EE 508
Lecture 10

Canonical Approximating Functions
Approximations

- Magnitude Squared Approximating Functions – $H_A(\omega^2)$
- Inverse Transform - $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation

Least Squares Approximations

Pade Approximations

- Other Analytical Optimizations
- Numerical Optimization
- Canonical Approximations
  - Butterworth
  - Chebyschev
  - Elliptic
  - Bessel
  - Thompson
Collocation

• Fitting an approximating function to a set of data or points (collocation points)
  – Closed-form matrix solution for fitting to a rational fraction in $\omega^2$
  – Can be useful when somewhat nonstandard approximations are required
  – Quite sensitive to collocation points
  – Although fit will be perfect at collocation points, significant deviation can occur close to collocation points
  – Inverse mapping to $T_A(s)$ may not exist
Least Squares Approximation

- Minimizes the function
  \[ C = \sum_{i=1}^{k} \left| T(\omega_k) - T_A(\omega_k) \right|^2 \]
  - Less sensitive to individual data points than collocation
  - Closed-form analytical expression does not exist for rational fraction approximations
  - Can find optimal zeros or with some modifications optimal poles if alternate roots are fixed
  - Cost function may not be a natural indicator of filter performance
  - Inverse mapping to TA(s) may not exist
Pade Approximations

- Approximation is already in s-domain
- With reflection of roots if necessary, provides minimum-phase approximation
- Useful for order reduction
- Particularly attractive for reducing good high-order all-pole or all-zero approximations to a more manageable order

Review from last time
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Other Analytical Approximations

• Numerous analytical strategies have been proposed over the years for realizing a filter
• Some focus on other characteristics (phase, time-domain response, group delay)
• Almost all based upon real function approximations
• Remember – inverse mapping must exist if a useful function $T(s)$ is to be obtained
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Numerical Optimization

• Optimization algorithms can be used to obtain approximations in either the s-domain or the real domain
• The optimization problem is often has a large number of degrees of freedom \((m+n+1)\)

\[
T(s) = \frac{\sum_{k=0}^{m} a_k s^k}{1 + \sum_{k=0}^{n} b_k s^k}
\]

• Need a good cost function to obtain good approximation
• Can work on either coefficient domain or root domain or other domains
• Rational fraction approximations inherently vulnerable to local minimums
• Can get very good results
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Canonical Approximations
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All special cases of analytical approximations
Approximations

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Butterworth Approximations

• Analytical Formulation

• Analytical formulation:
  – All pole approximation
  – Magnitude response is maximally flat at \( \omega = 0 \)
  – Goes to 0 at \( \omega = \infty \)
  – Assumes value \( \frac{1}{\sqrt{1+\varepsilon^2}} \) at \( \omega = 1 \)
  – Assumes value of 1 at \( \omega = 0 \)
  – Characterized by \{n, \varepsilon\}

• Emphasis almost entirely on performance at single frequency

Butterworth Approximations

- Analytical formulation:
  - Magnitude response is maximally flat at $\omega=0$
  - Goes to $0$ at $\omega=\infty$
  - Assumes value $\frac{1}{\sqrt{1+\varepsilon^2}}$ at $\omega=1$
  - Assumes value of $1$ at $\omega=0$
Stephen Butterworth (1885-1958) was a British physicist who invented the Butterworth filter[1], a class of electrical circuits that are used to filter electrical signals.

Stephen Butterworth was born on 11 August 1885 in Rochdale, England (a town located about 10 miles north of the city of Manchester). He was the son of Alexander Butterworth, a postman, and Elizabeth (maiden name unknown).[2] He was the second of four children.[3] In 1904, he entered the University of Manchester, from which he received, in 1907, both a Bachelor of Science degree in physics (first class) and a teacher's certificate (first class). In 1908 he received a Master of Science degree in physics.[4] For the next 11 years he was a physics lecturer at the Manchester Municipal College of Technology. He subsequently worked for several years at the National Physical Laboratory, where he did theoretical and experimental work for the determination of standards of electrical inductance. In 1921 he joined the Admiralty's Research Laboratory. Unfortunately, the classified nature of his work prohibited the publication of much of his research there. Nevertheless, it is known that he worked in a wide range of fields; e.g., he determined the electromagnetic field around submarine cables carrying a.c. current,[5] and he investigated underwater explosions and the stability of torpedos. In 1939, he was a "Principal Scientific Officer" at the Admiralty Research Laboratory in the Admiralty's Scientific Research and Experiment Department.[6] During World War II, he investigated both magnetic mines and the degaussing of ships (as a means of protecting them from magnetic mines).

