LECTURE 13

Review:

A nonlinear system time-invariant has the form

$$\frac{dx}{dt} = f(x, u) \quad ; \quad x \in R^n, \; u \in R^m$$

state vector

control vector.

LINEARIZATION:

Suppose we want to linearize the system about $(\bar{x}, \bar{u})$. Then the following is the linearized system

$$\frac{dx}{dt} = A(x - \bar{x}) + B(u - \bar{u})$$

where

$$A = \frac{\partial f}{\partial x}(\bar{x}, \bar{u}) \quad ; \quad B = \frac{\partial f}{\partial u}(\bar{x}, \bar{u}).$$

We can define new variables

$z = x - \bar{x}$; measuring all state variables with respect to the nominal state vector $\bar{x}$

$v = u - \bar{u}$; measuring the control variable with respect to the nominal control vector $\bar{u}$.

with the change we have

$$\frac{dz}{dt} = \frac{dx}{dt} - \frac{d\bar{x}}{dt} = \frac{dx}{dt}$$

$$\therefore \frac{dz}{dt} = \frac{dx}{dt} = A(x-\bar{x}) + B(u-\bar{u})$$

$$= Az + Bv.$$

where $A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}}$; $B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}}$.

EQUILIBRIUM POINTS:

Consider the nonlinear-system

$$\frac{dx}{dt} = f(x,u).$$

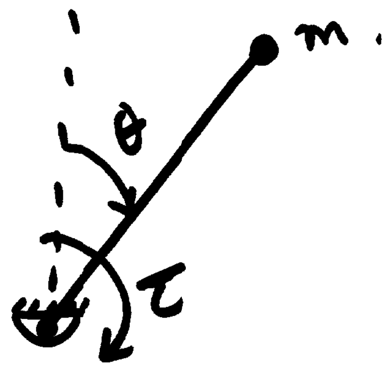
The equilibrium points ~~are~~ of the above dynamics are all ~~all~~ points x^* such that

$$f(x^*, 0) = 0$$

Control is set to 0.

EXAMPLE

INVERTED PENDULUM:



The equation of motion is given by

$$\ddot{\theta} = \frac{g}{\ell}\sin\theta + \frac{\tau}{m\ell^2} \quad ; \quad \tau \text{ is the torque applied.}$$

Define $x_1 = \theta$, $x_2 = \dot{\theta}$. Thus we have

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{g}{\ell}\sin x_1 + \frac{u}{m\ell^2} \quad ; \quad u \triangleq \tau$$

$\therefore$

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{g}{\ell}\sin x_1 + \frac{u}{m\ell^2} \end{bmatrix} =: f(x,u)$$

The equilibrium points are given as the solutions of

$$f(x,0) = 0$$

i.e. the solutions of

$$\begin{bmatrix} x_2 \\ \frac{g}{\ell}\sin x_1 + \frac{0}{m\ell^2} \end{bmatrix} = 0$$

The equilibrium points are

$\therefore$

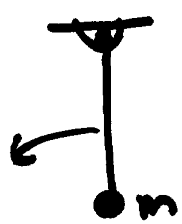
$$x_2 = 0$$

$$\frac{g}{l} \sin x_1 = 0$$

$$\Rightarrow \quad x_1 = \pm n\pi \quad ; n = 0, 1 \cdots$$

The two distinct equilibrium points are

(1) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$; (2) $\begin{bmatrix} \pi \\ 0 \end{bmatrix}$



$\theta = \pi$
$\dot{\theta} = 0$

$\theta = 0$
$\dot{\theta} = 0.$

We would like to study the behaviour of the pendulum near both these equilibrium points. We will now linearize the system about the equilibrium points:

Linearization about $[\bar{x}=0, \bar{u}=0]$

$$\dot{x} = Ax + Bu$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=0\\u=0}} \; ; \quad B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=0\\u=0}}$$

$$A = \left.\begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos x_1 & 0 \end{bmatrix}\right|_{\substack{x=0\\u=0}} \; ; \quad B = \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ g/l & 0 \end{bmatrix}$$

∴ The linearized dynamics is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ g/\ell & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m\ell^2 \end{bmatrix} u$$

<u>LINEARIZATION</u> ABOUT <u>$x = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$; $u = 0$.</u>

$$A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\binom{\pi}{0} \\ u=0}} = \left.\begin{bmatrix} 0 & 1 \\ \frac{g}{\ell}\cos x_1 & 0 \end{bmatrix}\right|_{\substack{x=\binom{\pi}{0} \\ u=0}}$$

$$= \begin{bmatrix} 0 & 1 \\ -g/\ell & 0 \end{bmatrix}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=0 \\ u=0}} = \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix}.$$

Thus the linearized-dynamics are described by

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -g/l & 0 \end{bmatrix} \left(x - \begin{pmatrix} \pi \\ 0 \end{pmatrix} \right) + \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix} u.$$

Defining

$$z = x - \begin{pmatrix} \pi \\ 0 \end{pmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

we have

$$\frac{dz}{dt} = \begin{bmatrix} 0 & 1 \\ -g/\ell & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u.$$

$$\therefore \quad \boxed{\dot{\underline{z}} = Az + Bu}$$