Filter Transformations

Lowpass to Bandpass
Lowpass to Highpass
Lowpass to Band-reject
Theorem: If the perimeter variations and contact resistance are neglected, the standard deviation of the local random variations of a resistor of area $A$ is given by the expression

$$\sigma_{R_{N}} = \frac{A \rho}{\sqrt{A}}$$

Theorem: If the perimeter variations are neglected, the standard deviation of the local random variations of a capacitor of area $A$ is given by the expression

$$\sigma_{C_{N}} = \frac{A_{C}}{\sqrt{A}}$$

Theorem: If the perimeter variations are neglected, the variance of the local random variations of the normalized threshold voltage of a rectangular MOS transistor of dimensions $W$ and $L$ is given by the expression

$$\sigma^{2}_{\frac{V_{T}}{V_{TN}}} = \frac{A^{2}_{VTO}}{V^{2}_{T_{N}} WL} \quad \text{or as} \quad \sigma^{2}_{\frac{V_{T}}{V_{TN}}} = \frac{A^{2}_{VT}}{WL}$$
Theorem: If the perimeter variations are neglected, the variance of the local random variations of the normalized $C_{OX}$ of a rectangular MOS transistor of dimensions $W$ and $L$ is given by the expression

$$\sigma^2_{C_{OX}} = \frac{A_{COX}^2}{WL}$$

Theorem: If the perimeter variations are neglected, the variance of the local random variations of the normalized mobility of a rectangular MOS transistor of dimensions $W$ and $L$ is given by the expression

$$\sigma^2_{\mu_R \mu_N} = \frac{A_{\mu}^2}{WL}$$

where the parameters $A_X$ are all constants characteristic of the process (i.e. model parameters)

- The effects of edge roughness on the variance of resistors, capacitors, and transistors can readily be included but for most layouts is dominated by the area dependent variations
- There is some correlation between the model parameters of MOS transistors but they are often ignored to simplify calculations
Statistical Modeling of dimensionless parameters - example

\[ K = 1 + \frac{R_2}{R_1} \]

Assume common centroid layout
area of \( R_1 \) is 100\( u^2 \) \( A_p = .01 \mu m \)

Determine the yield if the nominal gain is \( 10 \pm 1\% \)

\[ \frac{K}{K_N} \approx N(1, 0.00095) \]

\[ 9.9 < K < 10.1 \]

\[ .99 < \frac{K}{K_N} < 1.01 \]

\[ -.01 < \frac{K}{K_N} - 1 < .01 \]

The gain yield is essentially 100%.

Could substantially decrease area or increase gain accuracy if desired.
Statistical Modeling of dimensionless parameters - example

\[ K = 1 + \frac{R_2}{R_1} \]

\[ A_p = 0.025 \text{um} \quad A_{R_1} = 10 \text{um}^2 \]

\[ \sigma_{\frac{R_{PROC}}{R_{NOM}}} = 0.2 \]

Determine the yield if the nominal gain is 10 ±1%

\[ \frac{K}{K_N} \sim N(1, 0.0075) \]

\[ 9.9 < K < 10.1 \]

\[ 0.99 < \frac{K}{K_N} < 1.01 \]

\[ -0.01 < \frac{K}{K_N} - 1 < 0.01 \]

\[ Y = 2F_{N(0,1)}(1.33) - 1 = 2 \times 0.9082 - 1 = 0.8164 \]

Dramatic drop from 100% yield to about 82% yield!
Review from Last Time

Statistical Modeling of Filter Characteristics

The variance of dimensioned filter parameters (e.g. $\omega_0$, poles, band edges, ...) is often very large due to the process-level random variables which dominate.

The variance of dimensionless filter parameters (e.g. Q, gain, ...) are often quite small since in a good design they will depend dominantly on local random variations which are much smaller than process-level variations.

