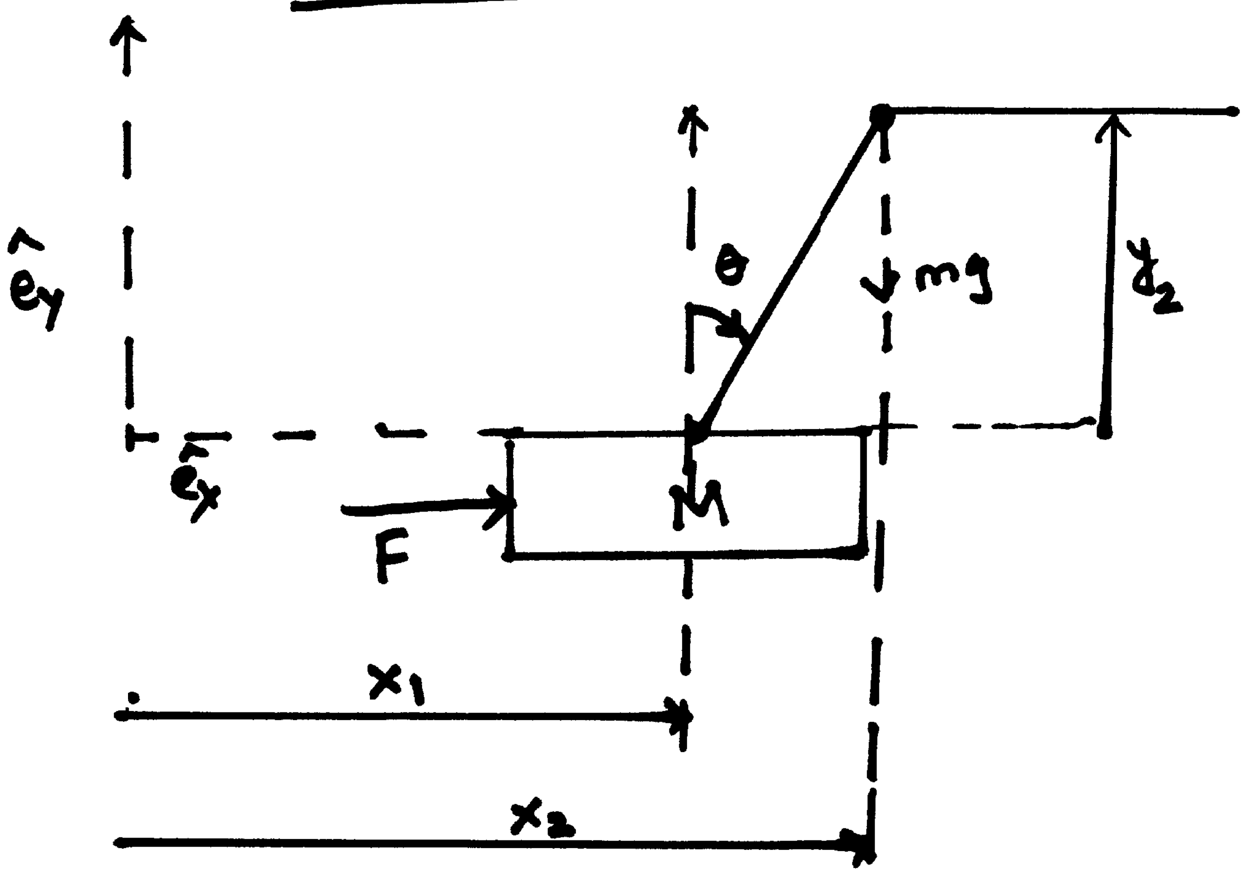
PENDULUM ON A CART



- Independent variables which completely determine the configuration are x_1 and θ

KINETIC ENERGY

$$KE = \frac{1}{2} M \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 + \frac{1}{2} m \dot{y}_2^2 ;$$

Note that

$$x_2 = x_1 + l \sin\theta \quad ; \quad y_2 = l \cos\theta$$

$$\therefore \quad \dot{x}_2 = \dot{x}_1 + l \cos\theta \, \dot{\theta}$$

$$\Rightarrow \quad \dot{x}_2^2 = \dot{x}_1^2 + l^2 \cos^2\theta \, \dot{\theta}^2 + 2 \dot{x}_1 l \cos\theta \, \dot{\theta}$$

$$\therefore \quad KE = \frac{1}{2} M \dot{x}_1^2 + \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m l^2 \cos^2\theta \, \dot{\theta}^2 + m \dot{x}_1 l \cos\theta \, \dot{\theta}$$
$$+ \frac{1}{2} m l^2 \sin^2\theta \, \dot{\theta}^2$$

$$KE = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}ml^2\dot{\theta}^2 + m\dot{x}_1 l\cos\theta\,\dot{\theta}$$

Potential Energy

$$PE = mgl\cos\theta.$$

Lagrangian

$$L = KE - PE$$

$$= \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}ml^2\dot{\theta}^2 + m\dot{x}_1\dot{\theta}\,l\cos\theta.$$

$$- mgl\cos\theta.$$

$$\frac{\partial L}{\partial \dot\theta} = ml^2\dot\theta + m\dot x_1 (l\cos\theta).$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = ml^2\ddot\theta + m\ddot x_1 (l\cos\theta) - m\dot x_1 l\sin\theta\,\dot\theta$$

$$\frac{\partial L}{\partial \theta} = -m\dot x_1 \dot\theta\, l\sin\theta + mgl\sin\theta.$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = ml^2\ddot\theta + m\ddot x_1 l\cos\theta - m\dot x_1 l\cancel{\sin\theta}\,\dot\theta + m\dot x_1\dot\theta\cancel{l\sin\theta} - mgl\sin\theta$$

$$= ml^2\ddot\theta + m\ddot x_1 l\cos\theta - mgl\sin\theta = 0.$$

$$ml^2\ddot\theta = mgl\sin\theta - m\ddot x_1 l\cos\theta.$$

$$\therefore\ \Rightarrow\quad \ddot\theta = \frac{g}{l}\sin\theta - \frac{\cancel{m}\ddot x_1}{l}\cos\theta.$$

$$\frac{\partial L}{\partial \dot{x}_1} = (M+m)\dot{x}_1 + m\dot{\theta}\, l\cos\theta .$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = (M+m)\ddot{x}_1 + m\ddot{\theta}\, l\cos\theta - m\dot{\theta}^2 l\sin\theta .$$

$$\frac{\partial L}{\partial x_1} = 0 .$$

$$\therefore \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = F$$

$$\Rightarrow \quad (M+m)\ddot{x}_1 \mp m\ddot{\theta}\, l\cos\theta - m\dot{\theta}^2 l\sin\theta = F .$$