

# EE 508

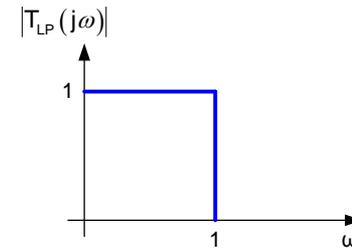
## Lecture 11

### The Approximation Problem

Classical Approximations

– the Chebyshev and Elliptic Approximations

# Butterworth Approximations



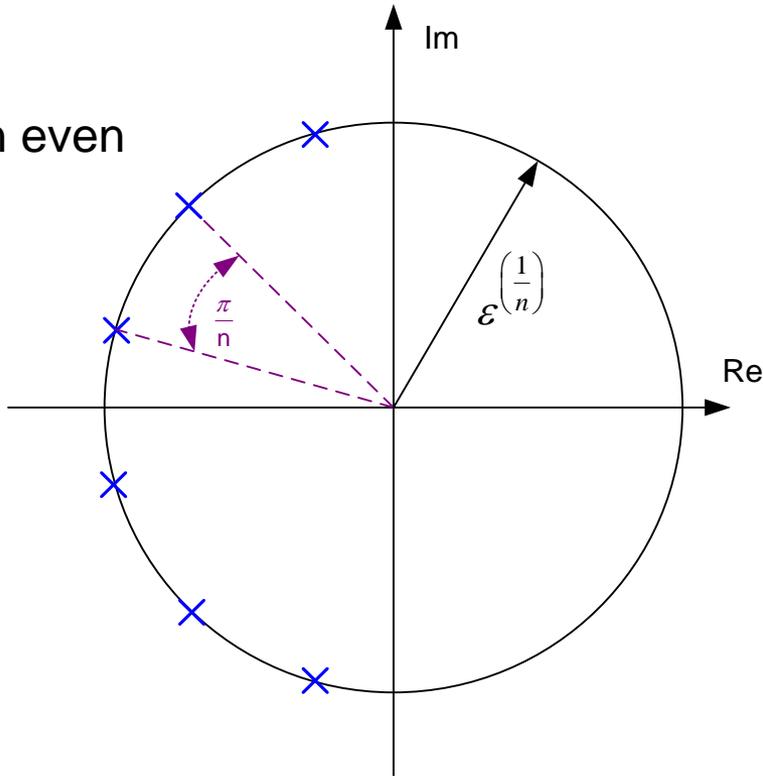
- Analytical formulation:
  - All pole approximation
  - Magnitude response is maximally flat at  $\omega=0$
  - Goes to 0 at  $\omega=\infty$
  - Assumes value  $\sqrt{\frac{1}{1+\varepsilon^2}}$  at  $\omega=1$
  - Assumes value of 1 at  $\omega=0$
  - Characterized by  $\{n,\varepsilon\}$
- Emphasis almost entirely on performance at single frequency

"On the Theory of Filter Amplifiers", Wireless Engineer (also called Experimental Wireless and the Radio Engineer), Vol. 7, 1930, pp. 536-541.

# Butterworth Approximation

Poles of  $T_{BW}(s)$

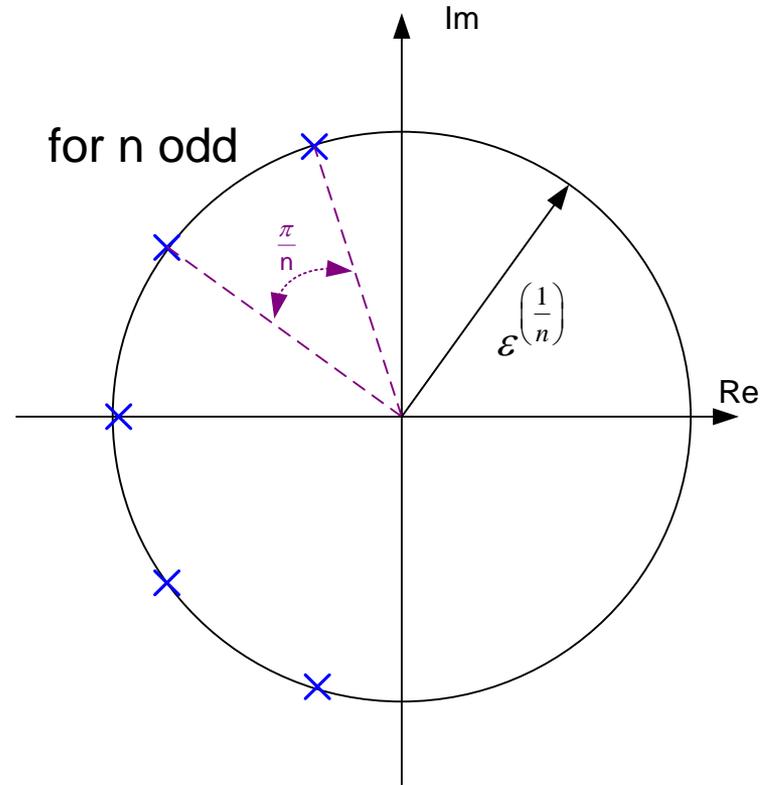
for  $n$  even



$$p_{k+1} = \epsilon^{1/n} \left[ -\sin\left(\left[1+2k\right]\frac{\pi}{2n}\right) \pm j \cos\left(\left[1+2k\right]\frac{\pi}{2n}\right) \right]$$

$$k=0, 1, \dots, \frac{n}{2}-1$$

for  $n$  odd



$$p_n = \epsilon^{1/n} [-1 + j0]$$

$$p_k = \epsilon^{1/n} \left[ -\sin\left(\left[1+2k\right]\frac{\pi}{2n}\right) \pm j \cos\left(\left[1+2k\right]\frac{\pi}{2n}\right) \right] \quad k=0, \dots, \frac{n-3}{2}$$

## Review from Last Time

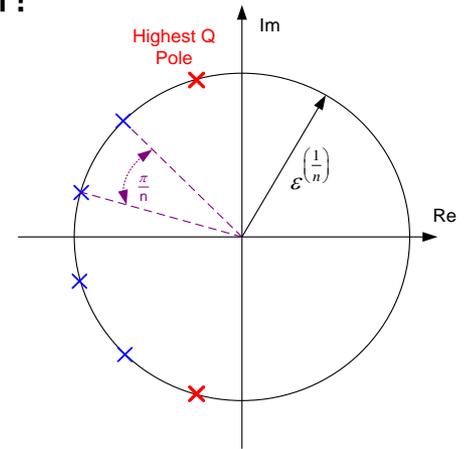
# Butterworth Approximation

What is the Q of the highest Q pole for the BW approximation?

$$p_0 = \varepsilon^{1/n} \left[ -\sin\left(\frac{\pi}{2n}\right) + j \cos\left(\frac{\pi}{2n}\right) \right] = \alpha + j\beta$$

$$Q_{MAX} = \frac{\sqrt{\alpha^2 + \beta^2}}{-2\alpha}$$

$$Q_{MAX} = \frac{\varepsilon^{1/n} \sqrt{\sin^2\left(\frac{\pi}{2n}\right) + \cos^2\left(\frac{\pi}{2n}\right)}}{2\varepsilon^{1/n} \sin\left(\frac{\pi}{2n}\right)} = \frac{1}{2 \sin\left(\frac{\pi}{2n}\right)}$$