He was a first-rate applied mathematician. He often solved problems that others had regarded as insoluble. For his successes, he employed judicious approximations, penetrating physical insight, ingenious experiments, and skillful use of models. He was a quiet and unassuming man. Nevertheless, his knowledge and advice were widely sought and readily offered. He was respected by his colleagues and revered by his subordinates.

In 1942 he was awarded the Order of the British Empire.[7] In 1945 he retired from the Admiralty Research Laboratory. He died on 28 October 1958 at his home in Cowes on the Isle of Wight, England.[8][9]
Butterworth Approximation

\[ H(\omega^2) = \frac{a_0}{\omega^{2n} + b_{n-1}\omega^{2n-2} + \ldots + b_1\omega^2 + b_0} \]

\[ H(1) = \frac{1}{1 + \varepsilon^2} \quad \text{H}(0) = 1 \quad \rightarrow \quad a_0 = b_0 \]

Let \( x = \omega^2 \)

\[ H(x) = \frac{a_0}{x^n + b_{n-1}x^{n-1} + \ldots + b_1x + b_0} \]

\[ \frac{\partial H}{\partial x} = -a_0 \frac{nx^{n-1} + b_{n-1}(n-1)x^{n-2} + \ldots + b_1}{\left(x^n + b_{n-1}x^{n-1} + \ldots + b_1x + b_0\right)^2} \]

\[ \left. \frac{\partial H}{\partial x} \right|_{x=0} = -a_0 \frac{nx^{n-1} + b_{n-1}(n-1)x^{n-2} + \ldots + b_1}{\left(x^n + b_{n-1}x^{n-1} + \ldots + b_1x + b_0\right)^2} \bigg|_{x=0} = 0 \quad \rightarrow \quad b_1 = 0 \]
Butterworth Approximation

\[ H(x) = \frac{a_0}{x^n + b_{n-1}x^{n-1} + \ldots + b_1x + b_0} \]

\[
\frac{\partial^2 H}{\partial x^2} = -a_0 \left( \frac{x^n + b_{n-1}x^{n-1} + \ldots + b_1x + b_0}{x^n + b_{n-1}x^{n-1} + \ldots + b_1x + b_0} \right)^2 \left( n(n-1)x^{n-2} + (b_{n-1}(n-1)(n-2)x^{n-2}) + \ldots + 6b_3x + 2b_2 \right) - \left( nx^{n-1} + b_{n-1}(n-1)x^{n-2} + \ldots + b_1 \right)^2 2 \left( x^n + b_{n-1}x^{n-1} + \ldots + b_1x + b_0 \right) \]

\[
\left( x^n + b_{n-1}x^{n-1} + \ldots + b_1x + b_0 \right)^4
\]

\[
\frac{\partial^2 H}{\partial x^2} \bigg|_{x=0} = 0
\]

\[
2b_0^2b_2 - 2b_0b_1^2 = 0
\]

\[
b_2 = 0
\]

Continuing this process obtain \( b_3 = 0, b_4 = 0, \ldots, b_{n-1} = 0 \)
Butterworth Approximation

\[ H(x) = \frac{b_0}{x^n + b_0} \]
\[ H(1) = \frac{1}{1 + \varepsilon^2} \]

\[ b_0 = \frac{1}{\varepsilon^2} \]

\[ H(\omega) = \frac{1}{1 + \varepsilon^2 \omega^{2n}} \]

Roots of \( H(\omega) \) are poles and are at

\[ \omega = \varepsilon^{1/n} (-1)^{1/(2n)} \]

The 2n roots of -1 are uniformly spaced on a circle of radius 1
Butterworth Approximation

\[ H(\omega) = \frac{1}{1 + \varepsilon^2 \omega^{2n}} \]

Roots of \( H(\omega) \) are poles and are at

\[ \omega = \varepsilon^{1/n} (-1)^{1/(2n)} \]

Poles of \( H(\omega) \) are a scaled version of the roots of -1

Roots of -1 are uniformly spaced around a unit circle with symmetry around real axis
Butterworth Approximation

Roots of -1 are uniformly spaced around a unit circle with symmetry around real axis.

