dynamics of

$$\mathring{x} = f(x,u)$$

about $(\bar{x},\bar{u})$ is given by

$$\mathring{x} = A(x-\bar{x}) + B(u-\bar{u})$$

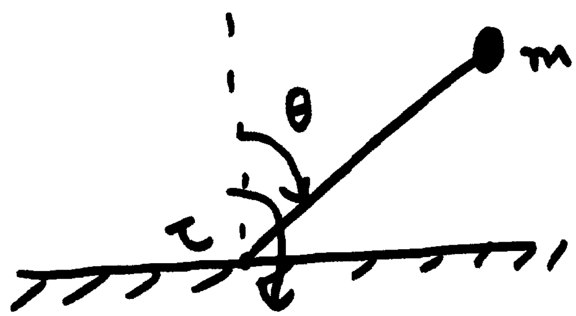
where

$$A = \frac{\partial f}{\partial x}(\bar{x},\bar{u})$$

$$B = \frac{\partial f}{\partial u}(\bar{x},\bar{u})$$

Example:

~~Let~~ Consider the example of a simple-pendulum



The Equation of motion is given by (See Page 8, Lect8].

$$\ddot{\theta} = \frac{g}{l} \sin\theta + \tau/ml^2$$

Defining $x_1 = \theta$, $x_2 = \dot{\theta}$, we have,

$$\dot{x}_1 = \dot{\theta} = x_2$$

$$\dot{x}_2 = \ddot{\theta} = \frac{g}{l}\sin x_1 + \frac{u}{ml^2} \quad ; \quad u := \tau$$

$$\therefore \quad \mathring{x} = f(x,u) = \begin{bmatrix} x_2 \\ \frac{g}{l}\sin x_1 + \frac{u}{ml^2} \end{bmatrix}$$

$$\therefore \quad f_1(x,u) = x_2$$

$$f_2(x,u) = \frac{g}{l}\sin x_1 + \frac{u}{ml^2}$$

$$\frac{\partial f_1}{\partial x_1} = 0 ; \quad \frac{\partial f_1}{\partial x_2} = 1$$

$$\frac{\partial f_2}{\partial x_1} = +\frac{g}{l}\cos x_1 ; \quad \frac{\partial f_2}{\partial x_2} = 0$$

$$\frac{\partial f_1}{\partial u_1} = 0 ; \quad \frac{\partial f_2}{\partial u_1} = \frac{1}{ml^2}.$$

Thus,

$$\frac{\partial f}{\partial x}(x,u) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell}\cos x_1 & 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial u}(x,u) = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \partial f_2/\partial u \end{bmatrix} = \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix}.$$

Suppose, we are interested in the state

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ ; with } \bar{u} = 0$$

Then the linearized dynamics are given by

$$\dot{x} = Ax + Bu$$

$$A = \frac{\partial f}{\partial x}(0,0) \quad ; \quad B = \frac{\partial f}{\partial x}(0,0)$$

$$\therefore A = \begin{bmatrix} 0 & 1 \\ g/l \cos x_1 & 0 \end{bmatrix}_{\substack{x=0 \\ u=0}} \quad ; \quad B = \frac{\partial f}{\partial u}(x,u)\Big|_{00} = \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix}_{\substack{x=0 \\ u=0}}$$

$$\therefore A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix}.$$

Thus

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ g/l & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix} u.$$

Example: Pendulum on a Cart:

$$\overset{\circ}{X} = \begin{bmatrix} x_2 \\ u_1/M \\ x_4 \\ \frac{g}{\ell}\sin x_3 - \frac{u_1}{M\ell}\cos x_3 + \frac{u_2}{m\ell^2} \end{bmatrix} = \begin{bmatrix} f_1(x, u) \\ f_2(x, u) \\ f_3(x, u) \\ f_4(x, u) \end{bmatrix}.$$

$$\frac{\partial f_1}{\partial x_1} = 0; \quad \frac{\partial f_1}{\partial x_2} = 1, \quad \frac{\partial f_1}{\partial x_3} = 0; \quad \frac{\partial f_1}{\partial x_4} = 0.$$

$$\frac{\partial f_2}{\partial x_1} = \frac{\partial f_2}{\partial x_2} = \frac{\partial f_2}{\partial x_3} = \frac{\partial f_2}{\partial x_4} = 0; \quad \frac{\partial f_2}{\partial u_1} = 1/M; \quad \frac{\partial f_2}{\partial u_2} = 0.$$

$$\frac{\partial f_3}{\partial x_1} = \frac{\partial f_3}{\partial x_2} = \frac{\partial f_3}{\partial x_3} = 0; \quad \frac{\partial f_3}{\partial x_4} = 1, \quad \frac{\partial f_4}{\partial u_1} = 0; \quad \frac{\partial f_4}{\partial u_2} = 0.$$

$$\frac{\partial f_4}{\partial x_1} = \frac{\partial f_4}{\partial x_2} = 0 \quad \frac{\partial f_4}{\partial x_3} = \frac{g}{\ell}\cos x_3 + \frac{u_1}{M\ell}\sin x_3; \quad \frac{\partial f_4}{\partial x_4} = 0; \quad \frac{\partial f_4}{\partial u_1} = \frac{-1}{M\ell}\cos x_3; \quad \frac{\partial f_4}{\partial u_2} = \frac{1}{m\ell^2}$$

$$\therefore \frac{\partial f}{\partial x}(x,u) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{l}\cos x_3 + \frac{u_1}{Ml}\sin x_3 & 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} \\ \frac{\partial f_4}{\partial u_1} & \frac{\partial f_4}{\partial u_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{1}{M} & 0 \\ 0 & 0 \\ -\frac{1}{Ml}\cos x_3 & \frac{1}{Ml^2} \end{bmatrix}.$$

Linearization about $(0,0) = (\bar{x}, \bar{u})$ is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{\ell} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ \frac{1}{M} & 0 \\ 0 & 0 \\ -\frac{1}{M\ell} & \frac{1}{M\ell^2} \end{bmatrix}$$

$\therefore$ <u>Linearized-dynamics is</u>

$$\overset{\circ}{X} = Ax + Bu$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & g/\ell & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{M} & 0 \\ 0 & 0 \\ -\frac{1}{ML} & \frac{1}{ML^2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$