$$Q_{MAX} = \frac{1}{2 \sin\left(\frac{\pi}{2n}\right)}$$

# Butterworth Approximation

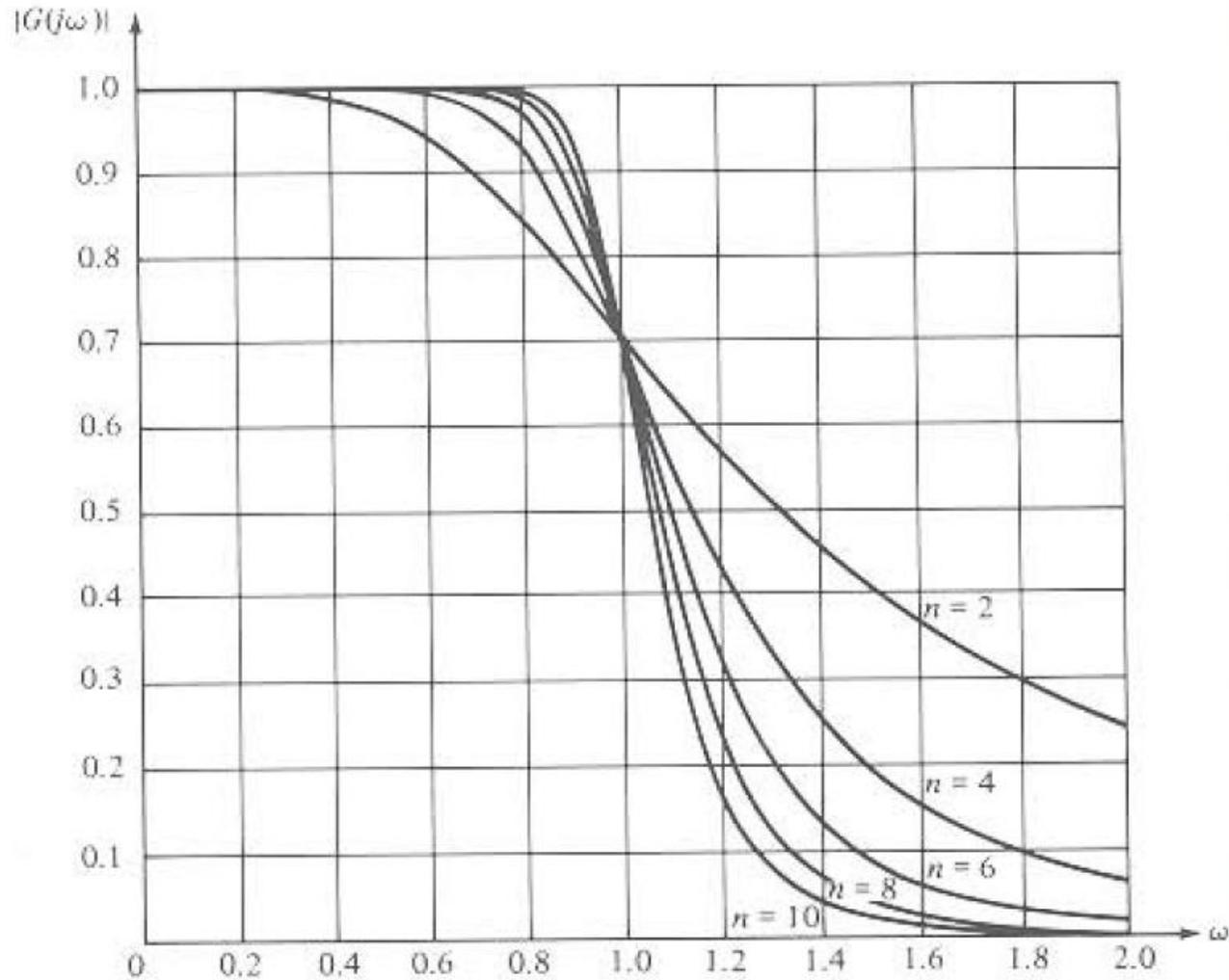


Fig. 17-3a Magnitude of the maximally flat approximation ( $\epsilon = 1$ )

Order needs to be rather high to get steep transition

Figure from Passive and Active Network Analysis and Synthesis, Budak

# Butterworth Approximation

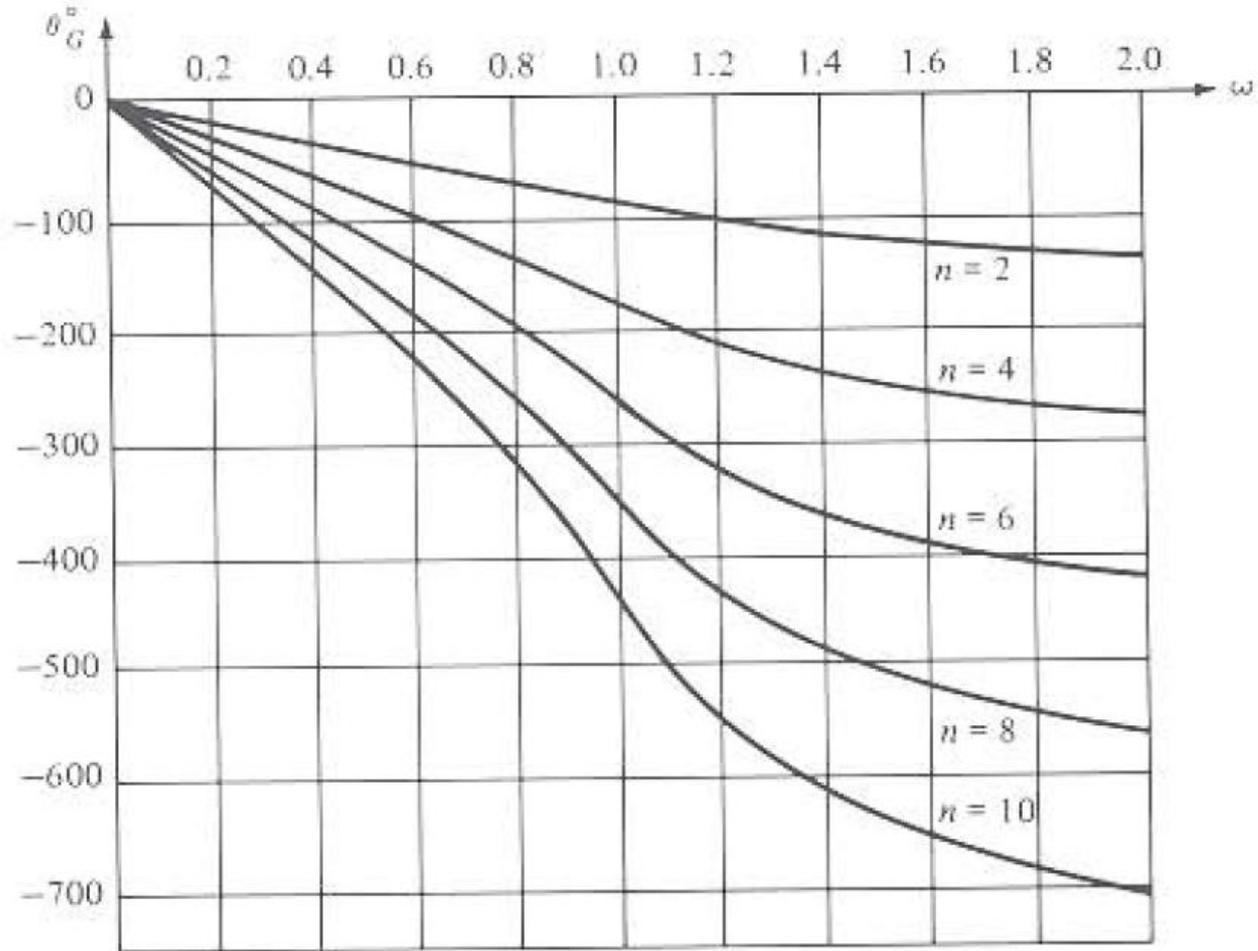


Fig. 17-3b Phase of the maximally flat approximation ( $\epsilon = 1$ )

Figure from Passive and Active Network Analysis and Synthesis, Budak

Phase is quite linear in passband (benefit unrelated to design requirements)

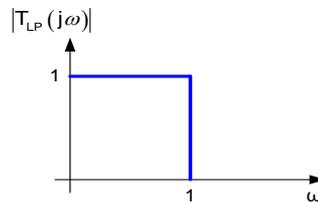
# Butterworth Approximation

## Summary

- Widely Used Analytical Approximation
- Characterized by  $\{\epsilon, n\}$
- Maximally flat at  $\omega=0$
- Almost all emphasis placed on characteristics at single frequency ( $\omega=0$ )
- Transition not very steep (requires large order for steep transition)
- Pole Q is quite low
- Pass-band phase is quite linear (no emphasis was placed on phase!)
- Poles lie on a circle
- Simple closed-form analytical expressions for poles and  $|T(j\omega)|$

# Approximations

- Magnitude Squared Approximating Functions –  $H_A(\omega^2)$
- Inverse Transform -  $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation
- Least Squares Approximations
- Pade Approximations
- Other Analytical Optimizations
- Numerical Optimization
- Canonical Approximations
  - Butterworth
  - Chebyshev
  - Elliptic
  - Bessel
  - Thompson





