LAGRANGE'S PRINCIPLE:

LECTURE 8:

- Let the instantaneous configuration of a system be completely determined by n- independent variables $q_1, q_2, \ldots q_n$.
- Let KE denote the total Kinetic energy of the system in terms of the variables

$$\left(q_1, q_2, \ldots, q_n, \frac{dq_1}{dt}, \frac{dq_2}{dt}, \ldots, \frac{dq_n}{dt}\right)$$

- Let PE denote the total potential energy of the system in terms of the variable

$$\left(q_1, q_2 \ldots, q_n, \frac{dq_1}{dt}, \frac{dq_2}{dt}, \ldots \frac{dq_n}{dt}\right)$$

- Let Q_j denote the "force" in the direction of the coordinate q_j.

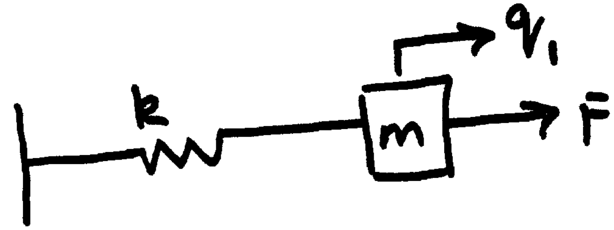
Then,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j; \quad j = 1 \cdots n.$$

where $L = KE - PE$

is the Lagrangian of the system.

Example:



- The coordinate q_1 completely determines the system's ~~in the~~ configuration at any instant.
- The $KE = \frac{1}{2} m \dot{q}_1^2$
- The $PE = \frac{1}{2} k q_1^2$
- $Q_1 = F$.

The Lagrangian L is given by

$$L = T - V$$

$$L(q_1, \dot{q}_1) = \frac{1}{2} m \dot{q}_1^2 - \frac{1}{2} k q_1^2 = \frac{1}{2} m \dot{q}_1^2 - \frac{1}{2} k q_1^2$$

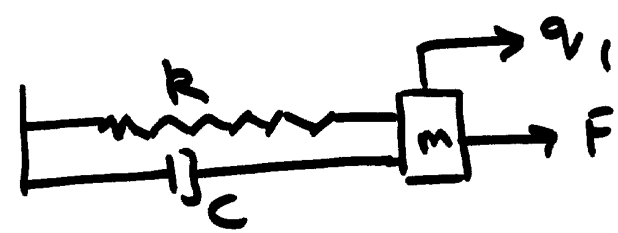
<u>The equation</u>:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = Q_1$$

$$\Rightarrow \frac{d}{dt}\left[m \dot{q}_1\right] + k q_1 = Q_1$$

$$\Rightarrow m \ddot{q}_1 + k q_1 = F.$$

Example:



This example, there is a dissipative force whose magnitude is $c\dot{q}_1$. Thus,

$$Q_1 = F - c\dot{q}_1$$

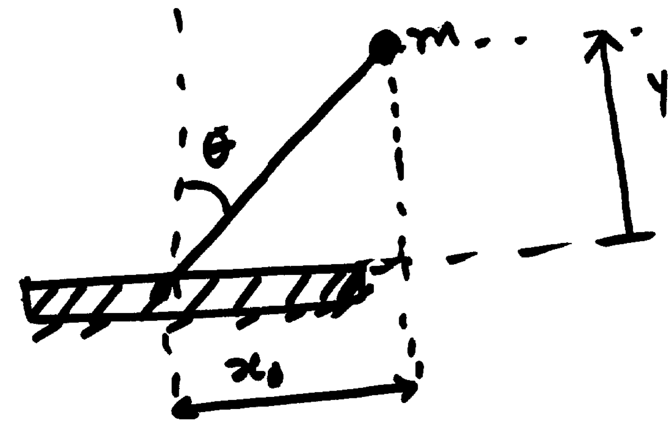
{Note that every force that can be derived by a potential is included in the PE term. Thus, the spring force $F_{sp} = -\frac{\partial V}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{1}{2}kx^2\right)$. }.

With the new force Q_1, the equation of motion is given by

$$m\ddot{q}_1 + kq_1 = F - c\dot{q}_1$$

$$\Rightarrow \quad m\ddot{q}_1 + kq_1 + c\dot{q}_1 = F.$$

Example:



Consider the example where a mass m is suspended at the end of a rigid-rod. The rod is assumed to be massless, and of length l.

- Note that θ determines the system configuration at any instant. Let $q_1 \equiv \theta$.

- The Kinetic-energy of the mass.

$$KE = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

where $x = l \sin\theta$, $y = l\cos\theta$

$$\therefore \quad \dot{x} = l\cos\theta \frac{d\theta}{dt}$$

$$\dot{y} = -l\sin\theta \frac{d\theta}{dt}$$

$$\therefore \quad KE = \frac{1}{2} m\left[(l\cos\theta)^2\left(\frac{d\theta}{dt}\right)^2 + (l\sin\theta)^2\left(\frac{d\theta}{dt}\right)^2\right]$$

$$= \frac{1}{2} m\left[l^2\cos^2\theta\, \dot{\theta}^2 + l^2\sin^2\theta\, \dot{\theta}^2\right]$$

$$= \frac{1}{2} m\, l^2 \dot{\theta}^2 (\cos^2\theta + \sin^2\theta)$$

$$= \frac{1}{2} m l^2 \dot{\theta}^2.$$

$$\therefore \quad KE = \frac{1}{2} m l^2 \dot{q}_1^2$$

The Potential Energy is given by

$$PE = mgy = mg\, l\cos\theta = mg\, l\cos q_1.$$

- The Lagrangian is:

$$L = KE - PE$$

$$= \frac{1}{2} m l^2 \dot{q}_1^2 - mgl\cos q_1$$

- The equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = 0$$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2} m l^2 \, 2\dot{q}_1\right] + mgl\sin q_1 = 0.$$

$$\Rightarrow m l^2 \ddot{q}_1 + mgl\sin q_1 = 0$$

$$\Rightarrow \ddot{q}_1 = \frac{g}{l}\sin q_1.$$