LECTURE 15

Identification based on step response of an underdamped second order System:

Consider the second order differential equation

$$\ddot{x} + \delta \dot{x} + \omega_0^2 x = f(t)$$

with zero initial conditions and

$$f(t) = u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

Then, the response $x(t)$ can be found using either variation of parameters formula, or Laplace transforms methods [see your EE 475 notes] as

$$x(t) = \frac{1}{\omega_0^2} - \frac{e^{-\frac{\delta}{2}t}}{\omega_0^2}\left[\cos\bar{\omega}t + \frac{\delta}{2\bar{\omega}}\sin\bar{\omega}t\right].$$

$$= \frac{1}{\omega_0^2} - \frac{e^{-\frac{\delta}{2}t}}{\omega_0^2}\left[\cos\bar{\omega}t + \frac{\delta}{2\bar{\omega}}\sin\bar{\omega}t\right]$$

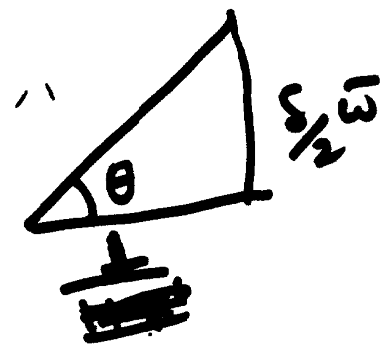
where

$\bar{\omega}^2 = \omega_0^2 - \frac{\delta^2}{4}$ is called the damped natural frequency.

(3)

$$\therefore\ x(t) = \frac{1}{\omega_0^2} - \frac{e^{-\frac{\delta}{2}t}}{\omega_0^2}\left\{\frac{1}{1+\frac{\delta^2}{4\bar{\omega}^2}}\cos\bar{\omega}t + \frac{\delta/2\bar{\omega}}{\left(1+\frac{\delta^2}{4\bar{\omega}^2}\right)}\sin\bar{\omega}t\right\}\left(1+\frac{\delta^2}{\bar{\omega}^2}\right)$$

$$= \frac{1}{\omega_0^2} - \frac{e^{-\delta/2\, t}}{\omega_0^2}\cdot\frac{\omega_0^2}{\bar{\omega}^2}\left\{\cos\theta\cos\bar{\omega}t + \sin\theta\sin\bar{\omega}t\right\}$$