Pafnuty Lvovich Chebyshev

**Born** May 16, 1821

**Died** December 8, 1894

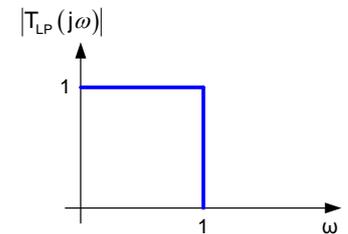
**Nationality** [Russian](#)

**Fields** [Mathematician](#)

# Chebyshev Approximations

## Type I Chebyshev Approximations

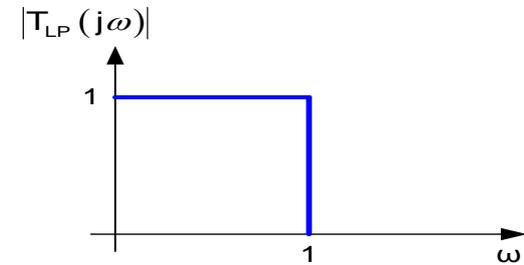
- Analytical formulation:
  - All pole approximation
  - Magnitude response bounded between 1 and  $\sqrt{\frac{1}{1+\varepsilon^2}}$  in the pass band
  - Assumes the value of  $\sqrt{\frac{1}{1+\varepsilon^2}}$  at  $\omega=1$
  - Goes to 0 at  $\omega=\infty$
  - Assumes extreme values maximum no times in  $[0, 1]$
  - Characterized by  $\{n, \varepsilon\}$
- Based upon Chebyshev Polynomials



Chebyshev polynomials were first presented in: P. L. Chebyshev (1854) "Théorie des mécanismes connus sous le nom parallélogrammes," *Mémoires des Savants étrangers présentés à l'Académie de Saint-Petersbourg*, vol. 7, pages 539-586.

# Chebyshev Approximations

## Type II Chebyshev Approximations (not so common)



- Analytical formulation:
  - Magnitude response bounded between 0 and  $\frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$  in the stop band
  - Assumes the value of  $\sqrt{\frac{1}{1+\varepsilon^2}}$  at  $\omega=1$
  - Value of 1 at  $\omega=0$
  - Assumes extreme values maximum times in  $[1 \infty]$
  - Characterized by  $\{n,\varepsilon\}$
- Based upon Chebyshev Polynomials

# Chebyshev Approximations

## Chebyshev Polynomials

The Chebyshev polynomials are characterized by the property that the polynomial assumes the extremum values of 0 and 1 a maximum number of times in the interval  $[0,1]$  and go to  $\infty$  for  $x$  large.

In polynomial form they can be expressed as

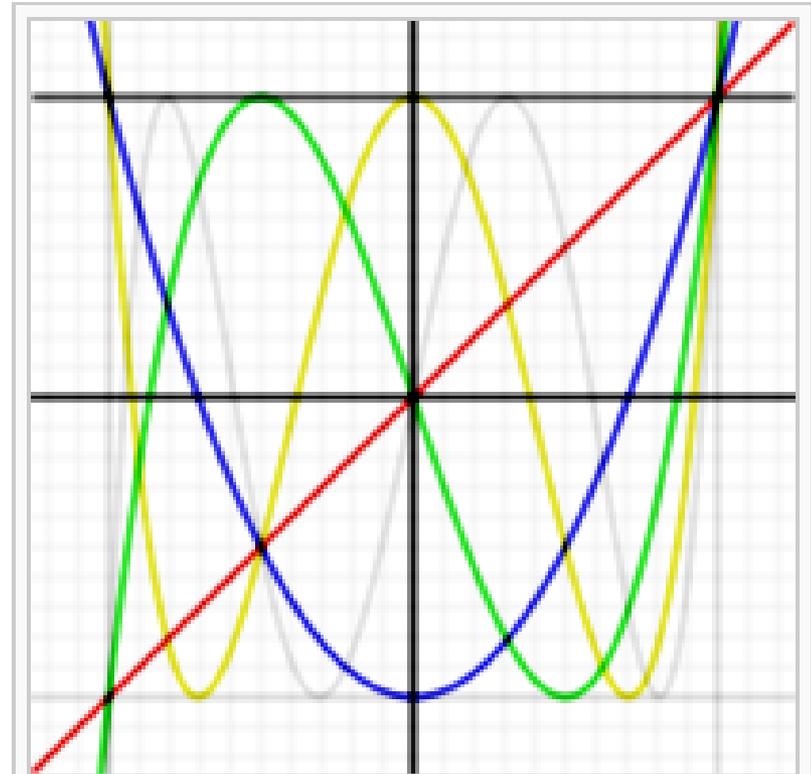
$$C_0(x)=1$$

$$C_1(x)=x$$

$$C_{n+1}(x)=2xC_n(x) - C_{n-1}(x)$$

Or, equivalently, in trigonometric form as

$$C_n(x) = \begin{cases} \cos(n \cdot \arccos(x)) & x \in [-1,1] \\ \cosh(n \cdot \operatorname{arcosh}(x)) & x \geq 1 \\ (-1)^n \cosh(n \cdot \operatorname{arcosh}(-x)) & x \leq -1 \end{cases}$$



This image shows the first few Chebyshev polynomials of the first kind in the domain  $-1 \leq x \leq 1$ ,  $-1 \leq y \leq 1$ ; the flat  $T_0$ , and  $T_1$ ,  $T_2$ ,  $T_3$ ,  $T_4$  and  $T_5$ .

Figure from Wikipedia

# Chebyshev Approximations

## Chebyshev Polynomials

The first 9 CC polynomials:

$$C_0(x) = 1$$

$$C_1(x) = x$$

$$C_2(x) = 2x^2 - 1$$

$$C_3(x) = 4x^3 - 3x$$

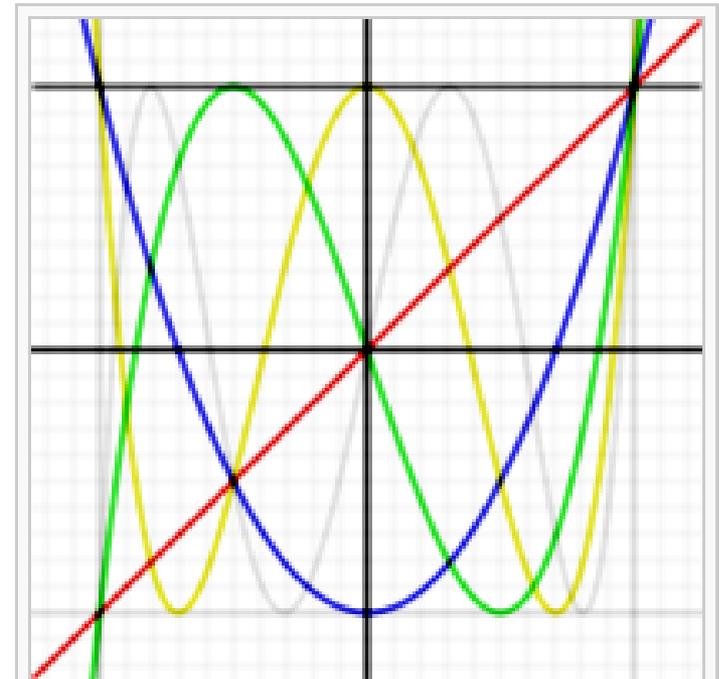
$$C_4(x) = 8x^4 - 8x^2 + 1$$

$$C_5(x) = 16x^5 - 20x^3 + 5x$$

$$C_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

$$C_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$$

$$C_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$$



This image shows the first few Chebyshev polynomials of the first kind in the domain  $-1 \leq x \leq 1$ ,  $-1 \leq y \leq 1$ ; the flat  $T_0$ , and  $T_1$ ,  $T_2$ ,  $T_3$ ,  $T_4$  and  $T_5$ . Figure from Wikipedia

- Even-indexed polynomials are functions of  $x^2$
- Odd-indexed polynomials are product of  $x$  and function of  $x^2$
- Square of all polynomials are function of  $x^2$  (i.e. an even function of  $x$ )

