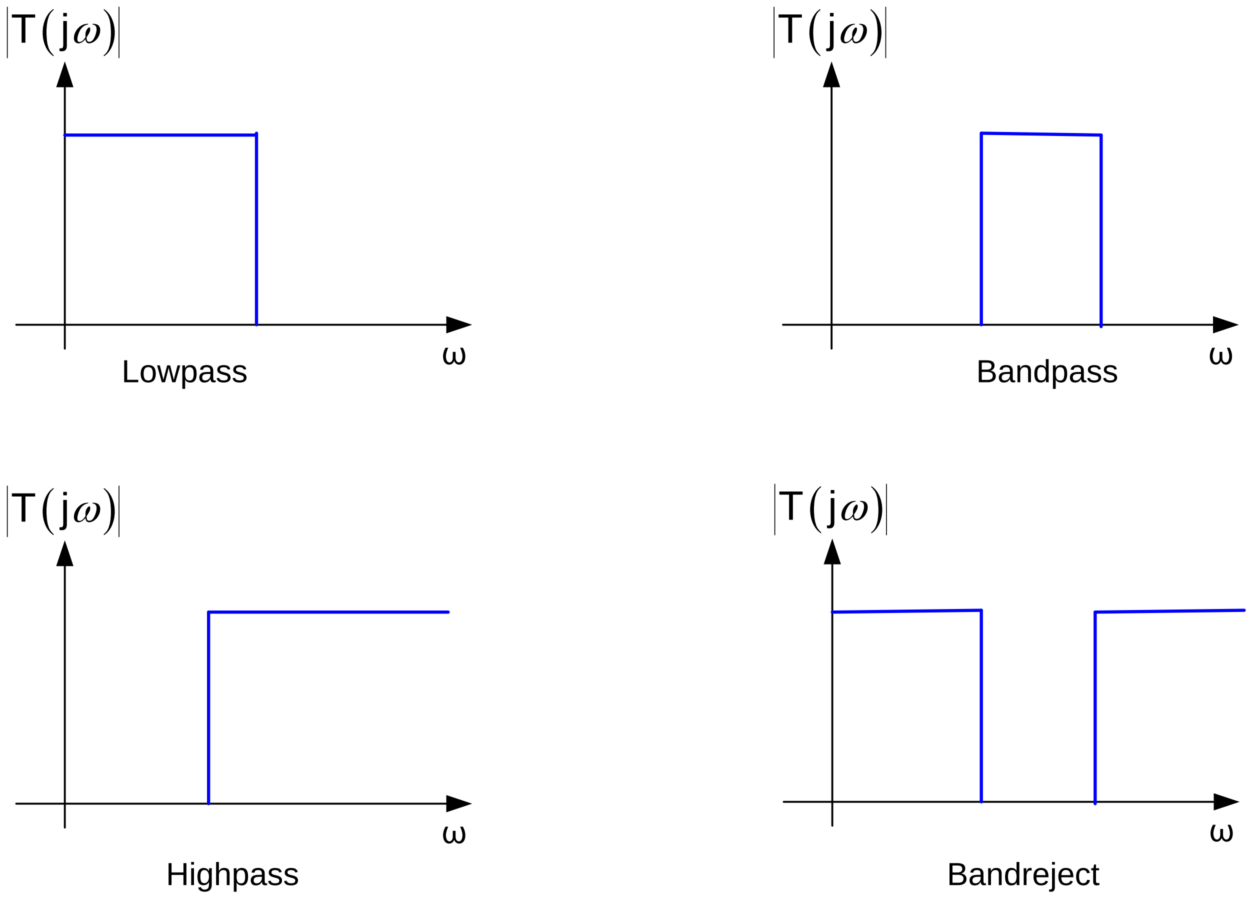
EE 508
Lecture 16

Filter Transformations

Lowpass to Bandpass
Lowpass to Highpass
Lowpass to Band-reject

Review from Last Time

Flat Passband/Stopband Filters



Review from Last Time

Standard LP to BP Transformation

s-domain

map s=0 to s= j1
map s=j1 to s=$j\omega_{BN}$
map s= –j1 to s= $j\omega_{AN}$

$T_{LPN}(f(s))$

ω-domain

map ω=0 to ω=1
map ω=1 to $\omega=\omega_{BN}$
map ω= –1 to $\omega=\omega_{AN}$

Verification of mapping Strategy:

$$s \rightarrow \frac{s^2+1}{s \bullet BW_N}$$

Image of Im axis:

$$j\omega = \frac{s^2+1}{s \bullet BW_N}$$

solving for s, obtain

$$s = \frac{j\omega \bullet BW_N \pm \sqrt{(BW_N \bullet j\omega)^2 - 4}}{2} = j\left(\frac{\omega \bullet BW_N \pm \sqrt{(BW_N \bullet \omega)^2 + 4}}{2}\right)$$

this has no real part so the imaginary axis maps to the imaginary axis

Can readily show this mapping maps PB to PB and SB to SB

The mapping $s \rightarrow \frac{s^2+1}{s \bullet BW_N}$ is termed the standard LP to BP transformation

