EE 508
Lecture 10

The Approximation Problem

Classical Approximations
– the Chebyschev and Elliptic Approximations
Butterworth Approximations

- Analytical formulation:
  - All pole approximation
  - Magnitude response is maximally flat at $\omega=0$
  - Goes to 0 at $\omega=\infty$
  - Assumes value $\sqrt{\frac{1}{1+\varepsilon^2}}$ at $\omega=1$
  - Assumes value of 1 at $\omega=0$
  - Characterized by \{n,\varepsilon\}

- Emphasis almost entirely on performance at single frequency

Butterworth Approximation

Poles of $T_{BW}(s)$

- For $n$ even:
  \[
  p_{k+1} = e^{1/n} \left[ -\sin \left( 1 + 2k \right) \frac{\pi}{2n} \right] \pm j \cos \left( 1 + 2k \right) \frac{\pi}{2n} \]
  \[k = 0, 1, \ldots, \frac{n}{2} - 1 \]

- For $n$ odd:
  \[
  p_n = e^{1/n} \left[ -1 + j0 \right]
  
  p_k = e^{1/n} \left[ -\sin \left( 1 + 2k \right) \frac{\pi}{2n} \right] \pm j \cos \left( 1 + 2k \right) \frac{\pi}{2n} \]
  \[k = 0, \ldots, \frac{n-3}{2} \]
Review from Last Time

Butterworth Approximation

What is the Q of the highest Q pole for the BW approximation?

\[ p_0 = \varepsilon^{1/n} \left[ -\sin \left( \frac{\pi}{2n} \right) + j \cos \left( \frac{\pi}{2n} \right) \right] = \alpha + j\beta \]

\[ Q_{MAX} = \frac{\sqrt{\alpha^2 + \beta^2}}{-2\alpha} \]

\[ Q_{MAX} = \frac{\varepsilon^{1/n} \sqrt{\sin^2 \left( \frac{\pi}{2n} \right) + \cos^2 \left( \frac{\pi}{2n} \right)}}{2\varepsilon^{1/n} \sin \left( \frac{\pi}{2n} \right)} = \frac{1}{2\sin \left( \frac{\pi}{2n} \right)} \]

\[ Q_{MAX} = \frac{1}{2\sin \left( \frac{\pi}{2n} \right)} \]
Butterworth Approximation

Order needs to be rather high to get steep transition

Figure from Passive and Active Network Analysis and Synthesis, Budak
Phase is quite linear in passband (benefit unrelated to design requirements)
Butterworth Approximation

Summary

- Widely Used Analytical Approximation
- Characterized by \( \epsilon, n \)
- Maximally flat at \( \omega = 0 \)
- Almost all emphasis placed on characteristics at single frequency (\( \omega = 0 \))
- Transition not very steep (requires large order for steep transition)
- Pole Q is quite low
- Pass-band phase is quite linear (no emphasis was placed on phase!)
- Poles lie on a circle
- Simple closed-form analytical expressions for poles and \( |T(j\omega)| \)
Approximations

- Magnitude Squared Approximating Functions – $H_A(\omega^2)$
- Inverse Transform - $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation
- Least Squares Approximations
- Pade Approximations
- Other Analytical Optimizations
- Numerical Optimization
- Canonical Approximations
  - Butterworth
  - Chebyshev
  - Elliptic
  - Bessel
  - Thompson

\[
|T_{LP}(j\omega)|
\]
Pafnuty Lvovich Chebyshev

Born    May 16, 1821
Died    December 8, 1894
Nationality    Russian
Fields    Mathematician
Chebyshev Approximations

Type I Chebyshev Approximations

- Analytical formulation:
  - All pole approximation
  - Magnitude response bounded between 1 and in the pass band
  - Assumes the value of \( \sqrt{\frac{1}{1+\varepsilon^2}} \) at \( \omega=1 \)
  - Goes to 0 at \( \omega=\infty \)
  - Assumes extreme values maximum no times in [0 1]
  - Characterized by \( \{n,\varepsilon\} \)