# Chebyshev Approximations

## Type 1

$$H_{BW}(\omega) = \frac{1}{1 + \varepsilon^2 \omega^{2n}}$$

Butterworth

$$H(\omega) = \frac{1}{1 + \varepsilon^2 F_n(\omega^2)}$$

A General Form

Observation:

$F_n(\omega^2)$  close to 1 in the pass band and gets very large in stop-band

The square of the Chebyshev polynomials have this property

$$H_{CC}(\omega) = \frac{1}{1 + \varepsilon^2 C_n^2(\omega)}$$

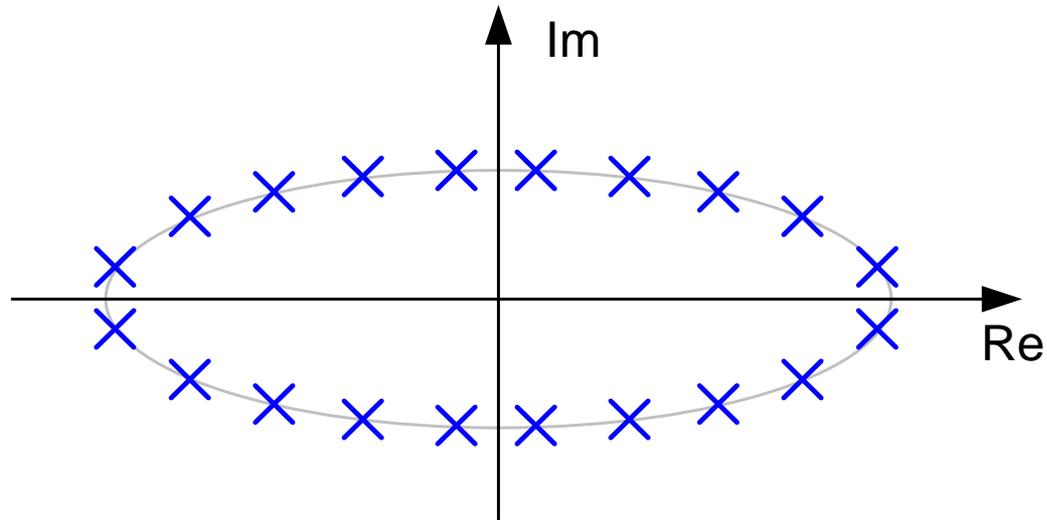
This is the magnitude squared approximating function of the Type 1 CC approximation

# Chebyshev Approximations

## Type 1

$$H_{CC}(\omega) = \frac{1}{1 + \varepsilon^2 C_n^2(\omega)}$$

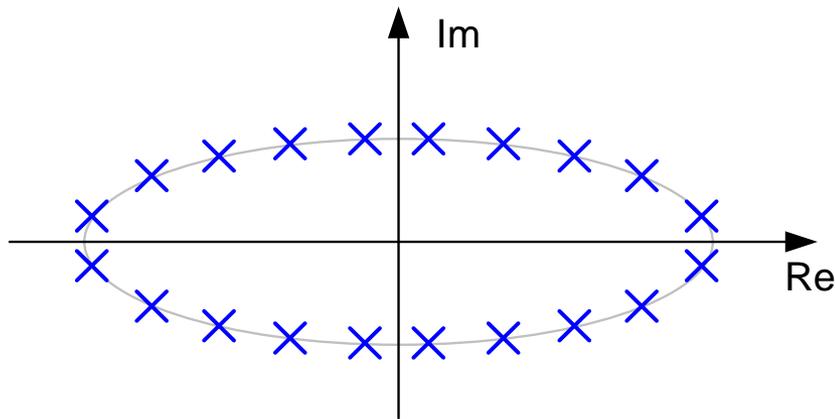
Poles of  $H_{CC}(\omega)$  lie on an ellipse with none on the real axis



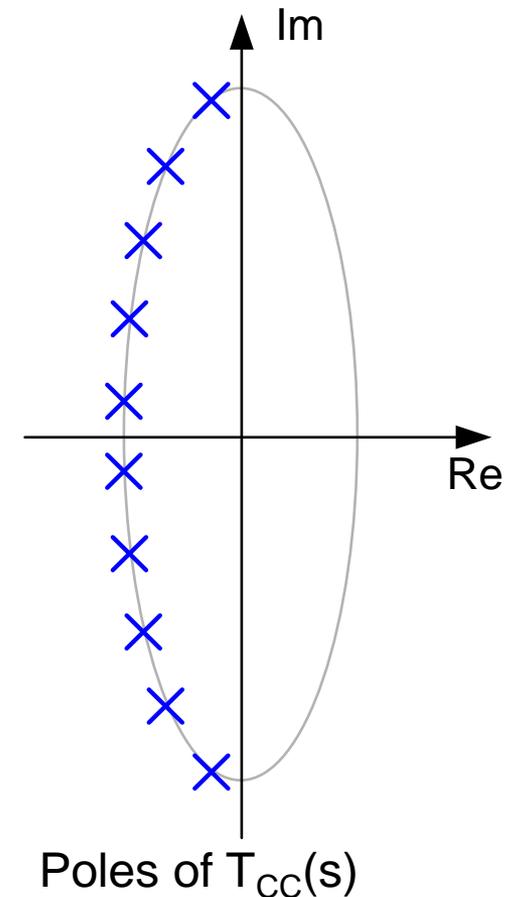
# Chebyshev Approximations

## Type 1

$$H_{CC}(\omega) = \frac{1}{1 + \varepsilon^2 C_n^2(\omega)}$$



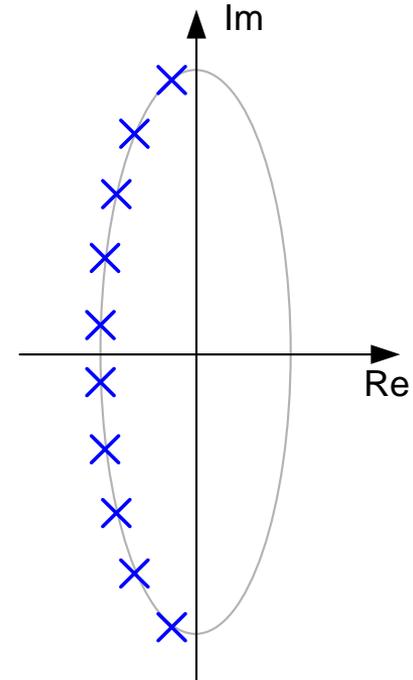
Inverse Mapping  
➔



# Chebyshev Approximations

## Type 1

$$\left[ \frac{\alpha_k}{\sinh \left[ \frac{1}{n} \operatorname{arcsinh} \left( \frac{1}{\varepsilon} \right) \right]} \right]^2 + \left[ \frac{\beta_k}{\cosh \left[ \frac{1}{n} \operatorname{arcsinh} \left( \frac{1}{\varepsilon} \right) \right]} \right]^2 = 1$$

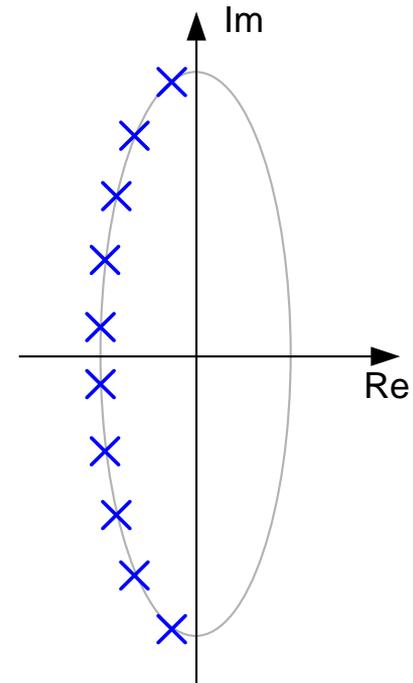


Ellipse Intersect Points for select n and  $\varepsilon$

n	$\varepsilon$	Y int	X int
2	1	1.099	0.455
2	0.25	1.600	1.250
2	0.1	2.351	2.127
2	0.05	3.242	3.084
4	1	1.024	0.222
4	0.25	1.140	0.548
4	0.1	1.294	0.822
4	0.05	1.456	1.059
6	1	1.011	0.147
6	0.25	1.062	0.356
6	0.1	1.127	0.521
6	0.05	1.195	0.654
8	1	1.006	0.110
8	0.25	1.034	0.265
8	0.1	1.071	0.384
8	0.05	1.108	0.478

