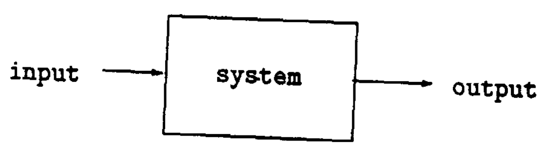
Lecture #2: Modeling

The first step in control design is the development of a mathematical model for the process or system under consideration. In the modeling of mechanical systems, we assume a cause and effect relationship described by the simple input/output diagram below. An input is applied to a mechanical system, and the system processes it to produce an output. In general, a mechanical system has the three basic components listed below.

1. Inputs: These represent the variables under the designer's disposal. The designer produces these signals directly and applies them to the system under consideration. For example, the voltage source to a motor and the external torque input to a robotic manipulator both represent inputs. Systems may have single or multiple inputs.

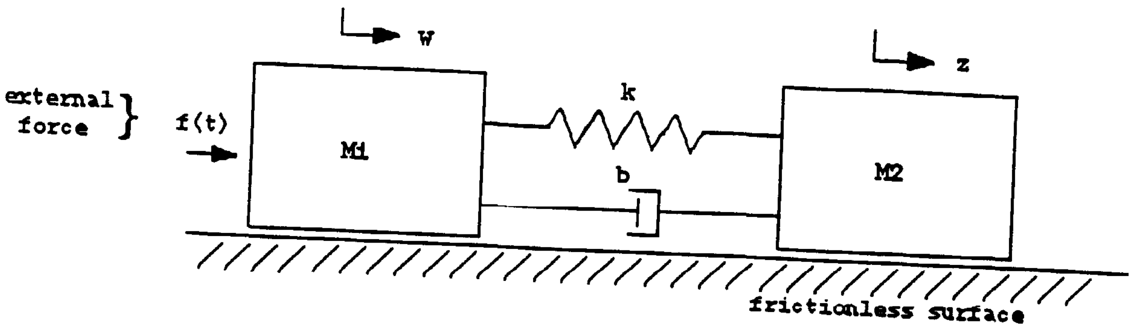
2. Outputs: These represent the variables which the designer ultimately wants to control and that can be measured by the designer. For example, in a flight control application an output may be the altitude of the aircraft, and in automobile cruise control the output is the speed of the vehicle.

3. System or Plant: This represents the dynamics of a physical process which relate the input and output signals. For example, in automobile cruise control, the output is the vehicle speed, the input is the supply of gasoline, and the system itself is the automobile. Similarly, an air conditioning system regulates the temperature in a room. The output from the system is the air temperature in the room. The input is cool air added to the room. The system itself is the room full of air with its air flow characteristics.



In this lecture we will show by examples how mathematical models for simple mechanical systems can be developed. Notice that in each example four steps are taken. First, a diagram of all system components and externally applied inputs is drawn. From this diagram, the inputs and outputs are identified. Then a free-body diagram is made of the system components in which all internal signals are shown. Such signals typically include forces or moments. Finally, the differential equations governing the system dynamics are obtained. These equations form the mathematical model of the mechanical system.

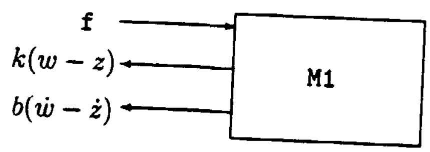
Example 2.0.1 *The modeling of a system containing two masses, a spring, and a dashpot.*



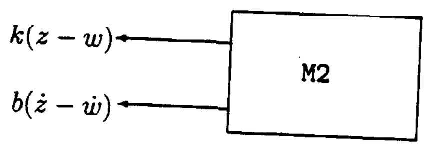
We first identify the input and output.

input: external force, $f(t)$, output: displacement of the second mass, $z(t)$

Next we find a set of differential equations describing this system:



$$M_1\ddot{w} = f - k(w-z) - b(\dot{w}-\dot{z})$$



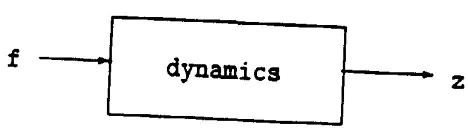
$$M_2\ddot{z} = -k(z-w) - b(\dot{z}-\dot{w})$$

Rearranging these equations we obtain:

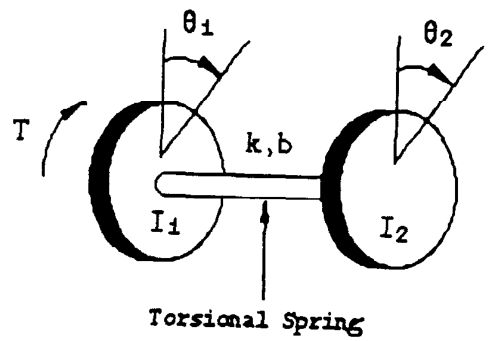
$$M_1\ddot{w} + k(w-z) + b(\dot{w}-\dot{z}) = f$$

$$M_2\ddot{z} + k(z-w) + b(\dot{z}-\dot{w}) = 0.$$

Notice how the system dynamics relate the input and output. Remember that f is the input and z is the output.