The variance of dimensionless filter parameters is invariably proportional to the reciprocal of the square root of the relevant area and thus can be managed with appropriate area allocation.
Linearization of Functions of a Random Variable

- Characteristics of most circuits of interest are themselves random variables
- Relationship between characteristics and the random variables often highly nonlinear
- Ad Hoc manipulations (repeated Taylor’s series expansions) were used to linearize the characteristics in terms of the random variables

\[ Y \approx Y_N + \sum_{i=1}^{n} (a_i x_R) \]

- This is important because if the random variables are uncorrelated the variance of the characteristic can be readily obtained

\[
\sigma_Y^2 \approx \sum_{i=1}^{n} (a_i^2 \sigma_{x_R}^2)
\]

\[
\sigma_{Y_N}^2 \approx \frac{1}{Y_N^2} \sum_{i=1}^{n} (a_i^2 \sigma_{x_R}^2)
\]

- This approach was applicable since the random variables are small
- These Ad Hoc manipulations can be formalized and this follows
Formalization of Statistical Analysis

Consider a function of interest $Y$

$$Y = f(x_{1N}, x_{2N}, \ldots, x_{nN}, x_{1R}, x_{2R}, \ldots, x_{nR}) = f([X_N], [X_R])$$

This can be expressed in a multi-variate power series as

$$Y = f([X_N], [X_R]) + \sum_{i=1}^{n} \left( \frac{\partial f}{x_{Ri}} \bigg|_{[X_N],[x_R]=[0]} \right) \cdot x_{Ri} + \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{\partial^2 f}{x_{Ri} x_{Rj}} \bigg|_{[X_N],[x_R]=[0]} \right) \cdot x_{Ri} x_{Rj} + \ldots$$

If the random variables are small compared to the nominal variables

$$Y \approx f([X_N], [X_R]) + \sum_{i=1}^{n} \left( \frac{\partial f}{x_{Ri}} \bigg|_{[X_N],[x_R]=[0]} \right) \cdot x_{Ri}$$

If the random variable are uncorrelated, it follows that

$$\sigma_y^2 = \sum_{i=1}^{n} \left( \left[ \frac{\partial f}{x_{Ri}} \bigg|_{[X_N],[x_R]=[0]} \right]^2 \cdot \sigma_{x_{Ri}}^2 \right)$$

$$\sigma_y^2 \frac{Y}{Y_N} = \frac{1}{Y_N} \sum_{i=1}^{n} \left( \left[ \frac{\partial f}{x_{Ri}} \bigg|_{[X_N],[x_R]=[0]} \right]^2 \cdot \sigma_{x_{Ri}}^2 \right)$$
Filter Transformations

Lowpass to Bandpass \hspace{2cm} \text{(LP to BP)}
Lowpass to Highpass \hspace{2cm} \text{(LP to HP)}
Lowpass to Band-reject \hspace{2cm} \text{(LP to BR)}

Approach will be to take advantage of the results obtained for the standard LP approximations

Will focus on flat passband and zero-gain stop-band transformations

Will focus on transformations that map passband to passband and stopband to stopband
Filter Transformations

If the imaginary axis in the s-plane is mapped to the imaginary axis in the s-plane with a variable mapping function, the basic shape of the function $T(s)$ will be preserved in the function $F(T(s))$ but the frequency axis may be warped and/or folded in the magnitude domain.

Claim:

If the imaginary axis in the s-plane is mapped to the imaginary axis in the s-plane with a variable mapping function, the basic shape of the function $T(s)$ will be preserved in the function $F(T(s))$ but the frequency axis may be warped and/or folded in the magnitude domain.

Preserving basic shape, in this context, constitutes maintaining features in the magnitude response of $F(T(s))$ that are in $T(s)$ including, but not limited to, the peak amplitude, number of ripples, peaks of ripples,
Example: Shape Preservation

Original Function

Shape Preserved

Shape Preserved
Example: Shape Preservation

Original Function

Shape Not Preserved
Flat Passband/Stopband Filters

\[
\begin{align*}
|T(j\omega)| & \quad \text{Lowpass} \\
|T(j\omega)| & \quad \text{Bandpass} \\
|T(j\omega)| & \quad \text{Highpass} \\
|T(j\omega)| & \quad \text{Bandreject}
\end{align*}
\]
Filter Transformations

- Lowpass to Bandpass (LP to BP)
- Lowpass to Highpass (LP to HP)
- Lowpass to Band-reject (LP to BR)