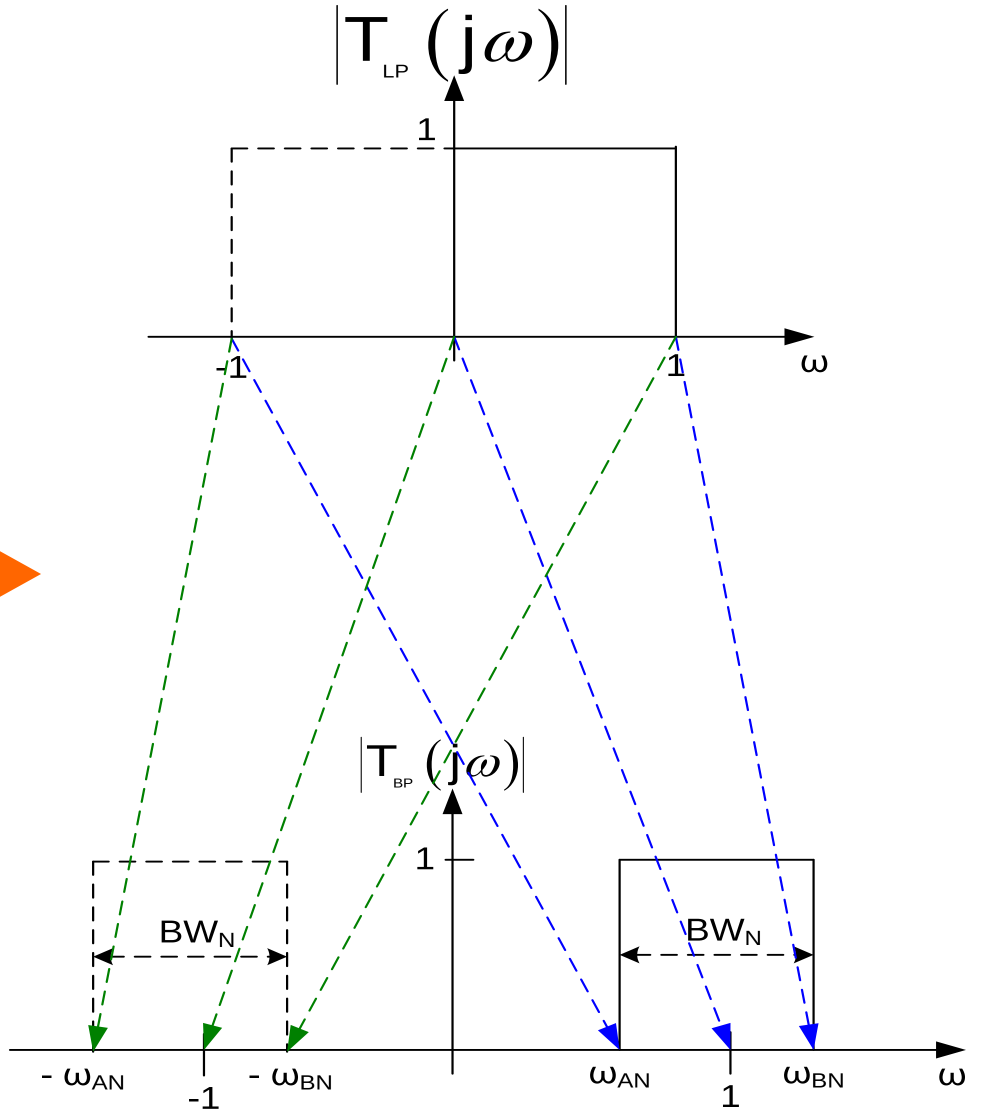
Review from Last Time

Standard LP to BP Transformation

$T_{LPN}(s)$

$$s \rightarrow \frac{s^2+1}{s \cdot BW_N}$$

$T_{BPN}(s)$

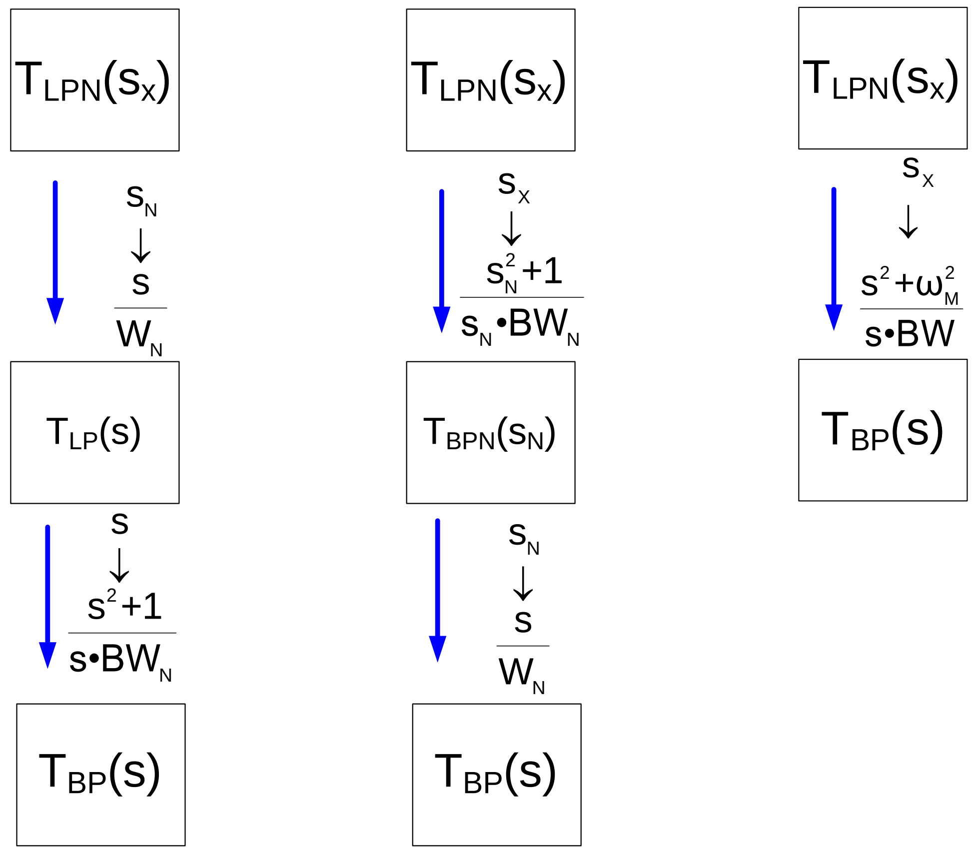


Review from Last Time

Standard LP to BP Transformation

Frequency and s-domain Mappings - Denormalized

(subscript variable in LP approximation for notational convenience)



All three approaches give same approximation

Which is most practical to use? Often none of them !

Review from Last Time

Standard LP to BP Transformation

Frequency and s-domain Mappings - Denormalized

(subscript variable in LP approximation for notational convenience)

$$T_{LPN}(s_X)$$

$$s_X \rightarrow \frac{s_N^2+1}{s_N \bullet BW_N}$$

$$T_{BPN}(s_N)$$

Often most practical to synthesize directly from the T_{BPN} and then do the frequency scaling of components at the circuit level rather than at the approximation level

Review from Last Time

Standard LP to BP Transformation

Pole Mappings

$$p \leftarrow \frac{p_X \bullet BW_N \pm \sqrt{\left(BW_N \bullet p_X\right)^2 - 4}}{2}$$