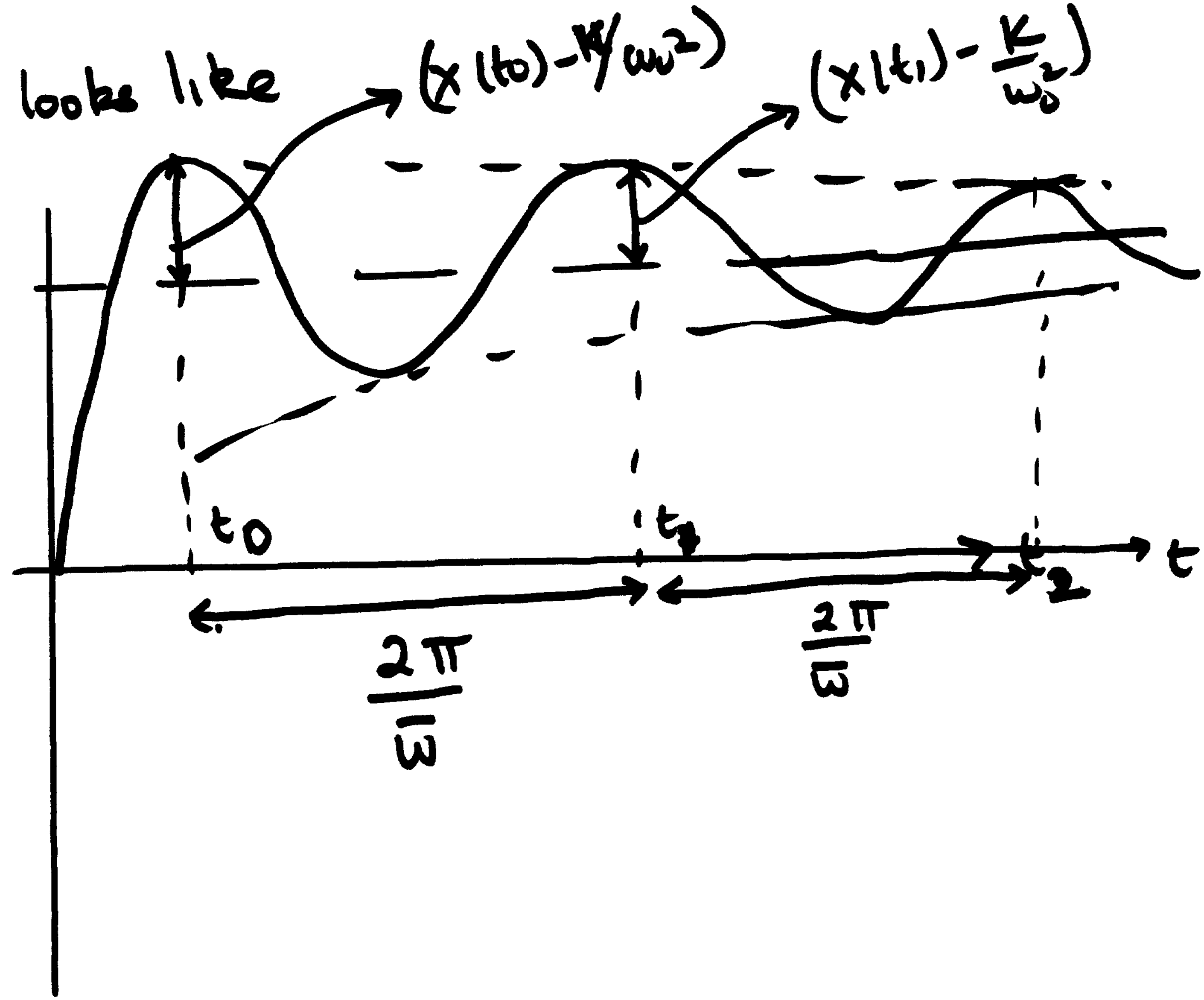


$$= \frac{1}{\omega_0^2} - \frac{e^{-\frac{\delta}{2}t}}{\bar{\omega}^2}\left\{\cos(\bar{\omega}t-\theta)\right\}.$$

For a step of magnitude K we have

$$x(t) = \frac{K}{\omega_0^2} - \frac{Ke^{-\delta/2\, t}}{\bar{\omega}^2}\cdot\cos(\bar{\omega}t-\theta).$$

The plot looks like



Thus, the steady-state value gives us $\frac{K}{\omega_0^2}$

Thus $\frac{K}{\omega_0^2}$ is determined. The period of the oscillations is $(2\pi/\bar{\omega})$ thus $\bar{\omega}$ is determined.

Let

$$t_1 = t_0 + \frac{2\pi}{\bar{\omega}}$$

$$t_2 = t_0 + 2\cdot\frac{2\pi}{\bar{\omega}} = \frac{4\pi}{\bar{\omega}}$$

$$\vdots$$

$$t_n = t_0 + \frac{2n\pi}{\bar{\omega}}.$$

Then

$$x(t_1) = \frac{k}{w_0^2} - \frac{ke^{-\frac{\delta}{2}(t_1)}}{\bar{\omega}^2} \cos(\bar{\omega} t_1 - \theta)$$

$$= \frac{k}{w_0^2} - \frac{ke^{-\frac{\delta}{2}(t_0 + \frac{2\pi}{\bar{\omega}})}}{\bar{\omega}^2} \cos(\bar{\omega} t_1 - \theta)$$

$$= \frac{K}{\omega_0^2} - \frac{K e^{-\frac{\delta}{2}\left(t_0 + \frac{2\pi}{\bar{\omega}}\right)}}{\bar{\omega}^2} \cos\left(\bar{\omega}\left(\frac{2\pi}{\bar{\omega}} + t_0\right) - \theta\right)$$

$$\therefore\ x(t_1) - K/\omega_0^2 = \frac{-K e^{-\delta/2\left(t_0 + 2\pi/\bar{\omega}\right)}}{\bar{\omega}^2} \cos\left(2\pi + \bar{\omega} t_0 - \theta\right)$$

$$= \frac{-e^{-\delta/2\left(t_0 + \frac{2\pi}{\bar{\omega}}\right)} \cdot K}{\bar{\omega}^2} \left[\cos\left(\bar{\omega} t_0 - \theta\right)\right].$$

$$= \frac{-e^{-\frac{\delta}{2} t_0} \cdot e^{-\frac{\delta}{2} \cdot \frac{2\pi}{\bar{\omega}}} \cdot K.}{\bar{\omega}^2} \cos\left(\bar{\omega} t_0 - \theta\right) \quad --- (1)$$

$$x(t_0) - K/\omega_0^2 = \frac{-e^{-\frac{\delta}{2} t_0} \cdot K.}{\bar{\omega}^2} \cos\left(\bar{\omega} t_0 - \theta\right) \quad --- (2)$$

$\therefore$ dividing (1) by (2) we have

$$\frac{x(t_0) - 1/\omega_0^2}{x(t_1) - 1/\omega_0^2} = \frac{1}{e^{-\frac{\delta}{2}\frac{2\pi}{\omega}}} = e^{+\frac{\delta}{2}\frac{2\pi}{\omega}}$$

In general

$$\frac{x(t_0) - 1/\omega_0^2}{x(t_n) - 1/\omega_0^2)} = e^{+\frac{\delta}{2}\frac{2n\pi}{\omega}}$$

$$\Rightarrow \quad \lg_e\left[\frac{x(t_0) - 1/\omega_0^2}{x(t_n) - 1/\omega_0^2}\right] = \frac{\delta}{2} \cdot \frac{2n\pi}{\omega}$$

$\bar{\omega}$ is known.

Thus,

$$\boxed{\delta = \frac{1}{n\pi}\bar{\omega} \; \lg_e \frac{(x(t_0) - 1/\omega_0^2)}{(x(t_n) - 1/\omega_0^2)}}$$

$\therefore$ δ is known.

now,

$$\omega_0^2 = \bar{\omega}^2 + \frac{\delta^2}{4}$$

Thus ω_0^2 is known.

$\frac{K}{\omega_0^2}$ is known. Thus, K is known.