# Chebyshev Approximations

Type 1



Poles of  $T_{CC}(s)$

$$\rho_k = -\sin\left[\frac{\pi}{2n}(1+2k)\right] \sinh\left[\frac{1}{n} \operatorname{arcsinh}\left(\frac{1}{\varepsilon}\right)\right] \pm j \cos\left[\frac{\pi}{2n}(1+2k)\right] \cosh\left[\frac{1}{n} \operatorname{arcsinh}\left(\frac{1}{\varepsilon}\right)\right] \quad k=0 \dots n-1$$

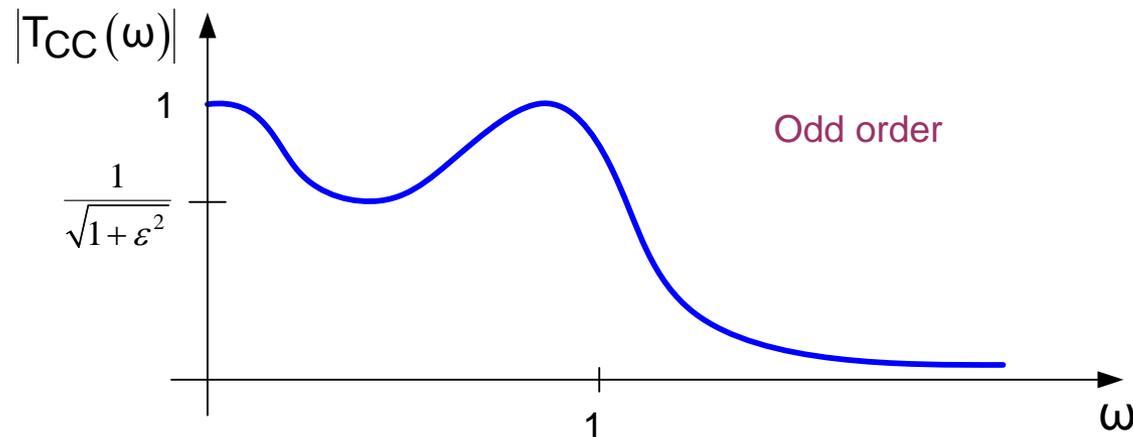
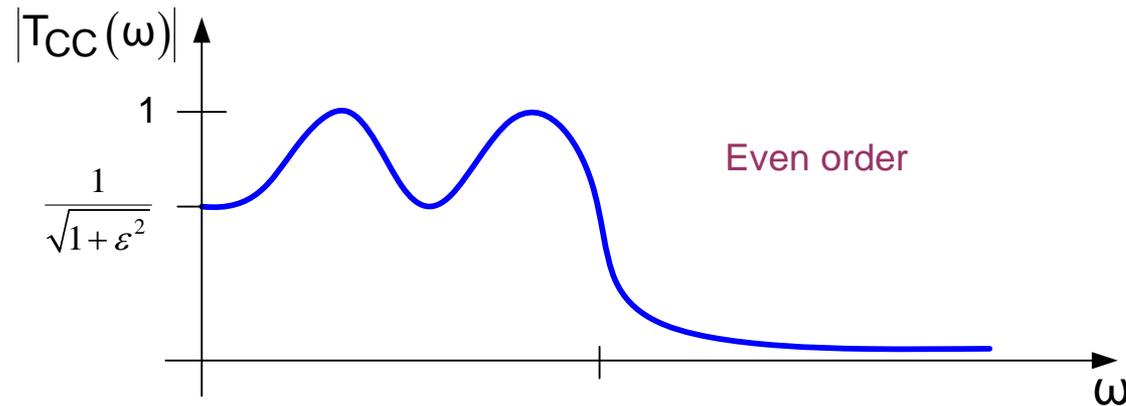
Properties of the ellipse

$$\rho_k = -\alpha_k \pm j\beta_k$$

$$\left[ \frac{\alpha_k}{\sinh\left[\frac{1}{n} \operatorname{arcsinh}\left(\frac{1}{\varepsilon}\right)\right]} \right]^2 + \left[ \frac{\beta_k}{\cosh\left[\frac{1}{n} \operatorname{arcsinh}\left(\frac{1}{\varepsilon}\right)\right]} \right]^2 = 1$$

# Chebyshev Approximations

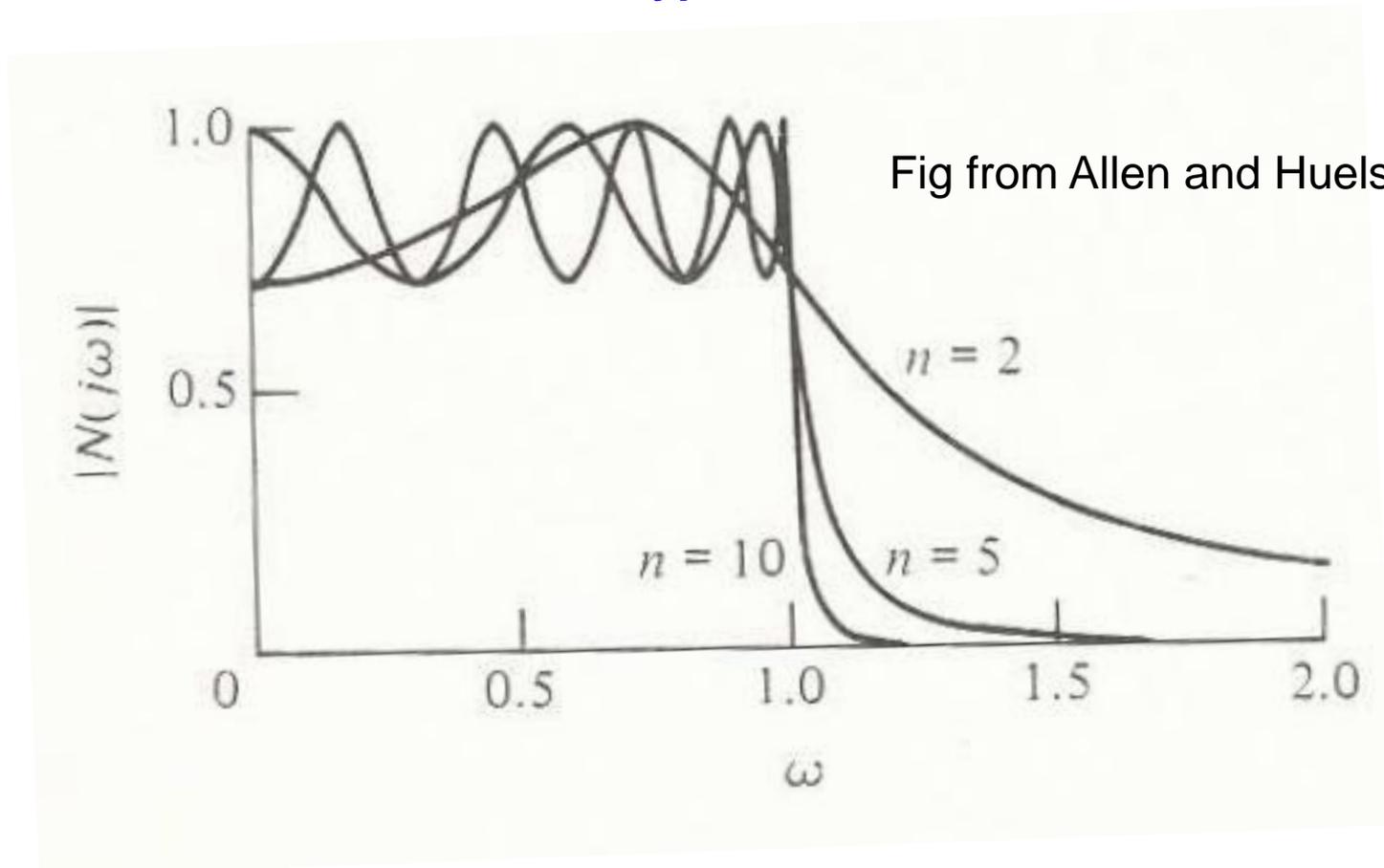
## Type 1



- $|T_{CC}(0)|$  alternates between 1 and  $\sqrt{\frac{1}{1+\epsilon^2}}$  with index number
- Substantial pass band ripple  $\sqrt{\frac{1}{1+\epsilon^2}}$
- Sharp transitions from pass band to stop band