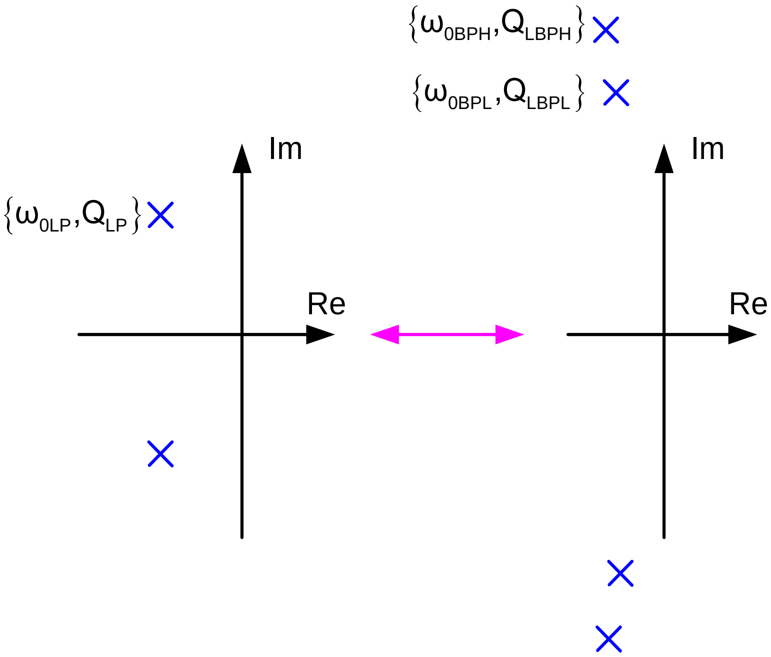
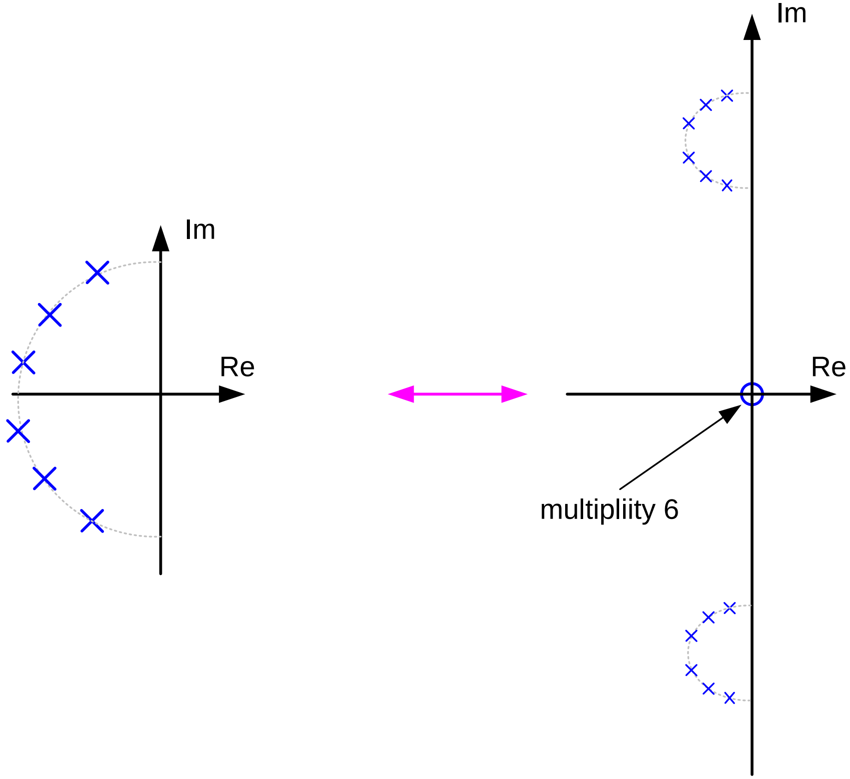


Image of the cc pole pair is the two pairs of poles

Standard LP to BP Transformation

Pole Mappings

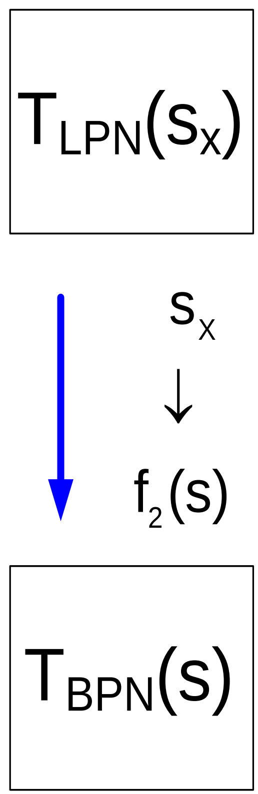
$$p \leftarrow \frac{p_X \bullet BW_N \pm \sqrt{\left(BW_N \bullet p_X\right)^2 - 4}}{2}$$



Note doubling of poles, addition of zeros, and likely Q enhancement

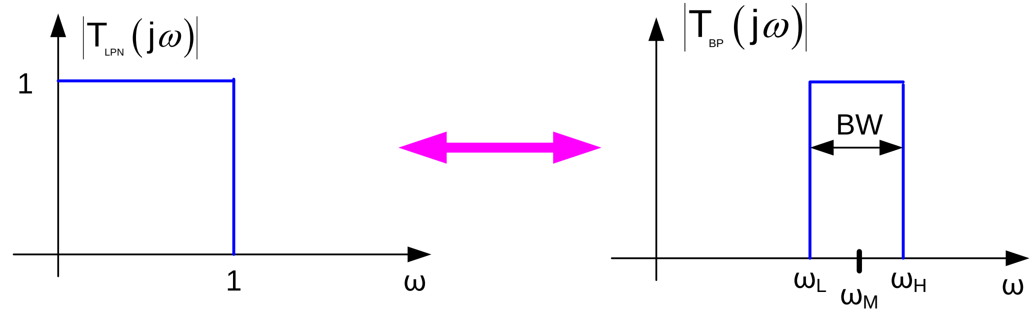
LP to BP Transformation

Claim: Other variable mapping transforms exist that satisfy the imaginary axis mapping properties needed to obtain the LP to BP transformation but are seldom, if ever, discussed. The Standard LP to BP transform Is by far the most popular and most authors treat it as if it is unique.