• Approach will be to take advantage of the results obtained for the standard LP approximations

• Will focus on flat passband and zero-gain stop-band transformations

• Will focus on transformations that map passband to passband, stopband to stopband, and Im axis to Im axis
LP to BP Filter Transformations

\[ T_{BP}(s) = T_{LPN}(f(s)) \]

Will consider rational fraction mappings

\[ f(s) = \sum_{i=0}^{m_T} a_{Ti}s^i - \sum_{i=0}^{n_T} b_{Ti}s^i \]

- Not all rational fraction mappings will map Im axis to the Im axis
- Not all rational fraction mappings will map passband to passband and stopband to stopband
- Consider only that subset of those mappings with these properties
LP to BP Transformation

Mapping Strategy: Consider first a mapping to a normalized BP approximation

\[ \left| T_{LPN}(j\omega) \right| \]

\[ \left| T_{BPN}(j\omega) \right| \]

\[ s \rightarrow f(s) \]

Normalized

\[ BW_N = \omega_{BN} - \omega_{AN} \]

\[ \sqrt{\omega_{AN} \omega_{BN}} = 1 \]
LP to BP Transformation

Mapping Strategy: Consider first a mapping to a normalized BP approximation

A mapping from \( s \rightarrow f(s) \) will map the entire imaginary axis

Thus, must consider both positive and negative frequencies. Since \( |T(j\omega)| \) is a function of \( \omega^2 \), the magnitude response on the negative \( \omega \) axis will be a mirror image of that on the positive \( \omega \) axis

\[
|T(j\omega)| \quad \text{s} \rightarrow f(s) \quad |T_{BP}(j\omega)|
\]

\[
\begin{align*}
|T(j\omega)| &= 1 \\
\omega &= -\omega_{AN} \quad -1 \rightarrow 1 \\
\text{Normalized} \\
T_{BP}(j\omega)| &= 1 \\
\omega &= \omega_{AN} \quad -1 \rightarrow 1 \\
\text{BW}_N \\
\text{BW}_N &= \omega_{BN} - \omega_{AN} \\
\sqrt{\omega_{AN} \omega_{BN}} &= 1
\end{align*}
\]
Standard LP to BP Transformation

Normalized LP to Normalized BP mapping Strategy:

- \( T(j\omega) \)
- \( T_{BP}(j\omega) \)

Variable Mapping Strategy to Preserve Shape of LP function:

Consider:

- s-domain
  - map \( s=j0 \) to \( s=j1 \)
  - map \( s=j1 \) to \( s=j\omega_{BN} \)
  - map \( s=-j1 \) to \( s=j\omega_{AN} \)

- \( T_{LPN}(f(s)) \)

- \( T_{BPN}(s) \)

This mapping will introduce 3 constraints.
**Standard LP to BP Transformation**

Mapping Strategy:

- **s-domain**
  - map $s=0$ to $s=j1$
  - map $s=j1$ to $s=j\omega_{BN}$
  - map $s=-j1$ to $s=j\omega_{AN}$

- **ω-domain**
  - map $\omega=0$ to $\omega=1$
  - map $\omega=1$ to $\omega=\omega_{BN}$
  - map $\omega=-1$ to $\omega=\omega_{AN}$

Consider variable mapping

$$f(s) = \frac{a_{T2}s^2 + a_{T1}s + a_{T0}}{b_{T1}s + b_{T0}}$$

With this mapping, there are 5 D.O.F and 3 mathematical constraints and the additional constraints that the Im axis maps to the Im axis and maps PB to PB and SB to SB