# Chebyshev Approximations

Type 1



Sharp transitions from pass band to stop band

# Chebyshev Approximations

## Type 1

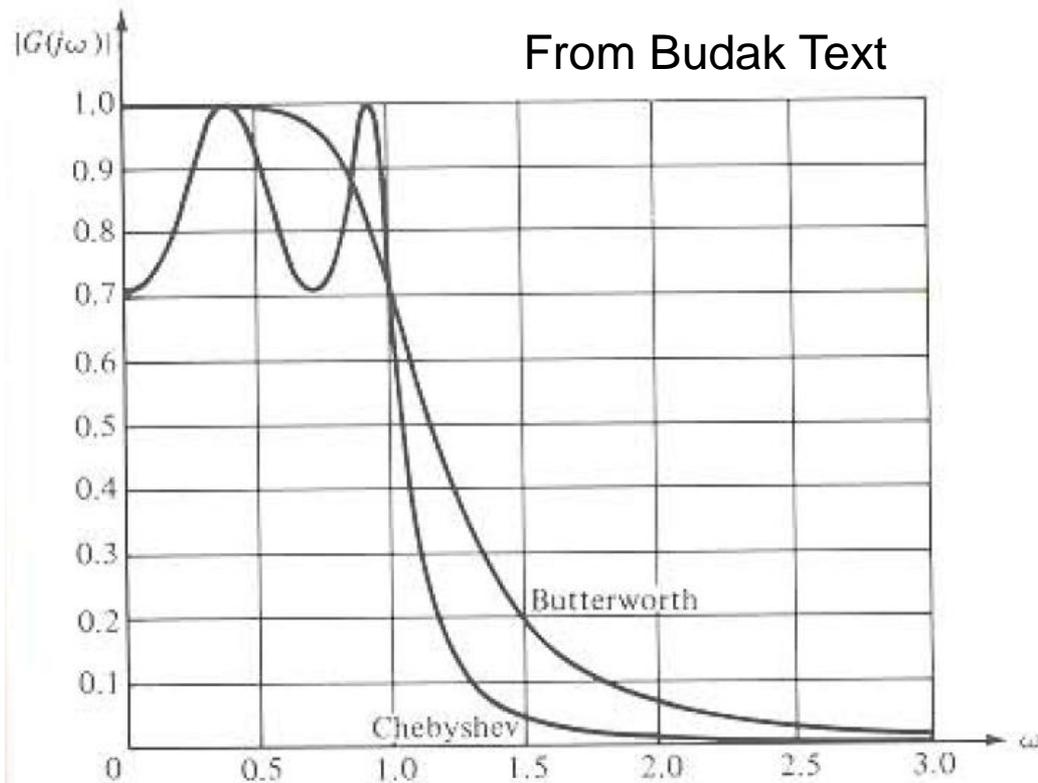


Fig. 17-6a Fourth-order Chebyshev and Butterworth magnitude characteristics

CC transition is much steeper than BW transition

# Comparison of BW and CC Responses

- CC slope at band edge much steeper than that of BW

$$Slope_{cc}(\omega = 1) = \left( \frac{-n}{2\sqrt{2}} \right) n = [Slope_{BW}(\omega = 1)]$$

- Corresponding pole Q of CC much higher than that of BW
- Lower-order CC filter can often meet same band-edge transition as a given BW filter
- Both are widely used
- Cost of implementation of BW and CC for same order is about the same

# Chebyshev Approximations

## Type 1

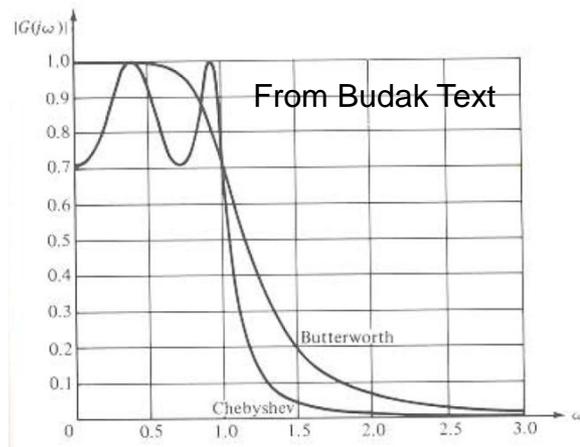


Fig. 17-6a Fourth-order Chebyshev and Butterworth magnitude characteristics

Analytically, it can be shown that, at the band-edge

$$\frac{d|T_{BW}(j\omega)|}{d\omega} = -n \frac{\epsilon^2}{(1 + \epsilon^2)^{3/2}}$$

$$\frac{d|T_{CC}(j\omega)|}{d\omega} = -n^2 \frac{\epsilon^2}{(1 + \epsilon^2)^{3/2}}$$

CC slope is n times steeper than that of the BW slope

# Chebyshev Approximations

## Type 1

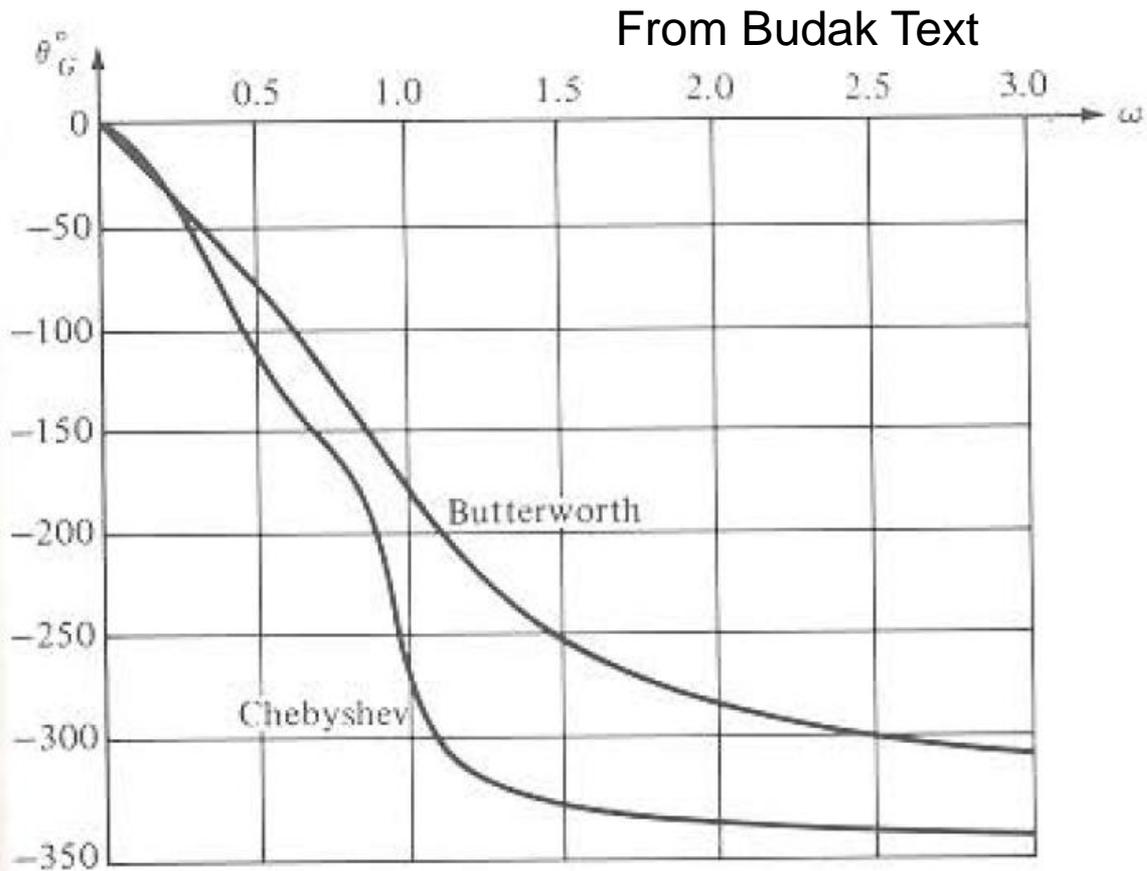


Fig. 17-6b Fourth-order Chebyshev and Butterworth phase characteristics

CC phase is much more nonlinear than BW phase

# Chebyshev Approximations

## Type 1

$$p_k = -\sin\left[\frac{\pi}{2n}(1+2k)\right] \sinh\left[\frac{1}{n} \operatorname{arcsinh}\left(\frac{1}{\varepsilon}\right)\right] \pm j \cos\left[\frac{\pi}{2n}(1+2k)\right] \cosh\left[\frac{1}{n} \operatorname{arcsinh}\left(\frac{1}{\varepsilon}\right)\right]$$