LP to BP Transformation

Pole Q of BP Approximations



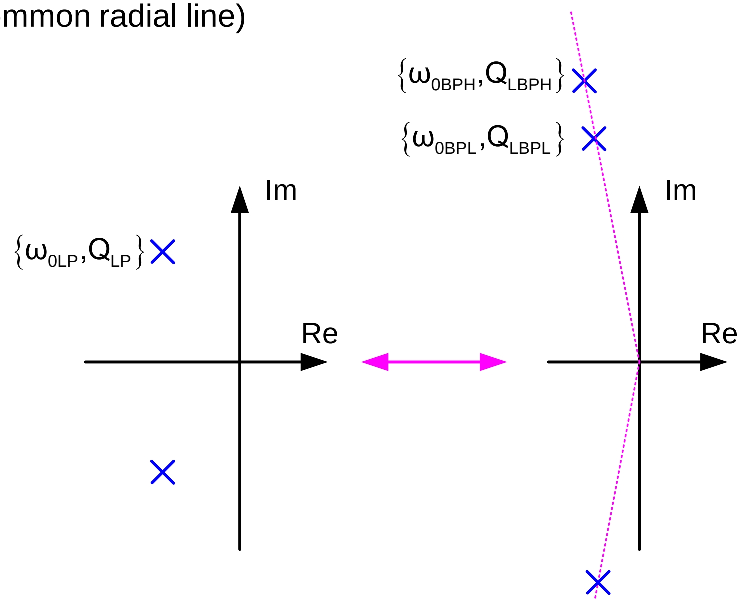
$$BW = \omega_H - \omega_L$$

$$\omega_M = \sqrt{\omega_H \omega_L}$$

Consider a pole in the LP approximation characterized by $\{\omega_{0LP}, Q_{LP}\}$

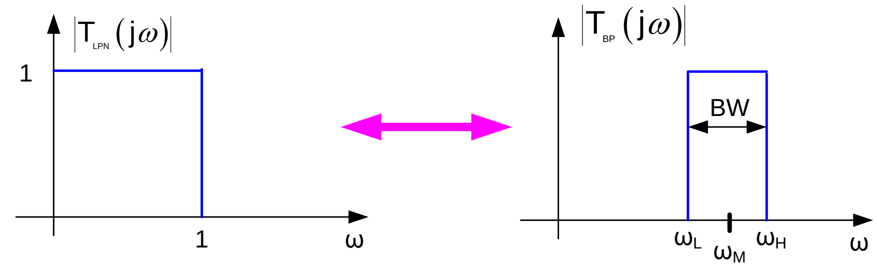
It can be shown that the corresponding BP poles have the same Q

(i.e. both bp poles lie on a common radial line)