- Based upon Chebyshev Polynomials

Chebyshev polynomials were first presented in: P. L. Chebyshev (1854) "Théorie des mécanismes connus sous le nom parallelogrammes," Mémoires des Savants étrangers présentées à l’Académie de Saint-Pétersbourg, vol. 7, pages 539-586.
Chebyshev Approximations

Type II Chebyshev Approximations (not so common)

- Analytical formulation:
  - Magnitude response bounded between 0 and in the stop band
  - Assumes the value of at $\omega=1$
    $$\sqrt{\frac{1}{1+\varepsilon^2}}$$
  - Value of 1 at $\omega=0$
  - Assumes extreme values maximum times in $[1, \infty]$  
  - Characterized by $\{n, \varepsilon\}$

- Based upon Chebyshev Polynomials
Chebyshev Approximations

Chebyshev Polynomials

The Chebyshev polynomials are characterized by the property that the polynomial assumes the extremum values of 0 and 1 a maximum number of times in the interval [0,1] and go to $\infty$ for $x$ large.

In polynomial form they can be expressed as

$$C_0(x) = 1$$
$$C_1(x) = x$$
$$C_{n+1}(x) = 2xC_n(x) - C_{n-1}(x)$$

Or, equivalently, in trigonometric form as

$$C_n(x) = \begin{cases} 
\cos(n \cdot \arccos(x)) & x \in [-1,1] \\
\cosh(n \cdot \arccosh(x)) & x \geq 1 \\
(-1)^n \cosh(n \cdot \arccosh(-x)) & x \leq -1 
\end{cases}$$

This image shows the first few Chebyshev polynomials of the first kind in the domain $-1 \leq x \leq 1$, $-1 \leq y \leq 1$; the flat $T_0$, and $T_1$, $T_2$, $T_3$, $T_4$ and $T_5$.

Figure from Wikipedia
Chebyshev Approximations

Chebyshev Polynomials

The first 9 CC polynomials:

\[ C_0(x) = 1 \]
\[ C_1(x) = x \]
\[ C_2(x) = 2x^2 - 1 \]
\[ C_3(x) = 4x^3 - 3x \]
\[ C_4(x) = 8x^4 - 8x^2 + 1 \]
\[ C_5(x) = 16x^5 - 20x^3 + 5x \]
\[ C_6(x) = 32x^6 - 48x^4 + 18x^2 - 1 \]
\[ C_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x \]
\[ C_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1 \]

- Even-indexed polynomials are functions of \( x^2 \)
- Odd-indexed polynomials are product of \( x \) and function of \( x^2 \)
- Square of all polynomials are function of \( x^2 \) (i.e. an even function of \( x \))

This image shows the first few Chebyshev polynomials of the first kind in the domain \(-1 \leq x \leq 1\), \(-1 \leq y \leq 1\); the flat \( T_0 \), and \( T_1, T_2, T_3, T_4 \) and \( T_5 \). Figure from Wikipedia
Chebyshev Approximations

Type 1

$$H_{BW}(\omega) = \frac{1}{1 + \varepsilon^2 \omega^{2n}}$$

Butterworth

$$H(\omega) = \frac{1}{1 + \varepsilon^2 F_n(\omega^2)}$$

A General Form

Observation:

- $F_n(\omega^2)$ close to 1 in the pass band and gets very large in stop-band
- The square of the Chebyshev polynomials have this property

$$H_{CC}(\omega) = \frac{1}{1 + \varepsilon^2 C_n^2(\omega)}$$

This is the magnitude squared approximating function of the Type 1 CC approximation
Chebyshev Approximations

Type 1

\[ H_{CC}(\omega) = \frac{1}{1 + \varepsilon^2 C_n^2(\omega)} \]