Will now show that the following mapping will meet these constraints

$$f(s) = \frac{s^2 + 1}{s \cdot BW_n}$$

or equivalently

$$s \rightarrow \frac{s^2 + 1}{s \cdot BW_n}$$

This is the lowest-order mapping that will meet these constraints and it doubles the order of the approximation.
Standard LP to BP Transformation

s-domain
map s=0 to s= j1
map s=j1 to s=jω_{BN}
map s= -j1 to s= jω_{AN}

ω-domain
T_{LPN}(f(s))
map ω=0 to ω=1
map ω=1 to ω=ω_{BN}
map ω= -1 to ω=ω_{AN}

Verification of mapping Strategy:

\[ \frac{s^2 + 1}{s \cdot BW} \bigg|_{s=j1} = 0 \]
\[ \frac{s^2 + 1}{s \cdot BW} \bigg|_{s=jω_{BN}} = \frac{1-ω_{BN}^2}{jω_{BN}(ω_{BN} - ω_{AN})} = j \frac{ω_{BN}^2 - 1}{ω_{BN}^2 - ω_{AN}ω_{BN}} = j ω_{BN}^2 - 1 = j \]
\[ \Rightarrow \quad j1 \rightarrow jω_{BN} \]

\[ \frac{s^2 + 1}{s \cdot BW} \bigg|_{s=jω_{AN}} = \frac{1-ω_{AN}^2}{jω_{AN}(ω_{BN} - ω_{AN})} = j \frac{ω_{AN}^2 - 1}{ω_{AN}^2 - ω_{BN}ω_{AN}} = j ω_{AN}^2 - 1 = -j \]
\[ \Rightarrow \quad -j1 \rightarrow jω_{AN} \]

Must still show that the Im axis maps to the Im axis and maps PB to PB and SB to SB
Standard LP to BP Transformation

s-domain
map $s=0$ to $s=j1$
map $s=j1$ to $s=j\omega_{BN}$
map $s=-j1$ to $s=j\omega_{AN}$

$T_{LPN}(f(s))$
map $\omega=0$ to $\omega=1$
map $\omega=1$ to $\omega=\omega_{BN}$
map $\omega=-1$ to $\omega=\omega_{AN}$

Verification of mapping Strategy:

Image of $\text{Im}$ axis:

\[
j\omega = \frac{s^2 + 1}{s \cdot BW_N}
\]

solving for $s$, obtain

\[
s = \frac{j\omega \cdot BW_N \pm \sqrt{(BW_N \cdot j\omega)^2 - 4}}{2}
= j\left(\frac{\omega \cdot BW_N \pm \sqrt{(BW_N \cdot \omega)^2 + 4}}{2}\right)
\]

this has no real part so the imaginary axis maps to the imaginary axis

Can readily show this mapping maps PB to PB and SB to SB

The mapping $s \rightarrow \frac{s^2 + 1}{s \cdot BW_N}$ is termed the standard LP to BP transformation
The standard LP to BP transformation

\[ s \rightarrow \frac{s^2 + 1}{s \cdot BW_N} \]

If we add a subscript to the LP variable for notational convenience, can express this mapping as

\[ s_x = \frac{s^2 + 1}{s \cdot BW_N} \]

**Question:** Is this mapping dimensionally consistent?

- The dimensions of the constant “1” in the numerator must be set so that this is dimensionally consistent.

- The dimensions of BW\_N must be set so that this is dimensionally consistent.
Standard LP to BP Transformation

\[ T_{LPN}(s) \]

\[ \frac{s}{s + 1} \]

\[ T_{BPN}(s) \]

\[ T_{LP}(j\omega) \]

\[ T_{BP}(j\omega) \]

\[ -\omega_{AN} \quad -1 \quad -\omega_{BN} \quad \omega_{AN} \quad 1 \quad \omega_{BN} \]
Standard LP to BP Transformation

Frequency and s-domain Mappings

(subscript variable in LP approximation for notational convenience)

\[ T_{LPN}(s_x) \]

\[ \frac{s_x}{s \cdot BW_N} \]

\[ T_{BPN}(s) \]

\[ \frac{s^2 + 1}{s \cdot BW_N} \]

\[ s_x \rightarrow \frac{s^2 + 1}{s \cdot BW_N} \]

\[ \omega_x \rightarrow \frac{\omega^2 - 1}{\omega \cdot BW_N} \]

solving for s or \( \omega \)

\[ s \leftarrow \frac{s_x \cdot BW_N \pm \sqrt{(BW_N \cdot s_x)^2 - 4}}{2} \]

\[ \omega \leftarrow \frac{\omega_x \cdot BW_N \pm \sqrt{(BW_N \cdot \omega_x)^2 + 4}}{2} \]