Maximum pole Q of CC approximation can be obtained by considering pole with index k=0

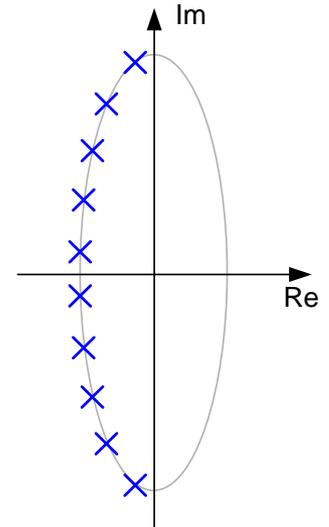
$$p_0 = -\sin\left[\frac{\pi}{2n}\right] \sinh\left[\frac{1}{n} \operatorname{arcsinh}\left(\frac{1}{\varepsilon}\right)\right] \pm j \cos\left[\frac{\pi}{2n}\right] \cosh\left[\frac{1}{n} \operatorname{arcsinh}\left(\frac{1}{\varepsilon}\right)\right]$$

$$p_0 = \alpha + j\beta$$

Recall

$$Q_{\text{MAX}} = \frac{\sqrt{\alpha^2 + \beta^2}}{-2\alpha}$$

$$Q_{\text{MAX,CC}} = \left( \frac{1}{2 \sin\left(\frac{\pi}{2n}\right)} \right) \sqrt{1 + \left[ \frac{\cos\left(\frac{\pi}{2n}\right)}{\sinh\left(\frac{1}{n} \operatorname{arcsinh}\left(\frac{1}{\varepsilon}\right)\right)} \right]^2}$$



# Chebyshev Approximations

## Type 1

Comparison of maximum pole Q of CC approximation with that of BW approximation

$$Q_{\text{MAX,BW}} = \frac{1}{2 \sin\left(\frac{\pi}{2n}\right)} \quad Q_{\text{MAX,CC}} = \left( \frac{1}{2 \sin\left(\frac{\pi}{2n}\right)} \right) \sqrt{1 + \left[ \frac{\cos\left(\frac{\pi}{2n}\right)}{\sinh\left(\frac{1}{n} \operatorname{arcsinh}\left(\frac{1}{\varepsilon}\right)\right)} \right]^2}$$

$$Q_{\text{MAX,CC}} = Q_{\text{MAX,BW}} \sqrt{1 + \left[ \frac{\cos\left(\frac{\pi}{2n}\right)}{\sinh\left(\frac{1}{n} \operatorname{arcsinh}\left(\frac{1}{\varepsilon}\right)\right)} \right]^2}$$

Example – compare the Q's for  $n=10$  and  $\varepsilon=1$

$$Q_{\text{BW}}=3.19$$

$$Q_{\text{CC}}=35.9$$

For large  $n$ , the CC filters have a very high pole Q !

# Chebyshev Approximations

## Type 2

$$H_{\text{BW}}(\omega) = \frac{1}{1 + \varepsilon^2 \omega^{2n}}$$

Butterworth

$$H(\omega) = \frac{1}{1 + \varepsilon^2 F_n(\omega^2)}$$

A General Form

Another General Form

$$H(\omega) = \frac{1}{1 + \frac{1}{\varepsilon^2 F_n(1/\omega^2)}}$$

$$H_{\text{CC2}}(\omega) = \frac{1}{1 + \frac{1}{\varepsilon^2 C_n^2(1/\omega)}}$$

Note: The second general form is not limited to use of the Chebyshev polynomials

# Chebyshev Approximations

## Type 2

$$H_{CC2}(\omega) = \frac{1}{1 + \frac{1}{\varepsilon^2 C_n^2\left(\frac{1}{\omega}\right)}}$$

- Equal-ripple in stop band
- Monotone in pass band
- Both poles and zeros present
- Poles of Type II CC are reciprocal of poles of Type I
- Zeros of Type II are inverse of the zeros of the CC Polynomials

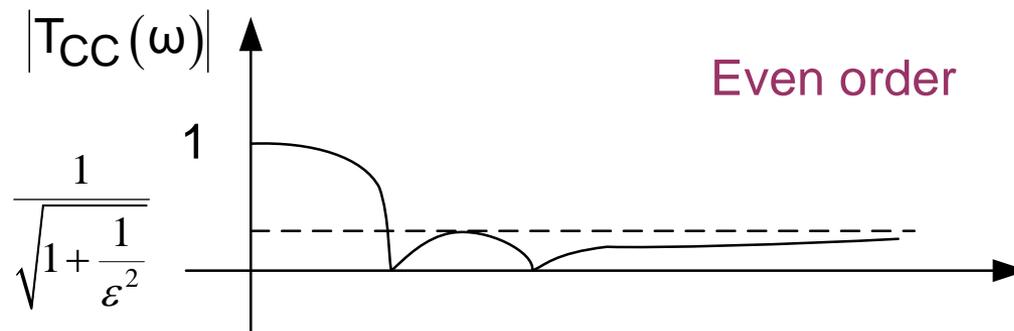
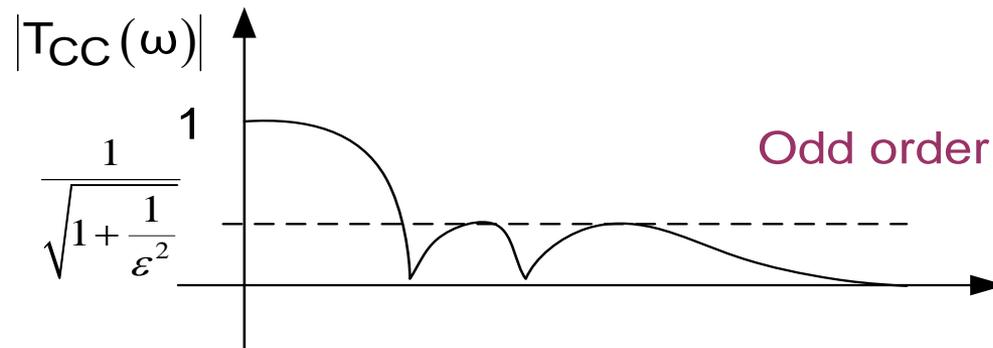
$$p_k = \frac{-1}{\sin\left[\frac{\pi}{2n}(1+2k)\right] \sinh\left[\frac{1}{n} \operatorname{arcsinh}\left(\frac{1}{\varepsilon}\right)\right] \pm j \cos\left[\frac{\pi}{2n}(1+2k)\right] \cosh\left[\frac{1}{n} \operatorname{arcsinh}\left(\frac{1}{\varepsilon}\right)\right]}$$

$$z_k = j \frac{1}{\cos\left(\frac{\pi(2k-1)}{2n}\right)}$$

# Chebyshev Approximations

## Type 2

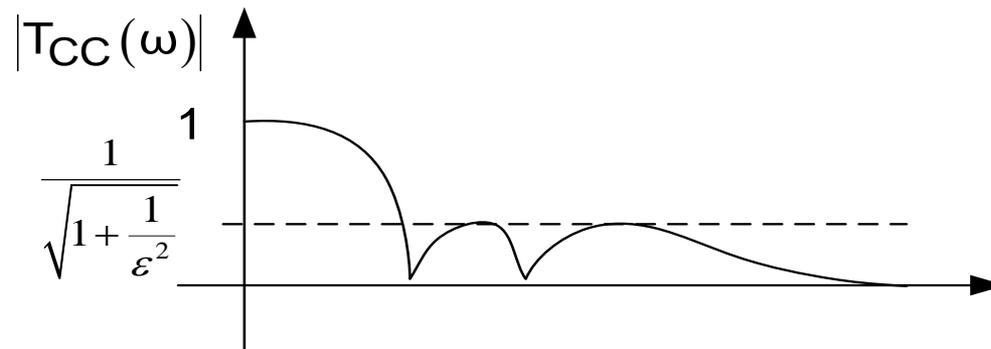
$$H_{CC2}(\omega) = \frac{1}{1 + \frac{1}{\varepsilon^2 C_n^2(1/\omega)}}$$



