LECTURE-14

SOLUTIONS OF LINEAR SYSTEMS:

Consider the following differential Equation

$$\frac{d^3y}{dt^3} + 3\frac{dy}{dt} + y = 0$$

In need to ~~t~~ solve the above equation we need the initial conditions $y(0), \dot{y}(0), \ddot{y}(0)$.

Let

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$x_3 = \ddot{y}$$

The state-space notation gives.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -x_1 - 3x_2$$

$$\Rightarrow \quad \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix};$$

$$x(0) = \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \end{bmatrix}.$$

This fits the general LTIC form

$$\frac{dx}{dt} = Ax + Bu; \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In general if x is a n-vector and u is a m-vector,

$$x(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} ; \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

then A is a $n \times n$ matrix and B is a $n \times m$ matrix.

Before arriving at the solution for a general case lets consider the scalar case

$$\frac{dx}{dt} = a x(t) + b u(t), \quad x(0) = x_0$$, a and b are scalars.

Thus

$$\frac{dx}{dt} - ax = bu$$

multiply both sides by e^{-at}

$$e^{-at}\frac{dx}{dt} - e^{-at}ax = bu\,e^{-at}.$$

$$\therefore \frac{d}{dt}\left[e^{-at}x(t)\right] = e^{-at}bu.$$

Integrating with respect to time we have

$$\int_0^t \frac{d}{d\tau}\left(e^{-a\tau}x(\tau)\right) = \int_0^t e^{-a\tau}\,b\,u(\tau)\,d\tau$$

$$\therefore \quad e^{-a\tau}x(\tau)\Big|_0^t = \int_0^t e^{-a\tau}\, b u(\tau)\, d\tau$$

$$\Rightarrow \quad e^{-at}x(t) - e^{-a\cdot 0}x(0) = \int_0^t e^{-a\tau}\, b u(\tau)\, d\tau$$

$$\Rightarrow \quad x(t) = e^{+at}x(0) + e^{+at}\int_0^t e^{-a\tau}\, b u(\tau)\, d\tau$$

$$\Rightarrow \quad \boxed{x(t) = e^{+at}x(0) + \int_0^t e^{+a(t-\tau)}\, b u(\tau)\, d\tau.}$$

SCALAR FORM OF VARIATION OF PARAMETERS FORMULA

$e^{-at}x_0$: transient response due to initial conditions

$\int_0^t e^{a(t-\tau)} b u(\tau)\, d\tau$: force response produced due to control input.

$x_0, u(t)$: given quantities

<u>Example</u>:

$$\frac{dx}{dt} = -3x + 2u(t)$$

where

$$u(t) = h(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

$$x(0) = 5$$

using the variation of parameter formula we have

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-z)}\, bu(z)\,dz$$

$$= e^{-3t}\cdot 5 + \int_0^t e^{-3(t-z)}\, 2\,dz$$

$$= 5e^{-3t} + 2e^{-3t}\int_0^t e^{3z}\,dz$$

$$= 5e^{-3t} + 2e^{-3t}\left.\frac{e^{3z}}{3}\right|_0^t$$

$$= 5e^{-3t} + 2e^{-3t}\left[\frac{e^{3t}}{3} - \frac{1}{3}\right] = 5e^{-3t} + \frac{2}{3} - \frac{2}{3}e^{-3t}$$

$$= \frac{13}{3}e^{-3t} + 2/3.$$

$\therefore\ x(t) = \frac{13}{3} e^{-3t} + \frac{2}{3}$.

$\therefore\ x(t) \longrightarrow 2/3$ as $t \to \infty$

$\therefore$ $x(t) = 2/3$ is the steady state response of the system. $\frac{13}{3} e^{-3t}$ affects only the transient response.

Consider

$$\dot{x} = Ax + Bu \; ; \quad x(0) = x_0$$

Let us define the matrix $\phi(t)$ which solves the following matrix differential equation

$$\frac{d\phi}{dt} = A\phi$$

$$\phi(0) = I$$

where I is the $n \times n$ identity matrix.

~~By this~~

By using Laplace-transforms we have

$$sI\,\hat{\phi}(s) - I = A\,\hat{\phi}(s)$$

where $\hat{\phi}(s) = \mathcal{L}\{\phi(t)\}$

and thus

$$(sI - A)\,\hat{\phi}(s) = I$$

$$\Rightarrow \quad \hat{\phi}(s) = (sI - A)^{-1}.$$

and

$$\phi(t) = \mathcal{L}^{-1}(sI - A)^{-1}$$

One can also verify that $\phi(t) = e^{tAt} = \sum_{j=0}^{\infty} \frac{(tA)^j}{j!} = \sum_{j=0}^{\infty} \frac{(tAt)^j}{j!}$

Note that

$$\frac{dx}{dt} - Ax = Bu$$

$$\Rightarrow \quad \phi^{-1}(t)\frac{dx}{dt} - \phi^{-1}(t)Ax = \phi^{-1}(t)Bu$$

$$\Rightarrow \quad \frac{d}{dt}\left(\phi^{-1}(t)x(t)\right) = \phi^{-1}(t)Bu.$$

$$\Rightarrow \quad \int_0^t \frac{d}{d\tau}\left(\phi^{-1}(\tau)\,x(\tau)\right)d\tau = \int_0^t \phi^{-1}(\tau)Bu(\tau)d\tau$$

$$\Rightarrow \quad \phi^{-1}(t)x(t) - \phi^{-1}(0)x(0) = \int_0^t \phi^{-1}(\tau)Bu(\tau)d\tau$$

scalar case

$$\frac{dx}{dt} - ax = bu.$$

$$e^{-at}\frac{dx}{dt} - e^{-at}ax = e^{-at}bu.$$

$$\frac{d}{dt}\left(e^{-at}x(t)\right) = e^{-at}bu.$$

$$\int_0^t \frac{d}{d\tau}\left(e^{-a\tau}x(\tau)\right)d\tau = \int_0^t e^{-a\tau}bu(\tau)\,d\tau$$

$$\therefore \quad x(t) = \left(\phi(t)\right)^{+1} x(0) + \int_0^t \left(\phi(t)\right)^{+1} \phi(\tau)\, B u(\tau)\, d\tau$$

$$\Rightarrow x(t) = e^{+At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau .$$

<u>Example</u>: $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t).$

Suppose $u(t) \equiv 0.$

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}.$$

Note that

$$e^{At} = \mathcal{L}^{-1}(sI-A)^{-1}$$

$$sI-A = s\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix} = \begin{bmatrix}s-1 & 0\\ 0 & s-3\end{bmatrix}$$

$$\therefore \quad (sI-A)^{-1} = \begin{bmatrix}1/s-1 & 0\\ 0 & 1/s-3\end{bmatrix}$$

$$\therefore \quad \phi(t) = \mathcal{L}^{-1}(sI-A)^{-1} = \begin{bmatrix}\mathcal{L}^{-1}1/(s-1) & 0\\ 0 & \mathcal{L}^{-1}\frac{1}{s-3}\end{bmatrix}$$

$$e^{At} = \begin{bmatrix}e^{t} & 0\\ 0 & e^{3t}\end{bmatrix}.$$

.6 From the variation of parameters formula

$$X(t) = e^{At}\,x(0)$$

$$= \begin{bmatrix} e^{t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 6e^{t} \\ 5e^{3t} \end{bmatrix}.$$