Exercise: Resolve the dimensional consistency in the last equation
Standard LP to BP Transformation

Denormalized Mapping

\[ T_{LPN}(s) = \frac{s^2 + \omega_M^2}{s \cdot BW} \]

\[ T_{BP}(s) = s + \omega \]

\[ |T_{LP}(j\omega)| \]

\[ |T_{BP}(j\omega)| \]
Standard LP to BP Transformation

Frequency and s-domain Mappings - Denormalized

(subscript variable in LP approximation for notational convenience)

\[ T_{LPN}(s_x) \]

\[ s_x \downarrow \]

\[ \frac{s^2 + \omega^2_M}{s \cdot BW} \]

\[ T_{BP}(s) \]

\[ s_x \rightarrow \frac{s^2 + \omega^2}{s \cdot BW} \]

\[ \omega_x \rightarrow \frac{\omega^2 - \omega^2_M}{\omega \cdot BW} \]

\[ \begin{align*}
    s &\leftarrow \frac{s_x \cdot BW \pm \sqrt{(BW \cdot s_x)^2 - 4\omega^2_M}}{2} \\
    \omega &\leftarrow \frac{\omega_x \cdot BW \pm \sqrt{(BW \cdot \omega_x)^2 + 4\omega^2_M}}{2}
\end{align*} \]

Exercise: Resolve the dimensional consistency in the last equation
Standard LP to BP Transformation

Frequency and s-domain Mappings - Denormalized

(subscript variable in LP approximation for notational convenience)

$T_{LPN}(s_x)$

$\frac{s_x}{s}$

$\frac{s_N}{W_N}$

$T_{LP}(s)$

$\frac{s}{s^2+1}$

$\frac{s}{s+N}$

$T_{BP}(s)$

$T_{LPN}(s_x)$

$\frac{s_x^2+1}{s_N\cdot BW_N}$

$\frac{s_N}{W_N}$

$T_{BPN}(s_N)$

$\frac{s}{s^2+\omega_m^2}$

$\frac{s}{BW}$

$T_{BP}(s)$

Which is most practical to use? Often none of them!
Standard LP to BP Transformation

Frequency and s-domain Mappings - Denormalized

(subscript variable in LP approximation for notational convenience)

\[ T_{LPN}(s_x) \]

\[ \frac{s^2 + 1}{s_N + 1} \]

\[ T_{BPN}(s_N) \]

Often most practical to synthesize directly from the \( T_{BPN} \) and then do the frequency scaling of components at the circuit level rather than at the approximation level.
Standard LP to BP Transformation

Frequency and s-domain Mappings

(subscript variable in LP approximation for notational convenience)

Poles and Zeros of the BP approximations

\[ s_x \rightarrow \frac{s^2 + 1}{s \cdot BW_N} \]

solving for \( s \)

\[ T_{BP}(s) = T_{LPN}(f(s)) \]

\[ T_{LPN}(p_x) = 0 \]

\[ T_{LPN}(f(p)) = 0 \]

\[ T_{BP}(p) = T_{LPN}(f(p)) = 0 \]

Since this relationship maps the complex plane to the complex plane, it also maps the poles and zeros of the LP approximation to the poles and zeros of the BP approximation.
Standard LP to BP Transformation

Pole Mappings

Claim: With a variable mapping transform, the variable mapping naturally defines the mapping of the poles of the transformed function

Exercise: Resolve the dimensional consistency in the last equation
Standard LP to BP Transformation

Pole Mappings

\[ p \leftarrow \frac{p_x \cdot BW_N \pm \sqrt{(BW_N \cdot p_x)^2 - 4}}{2} \]

Image of the cc pole pair is the two pairs of poles
Can show that the upper hp pole maps to one upper hp pole and one lower hp pole as shown. Corresponding mapping of the lower hp pole is also shown.
Standard LP to BP Transformation

Pole Mappings

\[ p \leftarrow \frac{p_x \cdot BW_N \pm \sqrt{(BW_N \cdot p_x)^2 - 4}}{2} \]

Note doubling of poles, addition of zeros, and likely Q enhancement
End of Lecture 15