# Chebyshev Approximations

## Type 2

$$H_{CC2}(\omega) = \frac{1}{1 + \frac{1}{\varepsilon^2 C_n^2(1/\omega)}}$$

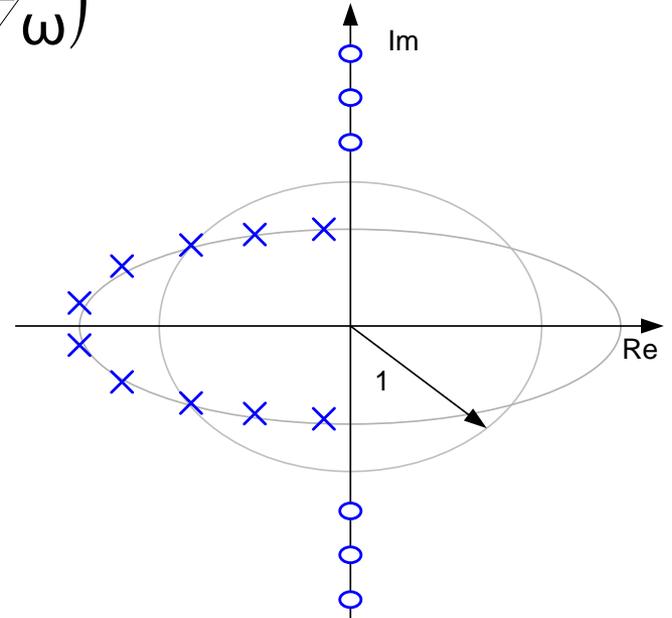
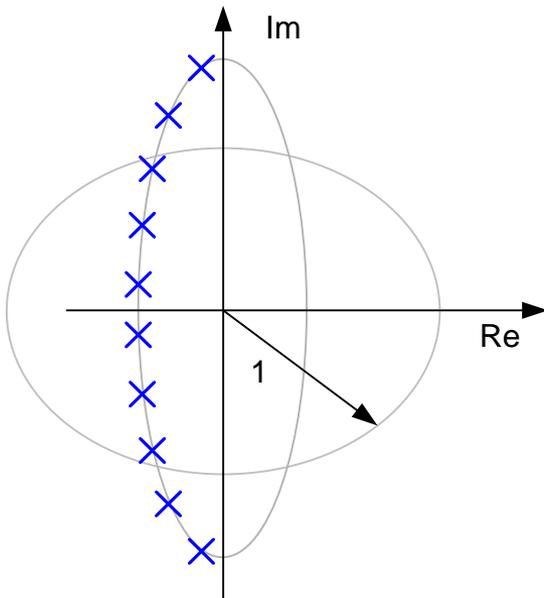


- Transition region not as steep as for Type 1
- Considerably less popular

# Chebyshev Approximations

## Type 2

$$H_{CC2}(\omega) = \frac{1}{1 + \frac{1}{\varepsilon^2 C_n^2(1/\omega)}}$$



- Pole Q expressions identical since poles are reciprocals
- Maximum pole Q is just as high as for Type 1

# Transitional BW-Chebyshev Approximations

$$H(\omega) = \frac{1}{1 + \varepsilon^2 F_n(\omega^2)}$$

General Form

Define  $F_{\text{BW}k} = \omega^{2k}$        $F_{\text{CC}k} = C_n^2(\omega)$

Consider:

$$H(\omega) = \frac{1}{1 + \varepsilon^2 F_{\text{BW}k} F_{\text{CC}(n-k)}} \quad 0 \leq k \leq n$$

$$H(\omega) = \frac{1}{1 + \varepsilon^2 \left[ (\theta) F_{\text{BW}k} + (1 - \theta) F_{\text{CC}(n-k)} \right]} \quad 0 \leq \theta \leq 1$$

- Other transitional approximations are possible
- Transitional approximations have some of the properties of both “parents”

# Transitional BW-CC filters

$$H_{ABW}(\omega^2) = \frac{1}{1 + \varepsilon^2 \omega^{2n}} \quad H_{ACC}(\omega^2) = \frac{1}{1 + \varepsilon^2 (C_n(\omega))^2}$$

$$H_{ATRAN1}(\omega^2) = \frac{1}{1 + \varepsilon^2 (\omega^{2k}) C_{n-k}^2(\omega)}$$

$$0 \leq k \leq n$$

$$H_{ATRAN2}(\omega^2) = \frac{1}{1 + \varepsilon^2 [\theta \omega^{2n} + (1 - \theta) C_n^2(\omega)]}$$

$$0 \leq \theta \leq 1$$

Other transitional BW-CC approximations exist as well

# Transitional BW-CC filters

$$H_{ATRAN1}(\omega^2) = \frac{1}{1 + \varepsilon^2 (\omega^{2k}) C_{n-k}^2(\omega)}$$

$$H_{ATRAN2}(\omega^2) = \frac{1}{1 + \varepsilon^2 [\theta \omega^{2n} + (1 - \theta) C_n^2(\omega)]}$$

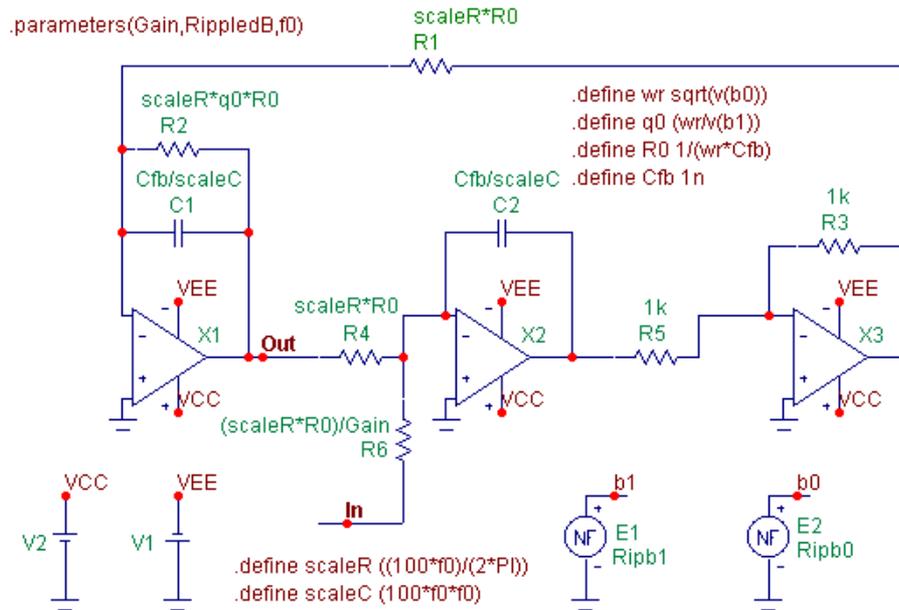
Transitional filters will exhibit flatness at  $\omega=0$ , passband ripple, and intermediate slope characteristics at band-edge

# Chebyshev Approximations

## from Spectrum Software:

### Chebyshev Filter Macro

Filters are a circuit element that seem to mesh perfectly with the macro capability of Micro-Cap. The macro capability is designed to produce components that can be varied through the use of parameters. Most filters consist of a basic structure whose component values can be modified through the use of well known equations. A macro component can be created that represents a specific filter's type, order, response, and implementation. The circuit below is the macro circuit for a low pass, 2nd order, Chebyshev filter with Tow-Thomas implementation.



- Note that this is introduced as a Chebyshev filter, the source correctly points out that it implements the CC filter in a specific filter topology
- It is important to not confuse the approximation from the architecture and this Tow-Thomas Structure can be used to implement either BW or CC functions only differing in the choice of the component values

**End of Lecture 11**