Poles of \( H_{CC}(\omega) \) lie on an ellipse with none on the real axis
Chebyshev Approximations

Type 1

\[ H_{CC}(\omega) = \frac{1}{1 + \varepsilon^2 C_n^2(\omega)} \]
Chebyshev Approximations

**Type 1**

\[
\left[ \frac{\alpha_k}{\sinh \left( \frac{1}{n} \arcsinh \left( \frac{1}{\varepsilon} \right) \right)} \right]^2 + \left[ \frac{\beta_k}{\cosh \left( \frac{1}{n} \arcsinh \left( \frac{1}{\varepsilon} \right) \right)} \right]^2 = 1
\]

**Ellipse Intersect Points for select n and \( \varepsilon \)**

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Chebyshev Approximations

Type 1

Poles of $T_{CC}(s)$

$$p_k = -\sin \left[ \frac{\pi}{2n} (1+2k) \right] \sinh \left[ \frac{1}{n} \arcsinh \left( \frac{1}{\varepsilon} \right) \right] \pm j \cos \left[ \frac{\pi}{2n} (1+2k) \right] \cosh \left[ \frac{1}{n} \arcsinh \left( \frac{1}{\varepsilon} \right) \right] \quad k=0\ldots n-1$$

Properties of the ellipse

$$p_k = -\alpha_k \pm j\beta_k$$

$$\left( \frac{\alpha_k}{\sinh \left[ \frac{1}{n} \arcsinh \left( \frac{1}{\varepsilon} \right) \right]} \right)^2 + \left( \frac{\beta_k}{\cosh \left[ \frac{1}{n} \arcsinh \left( \frac{1}{\varepsilon} \right) \right]} \right)^2 = 1$$
Chebyshev Approximations

Type 1

$|T_{CC}(\omega)|$

- $|T_{CC}(0)|$ alternates between 1 and $\sqrt{\frac{1}{1+\varepsilon^2}}$ with index number
- Substantial pass band ripple
- Sharp transitions from pass band to stop band
Chebyshev Approximations

Type 1

Fig from Allen and Huelsman

Sharp transitions from pass band to stop band
Chebyshev Approximations

Type 1

CC transition is much steeper than BW transition
Comparison of BW and CC Responses

• CC slope at band edge much steeper than that of BW
  \[ \text{Slope}_{cc} (\omega = 1) = \left( \frac{-n}{2\sqrt{2}} \right) n = [\text{Slope}_{bw} (\omega = 1)] \]

• Corresponding pole Q of CC much higher than that of BW

• Lower-order CC filter can often meet same band-edge transition as a given BW filter

• Both are widely used

• Cost of implementation of BW and CC for same order is about the same
Chebyshev Approximations

Type 1

From Budak Text

Analytically, it can be shown that, at the band-edge

\[
\frac{d |T_{BW} (j\omega)|}{d\omega} = -n \frac{\varepsilon^2}{(1 + \varepsilon^2)^{3/2}}
\]

\[
\frac{d |T_{CC} (j\omega)|}{d\omega} = -n^2 \frac{\varepsilon^2}{(1 + \varepsilon^2)^{3/2}}
\]

CC slope is n times steeper than that of the BW slope
Chebyshev Approximations

Type 1

CC phase is much more nonlinear than BW phase

From Budak Text
Chebyshev Approximations

Type 1

\[ p_k = -\sin \left[ \frac{\pi}{2n} (1+2k) \right] \sinh \left[ \frac{1}{n} \arcsinh \left( \frac{1}{\varepsilon} \right) \right] \pm j \cos \left[ \frac{\pi}{2n} (1+2k) \right] \cosh \left[ \frac{1}{n} \arcsinh \left( \frac{1}{\varepsilon} \right) \right] \]

