EE 508
Lecture 12

The Approximation Problem

Classical Approximating Functions
- Elliptic Approximations
- Thompson and Bessel Approximations
Chebyshev Approximations

Type II Chebyshev Approximations (not so common)

- Analytical formulation:
  - Magnitude response bounded between 0 and $\frac{1}{\sqrt{1+\varepsilon^2}}$ in the stop band
  - Assumes the value of $\frac{1}{\sqrt{1+\varepsilon^2}}$ at $\omega=1$
  - Value of $1$ at $\omega=0$
  - Assumes extreme values maximum times in $[1, \infty)$
  - Characterized by $\{n, \varepsilon\}$

- Based upon Chebyshev Polynomials

Review from Last Time

\[ |T_{LP}(j\omega)| \]

\[ \varepsilon \]

\[ \frac{1}{\sqrt{1+\varepsilon^2}} \]
Chebyshev Approximations

Chebyshev Polynomials

The first 9 CC polynomials:

\[ C_0(x) = 1 \]
\[ C_1(x) = x \]
\[ C_2(x) = 2x^2 - 1 \]
\[ C_3(x) = 4x^3 - 3x \]
\[ C_4(x) = 8x^4 - 8x^2 + 1 \]
\[ C_5(x) = 16x^5 - 20x^3 + 5x \]
\[ C_6(x) = 32x^6 - 48x^4 + 18x^2 - 1 \]
\[ C_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x \]
\[ C_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1 \]

- Even-indexed polynomials are functions of \( x^2 \)
- Odd-indexed polynomials are product of \( x \) and function of \( x^2 \)
- Square of all polynomials are function of \( x^2 \) (i.e. an even function of \( x \))
Chebyshev Approximations

Type 1

\[ H_{BW}(\omega) = \frac{1}{1 + \varepsilon^2 \omega^{2n}} \]
Butterworth

\[ H(\omega) = \frac{1}{1 + \varepsilon^2 F_n(\omega^2)} \]
A General Form

Observation:

\( F_n(\omega^2) \) close to 1 in the pass band and gets very large in stop-band

The square of the Chebyshev polynomials have this property

\[ H_{CC}(\omega) = \frac{1}{1 + \varepsilon^2 C_n^2(\omega)} \]

This is the magnitude squared approximating function of the Type 1 CC approximation
Chebyshev Approximations

Type 1

\[ H_{CC}(\omega) = \frac{1}{1 + \varepsilon^2 C_n^2(\omega)} \]
Review from Last Time

Chebyshev Approximations

Type 1

<table>
<thead>
<tr>
<th>$T_{CC}(\omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

Even order

\[
\frac{1}{\sqrt{1+\varepsilon^2}}
\]

Odd order

\[
\frac{1}{\sqrt{1+\varepsilon^2}}
\]

- $|T_{CC}(0)|$ alternates between 1 and $\frac{1}{\sqrt{1+\varepsilon^2}}$ with index number
- Substantial pass band ripple $\sqrt{1+\varepsilon^2}$
- Sharp transitions from pass band to stop band
Comparison of BW and Type 1 CC Responses

- CC slope at band edge much steeper than that of BW
  \[ \text{Slope}_{cc}(\omega = 1) = \left( \frac{-n}{2\sqrt{2}} \right) n = [\text{Slope}_{bw}(\omega = 1)] \]

- Corresponding pole Q of CC much higher than that of BW

- Lower-order CC filter can often meet same band-edge transition as a given BW filter

- Both are widely used

- Cost of implementation of BW and CC for same order is about the same
Review from Last Time

Chebyshev Approximations

Type 2

$H_{BW}(\omega) = \frac{1}{1 + \varepsilon^2 \omega^{2n}}$

Butterworth

$H(\omega) = \frac{1}{1 + \varepsilon^2 F_n(\omega^2)}$

A General Form

Another General Form

$H(\omega) = \frac{1}{1 + \frac{1}{1 + \frac{1}{\varepsilon^2 F_n(1/\omega^2)}}}$

$H_{CC2}(\omega) = \frac{1}{1 + \frac{1}{\varepsilon^2 C_n(1/\omega)}}$
Chebyshev Approximations

Type 2

\[ H_{CC2}(\omega) = \frac{1}{1 + \frac{\varepsilon^2 C_n^2}{1/\omega}} \]

Review from Last Time
Chebyshev Approximations

Type 2

\[ H_{CC2}(\omega) = \frac{1}{1 + \frac{\varepsilon^2 C_n^2}{1/\omega}} \]

- Transition region not as steep as for Type 1
- Considerably less popular
Chebyshev Approximations
Type 2

$H_{CC2}(\omega) = \frac{1}{1 + \frac{\varepsilon^2 C_n^2 \left(\frac{1}{\omega}\right)}{1}}$

