CONTROLLABILITY

A System is said to be controllable if for any value of T, x_0, and x_d there exists a control input $u(t)$; $0 \le t \le T$ to move the system from $x(0) = x_0$ to $x(T) = x_d$.

<u>i.e.</u>

given the linear System.

$$\dot{x} = Ax + Bu$$

$$x(0) = x_0$$

$$x_d = x(T) = e^{AT} x_0 + \int_0^T e^{A(t-\tau)} B u(\tau)\, d\tau.$$

When is a linear system controllable?

FACT: Kalman's Controllability Test - Single input case:

A system of the form $\dot{x} = Ax + Bu$ is controllable (i.e. a controller can be found to transfer the ~~system~~ state of the system from any initial state to any final state).

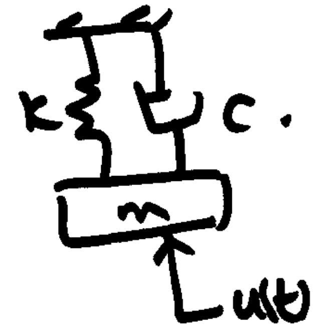
if $\det(C_{on}) \neq 0$ where

$$C_{on} = [B \;\; AB \;\; A^2B \;\cdots\; A^{n-1}B].$$

where $n \equiv$ no. of state variables

C_{on} is called the controllability matrix.

<u>Example</u>: Consider



The state-space description of the above system is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{c}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u$$

$$\therefore A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{c}{M} \end{bmatrix} ; \quad B = \begin{bmatrix} 0 \\ 1/M \end{bmatrix}.$$

note that

n = number of state variables = 2.

$\therefore \ Con = [B \quad AB]$.

$$AB = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{c}{M} \end{bmatrix} \begin{bmatrix} 0 \\ 1/M \end{bmatrix} = \begin{bmatrix} 1/M \\ -\frac{c}{M^2} \end{bmatrix}$$

$$\therefore \ Con = [B \quad AB]$$

$$= \begin{bmatrix} 0 & \frac{1}{M} \\ 1/M & -\frac{c}{M^2} \end{bmatrix}$$

$\det(\text{Con}) = -\frac{1}{M^2} \neq 0.$

$\therefore$ The system is controllable.

This implies from any given initial state and time T we can drive the system using $u(t)$ to reach any desired final state in a time T.

Example:

$$\frac{dx}{dt} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u.$$

$$\frac{dx_1}{dt} = x_1 + u$$

$$\frac{dx_2}{dt} = 2x_2.$$

$n = 2$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $AB = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$Con = [B \quad AB]$;

$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $AB = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$Con = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$; $\det(Con) = 0.$

System is not controllable.

Stabilization by Eigenvalue placement

Given: $\frac{dx}{dt} = Ax + Bu$

$y = Cx + Du.$

Objective: Find the vector K so that with

$u = -Kx + v$, the closed-loop system

$$\frac{dx}{dt} = (A-BK)x + Bv$$

$$y = (C-DK)x + Dv$$

is asymptotically stable. Furthermore place the eigenvalues of $(A-BK)$ in desired locations in the LHP by choosing K.

<u>FACT</u>: If the system is controllable then given any set of desired eigenvalue locations symmetric with respect to real axis, there exists a control law of the form

$$u = -Kx + v$$

Such that

Eigenvalues of $A-BK$ are the desired eigenvalues.

<u>Note</u>: A set of complex numbers $\Lambda = [\lambda_1 \dots \lambda_n]$ is symmetric with respect to the real axis if for λ in our set Λ, its complex conjugate $\bar{\lambda}$ is also in the set.

Pole placement Problem:

Given: a single input system with $\dot{x} = Ax + Bu$

~~where~~ which is controllable, and symmetric set

$\Lambda = [\lambda_1 \cdots \lambda_n]$ of complex numbers (n: number of states).

~~Find~~
Objective: Find K such that

$$\det(sI - (A - BK)) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n).$$

Pole Placement Algorithm

Assumption: $\dot{x} = Ax + Bu$ is controllable, u is a single input.

i.e. B is a $n \times 1$ matrix.

$\Lambda = [\lambda_1, \lambda_2 \cdots \lambda_n]$ is a symmetric set of eigenvalues.

Then let K be obtained by

Algo:

Step 1: Compute $C_{on} = [B \; AB \; A^2B \cdots A^{n-1}B]$.

Step 2: Compute C_{on}^{-1}

Step 3: Compute $\Delta^d(s) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)$

and $\Delta^d(A) = (A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I)$.

Step 4: $K = e_n^T C_{on}^{-1} \Delta^d(A)$ where $e_n^T = [0\;0 \cdots 0\;\underset{\uparrow}{1}]$ n^{th} place.

Example:

Consider:

$$\frac{dx}{dt} = \underbrace{\begin{bmatrix} -1 & -3 \\ 1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B.} u.$$

~~The poles of A ar~~

- The eigenvalues of A are $-1 \pm 3j$.
  (the system is stable).
- We would like the poles to be at $-4 \pm j$.
- n = number of state = 2

- Controllability matrix

$$Con = [B \;\; AB]$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$\det(Con) = 1 \neq 0$.

$\therefore$ system is controllable.

- $\Lambda = [-4+j \;\; -4-j]$;

(Λ is symmetric)

Step 1:

Step1: $Con = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$;

Step2: $Con^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Step3: $\Delta^d(s) = (s-\lambda_1)(s-\lambda_2)$

$= (s+4-j)(s+4+j)$

$= s^2 + 8s + 17$

Therefore

$\Delta^d(A) = A^2 + 8A + 17I$.

$= \begin{bmatrix} 7 & -18 \\ 6 & 7 \end{bmatrix}$.

Step4: $K = e_n^T Con^{-1} \Delta^d(A) = [0 \quad 1] \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 7 & -18 \\ 6 & 7 \end{bmatrix} = [6 \quad 7]$.

Hence

$$u = -Kx + v$$

$$= -[6 \quad 7]x + v.$$

which gives us

$$\frac{dx}{dt} = (A - BK)x + Bv$$

$$= \begin{bmatrix} -7 & 10 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v$$

$$y = \cancel{B}(C - DK)x + Bv.$$

$$= [1 \quad 2)x.$$

- $\underline{\text{Check}}$ $\text{Eig}(A - BK) = [-4+5, -4-5].$