\[
\omega_k = -\cos \left( \left[ 1 + 2k \right] \frac{\pi}{2n} \right) \pm j \sin \left( \left[ 1 + 2k \right] \frac{\pi}{2n} \right)
\]

\[k = 0, 1, \ldots, n-1\]
Butterworth Approximation

Roots of $T_{BW}(s)$

Take roots of $H(\omega)$, rotate by $90^\circ$ (i.e. multiply by $j$), keep those in LHP

$$p_k = j\varepsilon^{1/n} \left[ -\cos \left(\frac{1 + 2k}{2n} \pi \right) \pm j \sin \left(\frac{1 + 2k}{2n} \pi \right) \right]$$

(Actually denotes $2$ poles for each index)

for $n$ even

$$p_{k+1} = \varepsilon^{1/n} \left[ -\sin \left(\frac{1 + 2k}{2n} \pi \right) \pm j \cos \left(\frac{1 + 2k}{2n} \pi \right) \right] \quad k=0, 1, \ldots, \frac{n}{2} - 1$$

(k=0 for poles closest to Im axis)

for $n$ odd

$$p_n = \varepsilon^{1/n} [-1 + j0] \quad p_k = \varepsilon^{1/n} \left[ -\sin \left(\frac{1 + 2k}{2n} \pi \right) \pm j \cos \left(\frac{1 + 2k}{2n} \pi \right) \right] \quad k=0, \ldots, \frac{n-3}{2}$$
Butterworth Approximation

Poles of $T_{BW}(s)$

for $n$ even

$$p_{k+1} = e^{1/n} \left[ -\sin \left( \frac{1 + 2k}{2n} \right) \pm j \cos \left( \frac{1 + 2k}{2n} \right) \right]$$

$k=0,1, \ldots, \frac{n}{2} - 1$

for $n$ odd

$$p_{n} = e^{1/n} \left[ -1 + j0 \right]$$

$$p_{k} = e^{1/n} \left[ -\sin \left( \frac{1 + 2k}{2n} \right) \pm j \cos \left( \frac{1 + 2k}{2n} \right) \right]$$

$k=0, \ldots, \frac{n-3}{2}$
Butterworth Approximation

What is the Q of the highest Q pole for the BW approximation?

Highest Q pole corresponds to index $k=0$. Consider the Quadrant 2 high-Q pole

$$p_0 = e^{1/n} \left[ -\sin\left(\frac{\pi}{2n}\right) + j\cos\left(\frac{\pi}{2n}\right) \right] = \alpha + j\beta$$

But recall

$$s^2 + s\left(\frac{\omega_o}{Q}\right) + \omega_o^2 = s^2 + s(-2\alpha) + \left(\alpha^2 + \beta^2\right)$$

thus

$$Q = \frac{\sqrt{\alpha^2 + \beta^2}}{-2\alpha}$$
Butterworth Approximation

What order can be used if goal is to keep the highest Q BW pole less than 10?

\[ Q_{MAX} = \frac{1}{2 \sin \left( \frac{\pi}{2n} \right)} \]

\[ 10 = \frac{1}{2 \sin \left( \frac{\pi}{2n} \right)} \]

Solving for n, obtain \( n=31 \)

What order can be used if goal is to keep the highest Q BW pole less than 2?

\[ 2 = \frac{1}{2 \sin \left( \frac{\pi}{2n} \right)} \]

Solving for n, obtain \( n=6 \)

Observe the pole Q of the BW approximation is quite low, even for high order BW approximations!
Butterworth Approximation

What is the Q of the highest Q pole for the BW approximation?

\[
p_0 = \varepsilon^{1/n} \left[ -\sin\left(\frac{\pi}{2n}\right) + j\cos\left(\frac{\pi}{2n}\right) \right] = \alpha + j\beta
\]

\[
Q_{MAX} = \frac{\sqrt{\alpha^2 + \beta^2}}{-2\alpha}
\]

\[
Q_{MAX} = \frac{\varepsilon^{1/n}}{2\varepsilon^{1/n} \sin\left(\frac{\pi}{2n}\right)} = \frac{1}{2\sin\left(\frac{\pi}{2n}\right)}
\]
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Pafnuty Lvovich Chebyshev

**Born** May 16, 1821
**Died** December 8, 1894
**Nationality** Russian
**Fields** Mathematician
Review from last time

• Chebyschev approximation has following properties
  – Equal ripple in the pass band
  – Much higher maximum pole Q than for BW
  – Much steeper transition between pass-band and stop-band than for BW
  – Poles lie on ellipse