- Pole Q expressions identical since poles are reciprocals
- Maximum pole Q is just as high as for Type 1
Transitional BW-CC filters

$$H_{\text{ATRAN}1} (\omega^2) = \frac{1}{1 + \varepsilon^2 \left( \omega^{2k} \right) C_{n-k}^2 (\omega)}$$

$$H_{\text{ATRAN}2} (\omega^2) = \frac{1}{1 + \varepsilon^2 \left[ \theta \omega^{2n} + (1 - \theta) C_n^2 (\omega) \right]}$$

Transitional filters will exhibit flatness at $\omega=0$, passband ripple, and intermediate slope characteristics at band-edge.
Chebyshev Approximations

from Spectrum Software:

Chebyshev Filter Macro

Filters are a circuit element that seem to mesh perfectly with the macro capability of Micro-Cap. The macro capability is designed to produce components that can be varied through the use of parameters. Most filters consist of a basic structure whose component values can be modified through the use of well known equations. A macro component can be created that represents a specific filter’s type, order, response, and implementation. The circuit below is the macro circuit for a low pass, 2nd order, Chebyshev filter with Tow-Thomas implementation.

• Note that this is introduced as a Chebyshev filter, the source correctly points out that it implements the CC filter in a specific filter topology
• It is important to not confuse the approximation from the architecture and this Tow-Thomas Structure can be used to implement either BW or CC functions only differing in the choice of the component values
Approximations

- Magnitude Squared Approximating Functions – $H_A(\omega^2)$
- Inverse Transform - $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation
- Least Squares Approximations
- Pade Approximations
- Other Analytical Optimizations
- Numerical Optimization
- Canonical Approximations
  - Butterworth
  - Chebyshev
  - Elliptic
  - Bessel
  - Thompson
Elliptic Filters

Can be thought of as an extension of the CC approach by adding complex-conjugate zeros on the imaginary axis to increase the sharpness of the slope at the band edge.
Elliptic Filters

- Basic idea comes from the concept of a Chebyschev Rational Fraction
- Sometimes termed Cauer filters
Chebyshev Rational Fraction

A Chebyshev Rational Fraction is a rational fraction that is equal ripple in [-1,1] and equal ripple in [-∞,-1] and [1,∞]
Chebyshev Rational Fractions

Even-order CC rational fraction

Odd-order CC rational fraction
Chebyshev Rational Fractions

Even-order CC rational fraction

\[
C_{Rn}(x) = H \frac{\prod_{k=1}^{n/2} (x^2 - a_k)}{\prod_{k=1}^{n/2} (x^2 - b_k)}
\]

Odd-order CC rational fraction

\[
C_{Rn}(x) = H \frac{x \prod_{k=1}^{n/2} (x^2 - a_k)}{\prod_{k=1}^{n/2} (x^2 - b_k)}
\]
Elliptic Filters

Magnitude-Squared Elliptic Approximating Function

\[ H_{E}(\omega) = \frac{1}{1 + \varepsilon^2 C_{Rn}^2(\omega)} \]

Inverse mapping to \( T_{E}(s) \) exists

- For \( n \) even, \( n \) zeros on imaginary axis
- For \( n \) odd, \( n-1 \) zeros on imaginary axis
- Equal ripple in both pass band and stop band
- Analytical expression for poles and zeros not available
- Often choose to have less than \( n \) or \( n-1 \) zeros on imaginary axis
  (No longer based upon CC rational fractions)

Termed here “full order”
Elliptic Filters

- If of full-order, response completely characterized by \( \{n, \varepsilon, A_S, \Omega_S\} \)

- Any 3 of these parameters are independent

- Typically \( \varepsilon, \Omega_S, \) and \( A_S \) are fixed by specifications (i.e. must determine \( n \))
Elliptic Filters

\[
\frac{1}{\sqrt{1 + \varepsilon^2}}
\]

\[|T_E(j\omega)|\]

n odd

For full-order elliptic approximations

- (n-1)/2 peaks in pass band
- (n-1)/2 peaks in stop band
- Maximum occurs at \(\omega=0\)
- \(|T(j\infty)|=0\)
- \(|T(j0)|=1/\sqrt{1+\varepsilon^2}\)
- \(|T(j\infty)|=A_S\)

n even
Elliptic Filters

- Simple closed-form expressions for poles, zeros, and $|T_E(j\omega)|$ do not exist
- Simple closed form expressions for relationship between \{n,\epsilon,AS, and $\Omega_S$\} do not exist
- Simple expressions for max pole Q and slope at band edge do not exist
- Reduced-order elliptic approximations could be viewed as CC filters with zeros added to stop band
- General design tables not available though limited tables for specific characterization parameters do exist

\[ \frac{1}{\sqrt{1 + \epsilon^2}} \]
Elliptic Filters

Observations about Elliptic Filters

• Elliptic filters have steeper transitions than CC1 filters
• Elliptic filters do not roll off as quickly in stop band as CC1 or even BW
• Highest Pole-Q of elliptic filters is larger than that of CC filters
• For a given transition requirement, order of elliptic filter typically less than that of CC filter
• Cost of implementing elliptic filter is comparable to that of CC filter if orders are the same
• Cost of implementing a given filter requirement is often less with the elliptic filters
• Often need computer to obtain elliptic approximating functions though limited tables are available
• Some authors refer to elliptic filters as Cauer filters
Canonical Approximating Functions