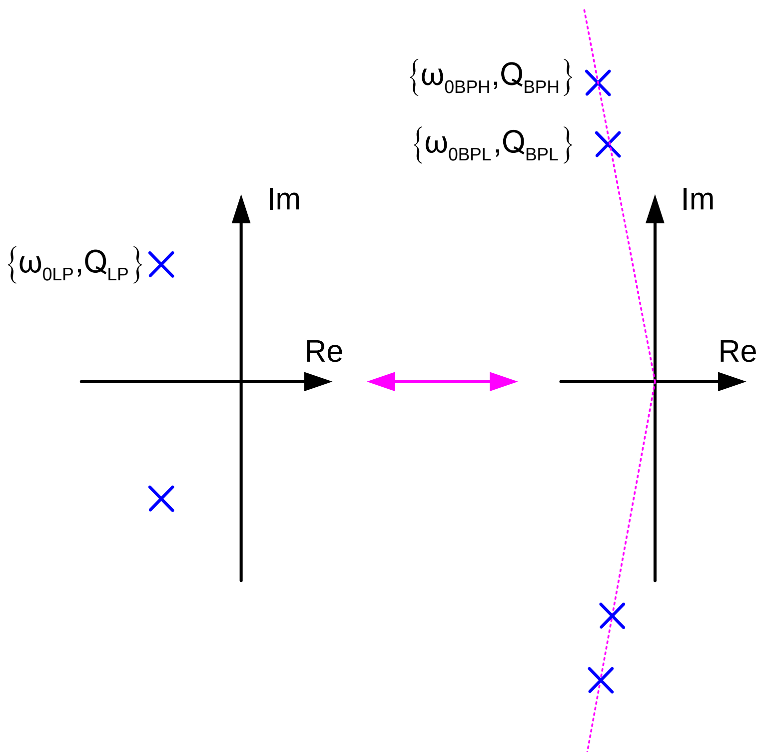
LP to BP Transformation

Pole Q of BP Approximations



$$BW = \omega_H - \omega_L$$

$$\omega_M = \sqrt{\omega_H \omega_L}$$



Define: $$\delta = \left(\frac{BW}{\omega_M}\right)\omega_{0LP}$$

It can be shown that

$$Q_{BPL} = Q_{BPH} = \frac{Q_{LP}}{\sqrt{2}}\sqrt{1+\frac{4}{\delta^2}+\sqrt{\left(1+\frac{4}{\delta^2}\right)^2-\frac{4}{\delta^2 Q_{2LP}^2}}}$$

