LECTURE 18

Stability:

Let x^* be an equilibrium point of

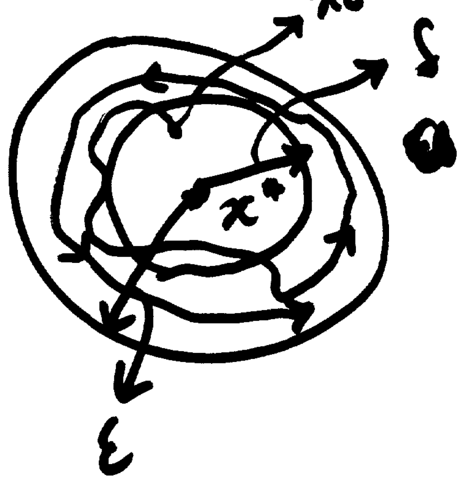
$$\frac{dx}{dt} = f(x,u)$$

[that is $f(x^*,0) = 0$].

Then x^* is a stable equilibrium point if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that $dist(x_0, x^*) < \delta$ implies that $dist(x(t), x^*) < \varepsilon$ for all time, where $x(t)$ is the solution to

$$\frac{dx}{dt} = f(x,0); \; x(0) = x_0.$$

Stability

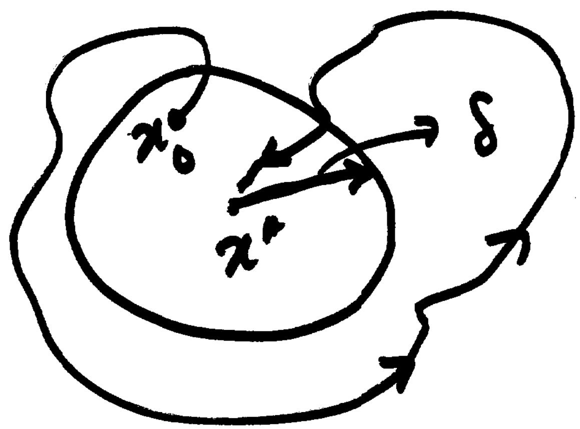


ε is given by somebody else; If δ can be found such that system starts at x_0 inside δ, then state forever remain in ε.

Any equilibrium point that is not stable is an unstable equilibrium point.

Asymptotic stability:

Consider $\dot{x} = f(x,u)$ and let x^* be such that $f(x^*,0) = 0$. Then x^* is an asymptotically stable equilibrium point if it is stable and there exists a $\delta > 0$ such that $dist(x(t), x^*) \longrightarrow 0$ if $dist(x(0), x^*) < \delta$.



Asymptotic stability: eventually the state goes to zero.

Normally one cannot solve for

$$\dot{x} = f(x, 0) \quad ; \quad x(0) = x_0.$$

as f is a nonlinear system.

How do we determine stability then?
we cannot use the definitions presented.

Theorem: (on stability)

Consider the nonlinear differential equation

$$\frac{dx}{dt} = f(x,u) \; \longrightarrow \; x(0) = x_0 \qquad \text{-(NL)}$$

with x^* as an equilibrium point.

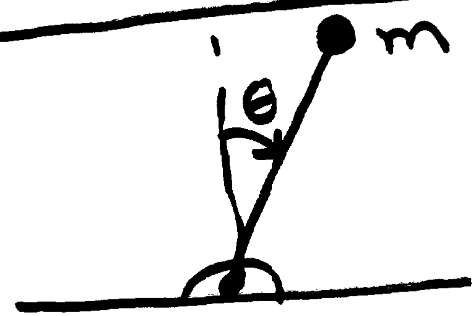
Consider the linearization of (NL) about $(x^*, 0) = (\bar{x}, \bar{u})$. given by

$$\frac{dx}{dt} = \left.\frac{\partial f}{\partial x}\right|_{\substack{x = x^* \\ u = 0}} (x - x^*) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x = x^* \\ u = 0}} (u)$$

$$=: A\,(x - x^*) + B\,u.$$

If A has all eigenvalues in the strict left half complex plane then the equilibrium point x^* of (NL) is asymptotically stable. If any eigenvalue in the strict right half plane then x^* is an unstable equilibrium point.

Example: Inverted Pendulum



The dynamics are given by

$$\ddot{\theta} = \frac{g}{\ell}\sin\theta + \tau/m\ell^2 .$$

Converting to state-space with $x_1 \equiv \theta$, $x_2 \equiv \dot{\theta}$

we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ +\frac{g}{\ell}\sin x_1 + \frac{u}{ml^2} \end{bmatrix} =: f(x,u).$$

Equilibrium points:

$$\begin{bmatrix} x_2 \\ +\frac{g}{\ell}\sin x_1 + \frac{u}{ml^2} \end{bmatrix}_{u=0} = 0 \Rightarrow x_2 = 0 \ ; \ \sin x_1 = 0 \Rightarrow x_1 = n\pi.$$

The distinct equilibrium points are

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ; \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}.$$

Linearization about Equilibrium point:

Case 1: $x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix};$

$$\dot{x} = \left.\frac{\partial f}{\partial x}\right|_{\substack{x = x^* \\ u = 0}} (x - x^*) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x = x^* \\ u = 0}} (u - {}^*)$$

$$= \left.\begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & 0 \end{bmatrix}\right|_{\substack{x_1^* = 0 \\ x_2^* = 0}} (x - x^*) + \begin{bmatrix} 0 \\ \frac{1}{Ml^2} \end{bmatrix} u.$$

$$= \begin{bmatrix} 0 & 1 \\ +g/l & 0 \end{bmatrix} (x - x^*) + \begin{bmatrix} 0 \\ 1/Ml^2 \end{bmatrix} u.$$

The eigen values of

$$A = \begin{bmatrix} 0 & 1 \\ +g/\ell & 0 \end{bmatrix}$$ are determined by

setting $\det(\lambda I - A) = 0$.

$$\therefore \det\left[\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ +g/\ell & 0 \end{bmatrix}\right] = 0$$

$$\Rightarrow \det\begin{bmatrix} \lambda & -1 \\ -g/\ell & \lambda \end{bmatrix} = 0$$

$$\Rightarrow \lambda^2 - g/\ell = 0 \quad \Rightarrow \lambda^2 = +g/\ell.$$

$$\Rightarrow \lambda = \pm\sqrt{g/\ell}. \quad \therefore$$

There there is an eigenvalue $+\sqrt{g/l}$ in the Right half complex plane

$\Rightarrow x^* \cancel{\text{is}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not stable.

<u>Case 2</u>: $\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$.

The linearization is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos x_1 & 0 \end{bmatrix}_{\substack{x_1^* = \pi \\ x_2^* = 0}} (x - x^*) + \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix} u.$$

$$= \begin{bmatrix} 0 & 1 \\ -g/l & 0 \end{bmatrix} (x - x^*) + \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix} u.$$

$$=: A(x - x^*) + Bu$$

The eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -g/l & 0 \end{bmatrix}$ are given by

$$\lambda^2 + g/l = 0 \Rightarrow \lambda^2 = -g/l \; \cancel{\Rightarrow \sqrt{g/l}}$$

$$\therefore \quad \lambda = \pm\left(\sqrt{g/l}\right)\sqrt{-1}$$

$$= \pm j\sqrt{g/l}$$

Thus, eigenvalues on the $j\omega$ axis. Cannot say about ~~stability or~~ asymptotic stability.