Example 2.0.2 *The modeling of a system containing two masses, a torsional spring, and a dashpot.*

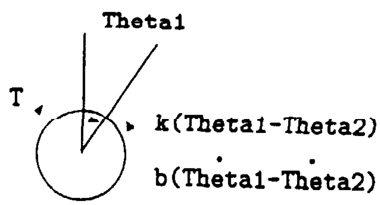


In the picture above, I_1 and I_2 are the moments of inertia of the two disks, k is the spring constant and b is the coefficient of damping of the torsional spring.

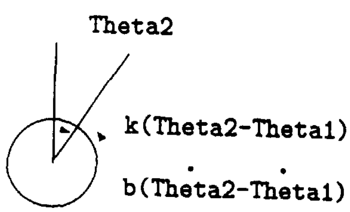
We first identify the input and output.

input: external torque, T, output: angular displacement of second disc, $\dot{\theta}_2$

Next we find a set of differential equations describing this system:



$$I_1\ddot{\theta}_1 = T - k(\theta_1 - \theta_2) - b(\dot{\theta}_1 - \dot{\theta}_2)$$



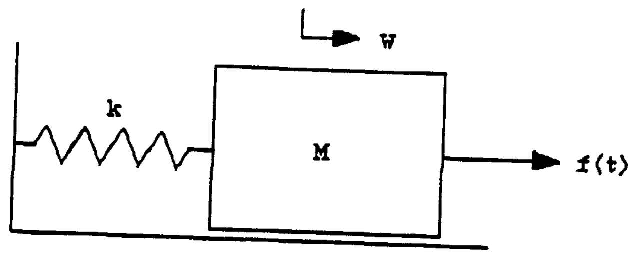
$$I_2\ddot{\theta}_2 = -k(\theta_2 - \theta_1) - b(\dot{\theta}_2 - \dot{\theta}_1)$$

Notice again how the input, $u = T$, and the output, $y = \dot{\theta}_2$, are related by the system dynamics which is described by the differential equations above.

2.1 State Variables

In deriving a mathematical model for a physical system one usually begins with a system of differential equations. It is often convenient to rewrite these equations as a system of first order differential equations. We will call this system of differential equations a *state space representation.* The solution to this system is a vector that depends on time and which contains enough information to completely determine the trajectory of the dynamical system. This vector is referred to as the *state of the system*, and the components of this vector are called the *state variables*. In order to illustrate these concepts consider the following example.

Example 2.1.1 *Finding a state space representation for a second order system.*



The equations of motion for this system are given by the following second order set of differential equations:

$$M\ddot{w} + kw = f \tag{2.1}$$

Suppose we specify the value of the input f as a function of time and give values to the system parameters k and M. For example, we could choose $F(t) = \sin(t), k = 1, andM = 1$. Let's choose the displacement $w(t)$ as our output. You can see that the output is simply a solution to the above differential equation. However, remember that such an equation can have many solutions depending upon the initial value of w and $\dot{w}$. In other words, if our initial time is $t = 0$, then we must specify $w(0)$ and $\dot{w}(0)$ in order to solve for the value of $w(t)$ for $t \geq 0$. Suppose $w(0) = 1$ and $\dot{w}(0) = 3$. These internal system variables, w and $\dot{w}$, are an example of system state variables. Using w and $\dot{w}$ we can rewrite the above equation in terms of a system of first order differential equations as follows.

We first define two new variables $x_1 = w$ and $x_2 = \dot{w}$ and rewrite the second order differential equation as an equivalent first order set of differential equations in our new variables.

$$\begin{aligned} \dot{x}_1 &= \dot{w} = x_2 \\ \dot{x}_2 &= \ddot{w} = \frac{f - kw}{m} = -\frac{k}{m}x_1 + \frac{1}{m}f \end{aligned}$$

So the state space representation of the above system is:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 + \frac{1}{m}f \end{aligned}$$

$$y(t) = w(t)$$

where $x_1(0) = 1$ and $x_2(0) = 3$.

Notice that the state space representation for our system includes the initial conditions for each of the state variables. This information is required in order to solve for the output $w(t) = x_1(t)$ for $t \geq 0$. Notice also that our system state is the vector

$$\begin{pmatrix} w(t) \\ \dot{w}(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}. \tag{2.2}$$