Maximum pole \( Q \) of CC approximation can be obtained by considering pole with index \( k=0 \)

\[ p_0 = -\sin \left[ \frac{\pi}{2n} \right] \sinh \left[ \frac{1}{n} \arcsinh \left( \frac{1}{\varepsilon} \right) \right] \pm j \cos \left[ \frac{\pi}{2n} \right] \cosh \left[ \frac{1}{n} \arcsinh \left( \frac{1}{\varepsilon} \right) \right] \]

\[ p_0 = \alpha + j \beta \]

Recall

\[ Q_{\text{MAX}} = \frac{\sqrt{\alpha^2 + \beta^2}}{-2\alpha} \]

\[ Q_{\text{MAX,CC}} = \left( \frac{1}{2 \sin \left( \frac{\pi}{2n} \right)} \right) \sqrt{1 + \left[ \frac{\cos \left( \frac{\pi}{2n} \right)}{\sinh \left( \frac{1}{n} \arcsinh \left( \frac{1}{\varepsilon} \right) \right)} \right]^2} \]
Chebyshev Approximations

Type 1

Comparison of maximum pole $Q$ of CC approximation with that of BW approximation

$$Q_{\text{MAX, BW}} = \frac{1}{2 \sin \left( \frac{\pi}{2n} \right)}$$

$$Q_{\text{MAX, CC}} = \left( \frac{1}{2 \sin \left( \frac{\pi}{2n} \right)} \right)^2 \sqrt{1 + \left[ \frac{\cos \left( \frac{\pi}{2n} \right)}{\sinh \left( \frac{1}{n \arcsinh \left( \frac{1}{\epsilon} \right)} \right)} \right]^2}$$

$$Q_{\text{MAX, CC}} = Q_{\text{MAX, BW}} \sqrt{1 + \left[ \frac{\cos \left( \frac{\pi}{2n} \right)}{\sinh \left( \frac{1}{n \arcsinh \left( \frac{1}{\epsilon} \right)} \right)} \right]^2}$$

Example – compare the $Q$’s for $n=10$ and $\epsilon=1$

$$Q_{\text{BW}} = 3.19 \quad Q_{\text{CC}} = 35.9$$

For large $n$, the CC filters have a very high pole $Q$!
Chebyshev Approximations

Type 2

\[ H_{BW}(\omega) = \frac{1}{1 + \varepsilon^2 \omega^{2n}} \]
Butterworth

\[ H(\omega) = \frac{1}{1 + \varepsilon^2 F_n(\omega^2)} \]
A General Form

Another General Form

\[ H(\omega) = \frac{1}{\frac{1}{1 + \varepsilon^2 F_n(1/\omega^2)}} \]

\[ H_{CC2}(\omega) = \frac{1}{\frac{1}{1 + \varepsilon^2 C_n^2(1/\omega)}} \]

Note: The second general form is not limited to use of the Chebyshev polynomials.
Chebyshev Approximations

Type 2

$$H_{CC2}(\omega) = \frac{1}{1 + \frac{1}{\varepsilon^2 C_n^2 \left( \frac{1}{\omega} \right)}}$$

- Equal-ripple in stop band
- Monotone in pass band
- Both poles and zeros present
- Poles of Type II CC are reciprocal of poles of Type I
- Zeros of Type II are inverse of the zeros of the CC Polynomials

$$p_k = \frac{-1}{\sin \left[ \frac{\pi}{2n} (1+2k) \right] \sinh \left[ \frac{1}{n} \text{arcsinh} \left( \frac{1}{\varepsilon} \right) \right] \pm j \cos \left[ \frac{\pi}{2n} (1+2k) \right] \cosh \left[ \frac{1}{n} \text{arcsinh} \left( \frac{1}{\varepsilon} \right) \right]}$$

$$z_k = j \frac{1}{\cos \left[ \frac{\pi (2k-1)}{2} \frac{1}{n} \right]}$$
Chebyshev Approximations