For δ small, $$Q_{BP} \simeq \frac{2Q_{LP}}{\delta}$$

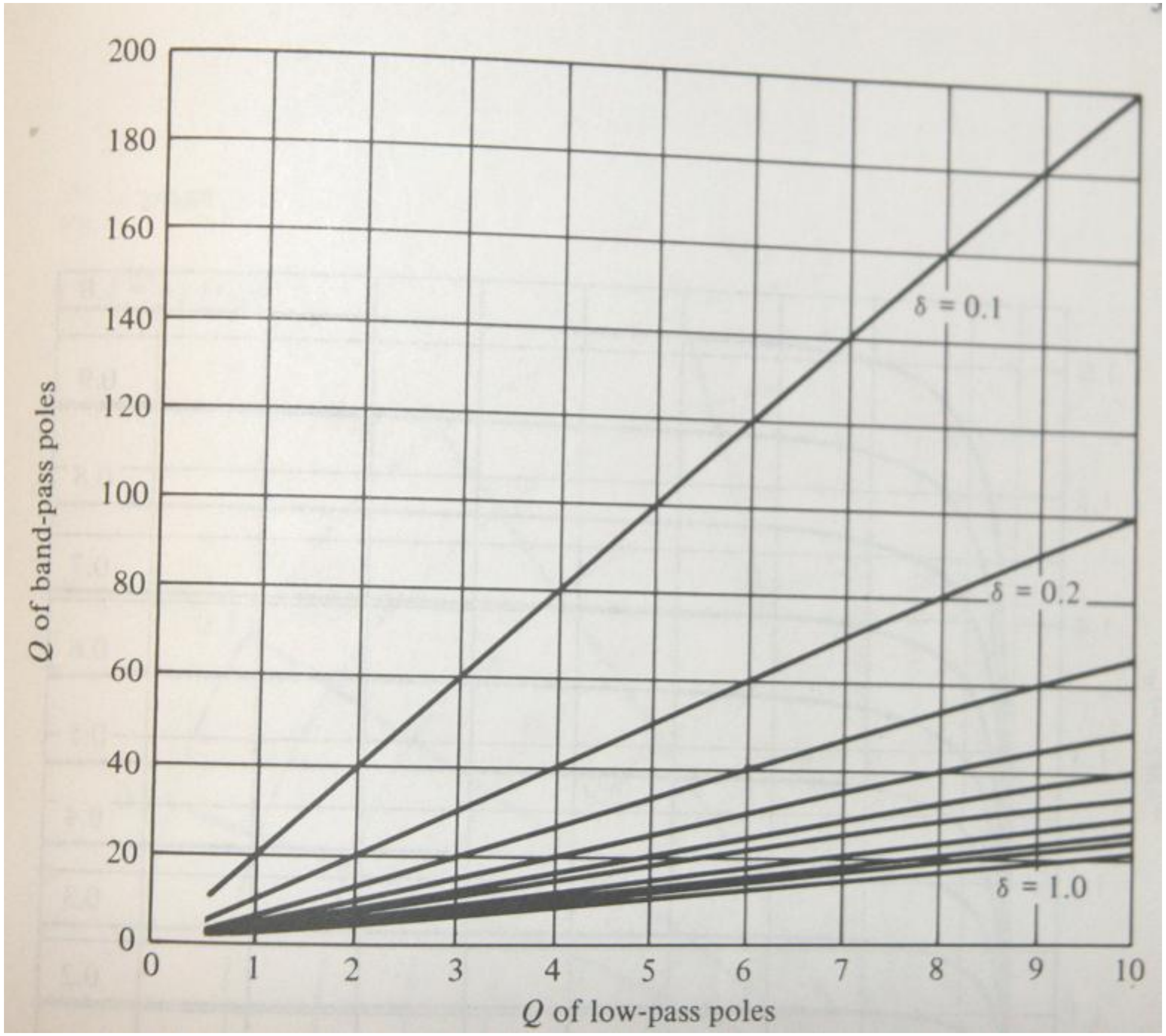
It can be shown that

$$\omega_{0BP} = \frac{\omega_M}{2}\left[\delta\frac{Q_{BP}}{Q_{LP}} \pm \sqrt{\left(\delta\frac{Q_{BP}}{Q_{LP}}\right)^2 - 4}\right]$$

Note for δ **small, Q_{BP} can get very large**

LP to BP Transformation

Pole Q of BP Approximations



