STATE SPACE VS. INPUT OUTPUT

State-Space

$$\left.\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx + Du. \end{aligned}\right\} \text{Initial condition } x(0) = x_0.$$

Input-Output:

Transfer function $T(s)$ between ~~the~~ input u and output y is given by

$$T(s) = C(sI - A)^{-1}B + D.$$

Recall that

$$M^{-1} = \frac{\mathrm{Adj}(M)}{\det(M)}.$$

$$\therefore (sI-A)^{-1} = \frac{\mathrm{Adj}(sI-A)}{\det(sI-A)}.$$

$$\therefore T(s) = \frac{C\,\mathrm{Adj}(sI-A)B}{\det(sI-A)} + D$$

$$= \frac{C\,\mathrm{Adj}(sI-A)B + D\det(sI-A)}{\det(sI-A)}.$$

Example:

$$\mathring{x} = Ax + Bu$$
$$y = Cx + Du$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$C = [1 \quad 1 \quad 1]; \qquad D = 0$$

We will first calculate the Adjoint of matrix A.

$$M = (SI - A) = \begin{bmatrix} s-1 & 0 & 0 \\ 0 & s & -3 \\ 0 & -2 & s \end{bmatrix}$$

$$m_{11} = s^2 - 6; \quad m_{12} = (-1)^{1+2} \det \begin{bmatrix} 0 & -3 \\ 0 & s \end{bmatrix} = 0$$

$$m_{13} = (-1)^{1+3} \det \begin{bmatrix} 0 & s \\ 0 & -2 \end{bmatrix} = 0; \quad m_{21} = (-1)^{2+1} \det \begin{bmatrix} 0 & 0 \\ -2 & s \end{bmatrix} = 0$$

Similarly; $m_{22} = s(s-1)$; $m_{23} = 2(s-1)$; $m_{31} = 0$; $m_{32} = 3(s-1)$

$m_{33} = s(s-1)$.

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$$\therefore \ \mathrm{Adj}\,(SI-A) = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} s^2-6 & 0 & 0 \\ 0 & s(s-1) & 3(s-1) \\ 0 & 2(s-1) & s(s-1) \end{bmatrix}$$

$$\det(SI-A) = (s-1)\left[s^2 - 6 \right] = s^3 - s^2 - 6s + 6.$$

$$\therefore \quad T(s) = \frac{N(s)}{D(s)}$$

$$N(s) = C\,\mathrm{Adj}(sI-A)B + D\det(sI-A).$$

$$= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} s^2-6 & 0 & 0 \\ 0 & s(s-1) & 3(s-1) \\ 0 & 2(s-1) & s(s-1) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$+ 0(s^3 - s^2 - 6s + 6)$$

$$= 3s^2 + 3s + 1$$

$$D(s) = \det(sI-A).$$

$$\therefore \quad T(s) = \frac{3s^2+3s+1}{s^3-s^2-6s+6}.$$

For a single input single output system with $A \in R^{n\times n}$, $B \in R^{n\times 1}$, $C \in R^{1\times n}$; $\det(sI-A)$ ~~has~~ ~~order~~ is a polynomial of degree n.

$Adj(sI-A)$ will be a polynomial of degree $(n-1)$ or smaller.

$\therefore$ $N(s) = C\,Adj(sI-A)B + D\det(sI-A)$ has degree n or smaller. It will have degree n only if $D \neq 0$. Otherwise $N(s)$ has degree $(n-1)$ or less.

If $\deg N(s) < \deg(D(s))$ then

$T(s)$ is said to strictly proper.

~~Roots of a S~~

Zeros of a System:

The roots of $N(s)$ are called the roots of the system

Poles of a System:

The roots of $D(s)$ which are roots of $\det(sI-A)$ are called the poles of the system.

Suppose, the zeros of the system are $z_1, z_2, \ldots, z_n$ and poles are $p_1, \ldots, p_m$.

Then

$$T(s) = \frac{N(s)}{D(s)} = \frac{(s-z_1)(s-z_2)\cdots(s-z_n)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

It is possible that ~~N(s) has~~ some ~~roots~~ zeros coincide with the poles. Suppose

$$z_i = p_i \; ; \quad i = 1 \ldots J$$

Then
$$T(s) = \frac{(s-z_1)\cdots(s-z_J)}{(s-p_1)\cdots(s-p_J)} \frac{N_o(s)}{D_o(s)} = \frac{N_o(s)}{D_o(s)}.$$

The eigenvalues of the matrix A are the roots of $\det(sI - A) = D(s)$. Thus the poles of the transfer function are included in the eigenvalues of the matrix A.

The poles of a system are equal to the eigenvalues of the A matrix if there are no pole-zero cancellations.