Type 2

\[ H_{CC2}(\omega) = \frac{1}{1 + \frac{\varepsilon^2 C_n^2}{\varepsilon^2 C_n^2 (\frac{1}{\omega})}} \]

\[ |T_{CC}(\omega)| \]

Odd order

Even order
Chebyshev Approximations

Type 2

\[ H_{CC2}(\omega) = \frac{1}{1 + \varepsilon^2 C_n^2 \left(\frac{1}{\omega}\right)} \]

- Transition region not as steep as for Type 1
- Considerably less popular
Chebyshev Approximations

Type 2

\[ H_{CC2}(\omega) = \frac{1}{1 + \frac{1}{\varepsilon^2 C_n^2 \left( \frac{1}{\omega} \right)}} \]

- Pole Q expressions identical since poles are reciprocals
- Maximum pole Q is just as high as for Type 1
Transitional BW-Chebyshev Approximations

\[ H(\omega) = \frac{1}{1 + \varepsilon^2 F_n(\omega^2)} \]

General Form

Define \( F_{BWk} = \omega^{2k} \) \( F_{CCk} = C_n^2(\omega) \)

Consider:

\[ H(\omega) = \frac{1}{1 + \varepsilon^2 F_{BWk} F_{CC(n-k)}} \quad 0 \leq k \leq n \]

\[ H(\omega) = \frac{1}{1 + \varepsilon^2 \left[ (\theta) F_{BWk} + (1 - \theta) F_{CC(n-k)} \right]} \quad 0 \leq \theta \leq 1 \]

• Other transitional approximations are possible
• Transitional approximations have some of the properties of both “parents”
Transitional BW-CC filters

\[ H_{ABW}(\omega^2) = \frac{1}{1 + \varepsilon^2 \omega^{2n}} \]

\[ H_{ACC}(\omega^2) = \frac{1}{1 + \varepsilon^2 \left( C_n(\omega) \right)^2} \]

\[ H_{ATRAN1}(\omega^2) = \frac{1}{1 + \varepsilon^2 \left( \omega^{2k} \right) C_{n-k}^2(\omega)} \]

\[ 0 \leq k \leq n \]

\[ H_{ATRAN2}(\omega^2) = \frac{1}{1 + \varepsilon^2 \left[ \theta \omega^{2n} + (1 - \theta) C_n^2(\omega) \right]} \]

\[ 0 \leq \theta \leq 1 \]

Other transitional BW-CC approximations exist as well
Transitional BW-CC filters

\[
H_{\text{ATRAN}1} (\omega^2) = \frac{1}{1 + \varepsilon^2 (\omega^{2k}) C_{n-k}^2 (\omega)}
\]

\[
H_{\text{ATRAN}2} (\omega^2) = \frac{1}{1 + \varepsilon^2 \left[ \theta \omega^{2n} + (1 - \theta) C_n^2 (\omega) \right]}
\]

Transitional filters will exhibit flatness at \(\omega=0\), passband ripple, and intermediate slope characteristics at band-edge.
Distinguish Between Circuit and Approximation

Note that what distinguishes between different filter approximations having the same number of cc poles and zeros and the same number of real axis poles and zeros is the component values of a given circuit, not the filter architecture itself.
Chebyshev Approximations

from Spectrum Software:

Chebyshev Filter Macro

Filters are a circuit element that seem to mesh perfectly with the macro capability of Micro-Cap. The macro capability is designed to produce components that can be varied through the use of parameters. Most filters consist of a basic structure whose component values can be modified through the use of well known equations. A macro component can be created that represents a specific filter's type, order, response, and implementation. The circuit below is the macro circuit for a low pass, 2nd order, Chebyshev filter with Tow-Thomas implementation.

• Note that this is introduced as a Chebyshev filter, the source correctly points out that it implements the CC filter in a specific filter topology
• It is important to not confuse the approximation from the architecture and this Tow-Thomas Structure can be used to implement either BW or CC functions only differing in the choice of the component values
End of Lecture 10