$$\delta = \left(\frac{BW}{\omega_M}\right)\omega_{0LP}$$

$$Q_{BPL} = Q_{BPH} = \frac{Q_{LP}}{\sqrt{2}}\sqrt{1+\frac{4}{\delta^2}+\sqrt{\left(1+\frac{4}{\delta^2}\right)^2-\frac{4}{\delta^2 Q_{2LP}^2}}}$$

LP to BP Transformation

Pole locations vs Q_{LP} and δ

$$\delta = \left(\frac{BW}{\omega_M}\right)\omega_{0LP}$$

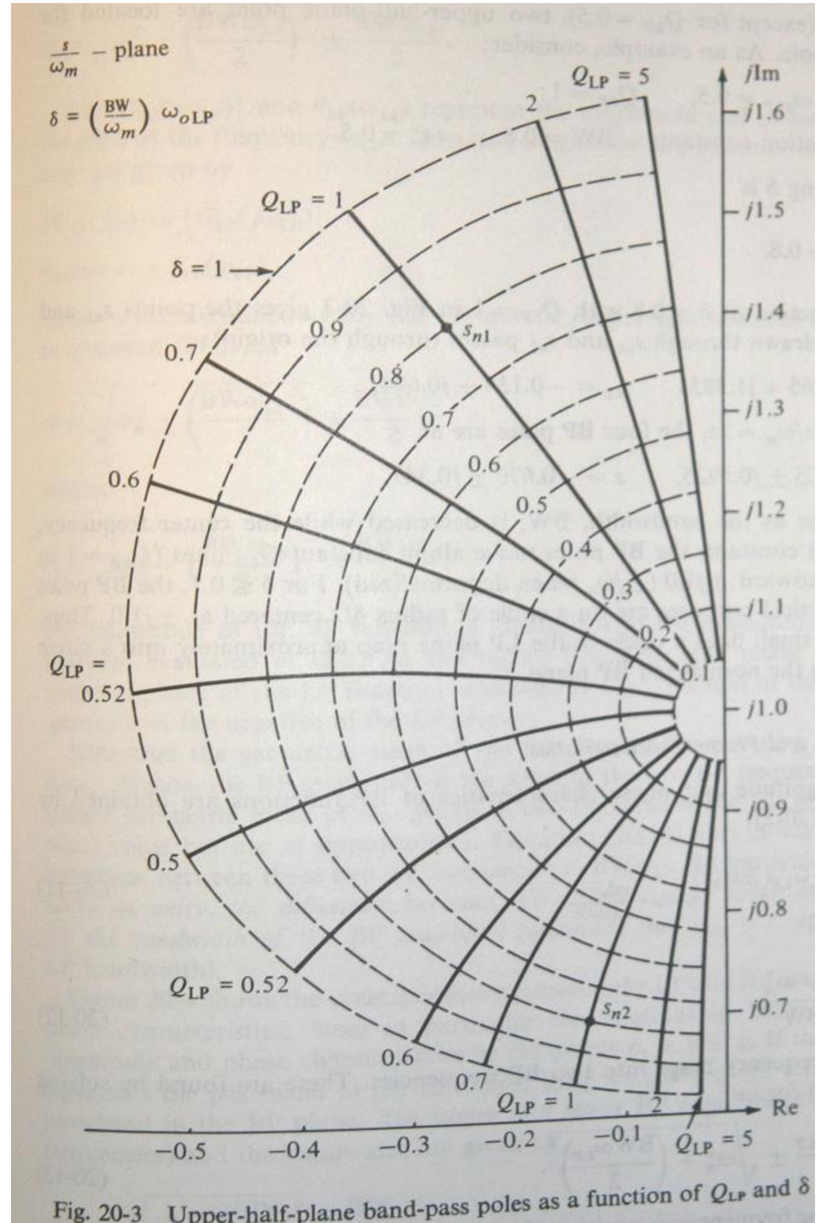
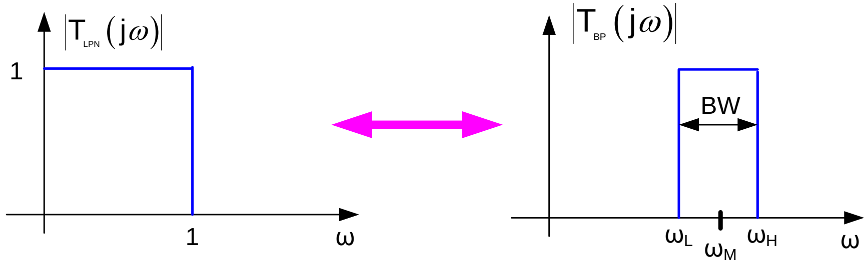