Butterworth
Chebyshev
Transitional BW-CC
Elliptic
Thompson
Bessel

Thompson and Bessel Approximating Functions are Two Different Names for the Same Approximation
Thompson and Bessel Approximations

- All-pole filters
- Maximally linear phase at $\omega=0$
Thompson and Bessel Approximations

Consider $T(j\omega)$

$$T(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{N_R(j\omega) + jN_{IM}(j\omega)}{D_R(j\omega) + jD_{IM}(j\omega)}$$

$$\text{phase} = \angle(T(j\omega)) = \tan^{-1}\left(\frac{N_I(j\omega)}{N_R(j\omega)}\right) - \tan^{-1}\left(\frac{D_I(j\omega)}{D_R(j\omega)}\right)$$

- Phase expressions are difficult to work with
- Will first consider group delay and frequency distortion
Linear Phase

Consider $T(j\omega)$

$$T(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{N_R(j\omega) + jN_{IM}(j\omega)}{D_R(j\omega) + jD_{IM}(j\omega)}$$

$$\angle(T(j\omega)) = \tan^{-1}\left(\frac{N_I(j\omega)}{N_R(j\omega)}\right) - \tan^{-1}\left(\frac{D_I(j\omega)}{D_R(j\omega)}\right)$$

Defn: A filter is said to have linear phase if the phase is given by the expression

$$\angle(T(j\omega)) = \theta \omega$$ where $\theta$ is a constant that is independent of $\omega$
Distortion in Filters

Types of Distortion

**Frequency Distortion**
- Amplitude Distortion
- Phase Distortion

**Nonlinear Distortion**

Although the term “distortion” is used for these two basic classes, there is little in common between these two classes.
Distortion in Filters

Frequency Distortion
- Amplitude Distortion
  A filter is said to have frequency (magnitude) distortion if the magnitude of the transfer function changes with frequency
- Phase Distortion
  A filter is said to have phase distortion if the phase of the transfer function is not equal to a constant times $\omega$

Nonlinear Distortion
A filter is said to have nonlinear distortion if there is one or more spectral components in the output that are not present in the input
Distortion in Filters

- Phase and frequency distortion are concepts that apply to linear circuits.

- If frequency distortion is present, the relative magnitude of the spectral components that are present will be different than the spectral components in the output.

- If phase distortion is present, at least for some inputs, waveshape will not be preserved.

- Nonlinear distortion does not exist in linear networks and is often used as a measure of the linearity of a filter.

- No magnitude distortion will be present in a specific output of a filter if all spectral components that are present in the input are in a flat passband.

- No phase distortion will be present in a specific output of a filter if all spectral components that are present in the input are in a linear phase passband.

- Linear phase can occur even when the magnitude in the passband is not flat.

- Linear phase will still occur if the phase becomes nonlinear in the stopband.
Filter Passband and Stopband

Flat Passband

Non-Flat Passband

- Frequencies where gain is ideally 0 or very small is termed the stopband
- Frequencies where the gain is ideally not small is termed the passband
- Passband is often a continuous region in $\omega$ though could be split
Linear and Nonlinear Phase

Linear Passband Phase

Nonlinear Passband Phase
Preserving the Waveshape:

Example: Consider a signal $x(t) = \sin(\omega_1 t) + 0.25\sin(3\omega_1 t)$

Note the wave shape and spectral magnitude of $x(t)$
Preserving the waveshape

A filter has no frequency distortion for a given input if the output wave shape is preserved (i.e. the output wave shape is a magnitude scaled and possibly time-shifted version of the input).

Mathematically, no frequency distortion for $V_{IN}(t)$ if

$$V_{OUT}(t) = KV_{IN}(t-t_{shift})$$

Could have frequency distortion for other inputs.
Example of No Frequency Distortion

Input

Output

No Frequency Distortion
Example with Frequency Distortion

Input

Output

Phase Distortion (No Amplitude Distortion)
Example with Frequency Distortion

Input

```
|ω| 1 3
|---|
```

Output

```
|S| \frac{1}{\sqrt{2}}
|---|
```

Amplitude Distortion

(No Phase Distortion)
Example with Frequency Distortion

**Input**

**Output**

Amplitude and Phase Distortion
Example with Nonlinear Distortion and Frequency Distortion

Nonlinear distortion evidenced by presence of spectral components in output that are not in the input
Frequency Distortion

In most audio applications (and many other signal processing applications) there is little concern about phase distortion but some applications do require low phase distortion.

In audio applications, any substantive magnitude distortion in the pass band is usually not acceptable.

Any substantive nonlinear distortion in the pass band is unacceptable in most audio applications.
End of Lecture 12