Fig. 20-3 Upper-half-plane band-pass poles as a function of Q_{LP} and δ

LP to BP Transformation



Classical BP Approximations

Butterworth
Chebyschev
Elliptic
Bessel

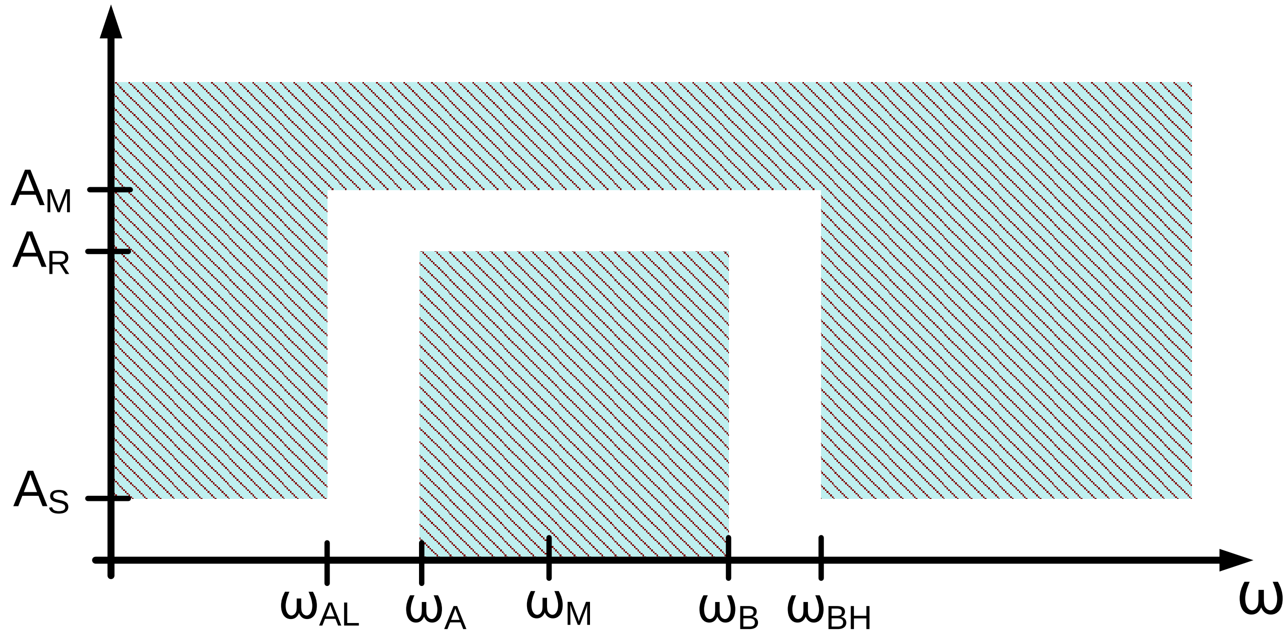
Obtained by the LP to BP transformation of the corresponding LP approximations

Standard LP to BP Transformation

$$s \rightarrow \frac{s^2 + 1}{s \bullet BW_N}$$

- Standard LP to BP transform is a variable mapping transform
- Maps jω axis to jω axis
- Maps LP poles to BP poles
- Preserves basic shape but warps frequency axis
- Doubles order
- Pole Q of resultant band-pass functions can be very large for narrow pass-band
- Sequencing of frequency scaling and transformation does not affect final function

Example 1: Obtain an approximation that meets the following specifications



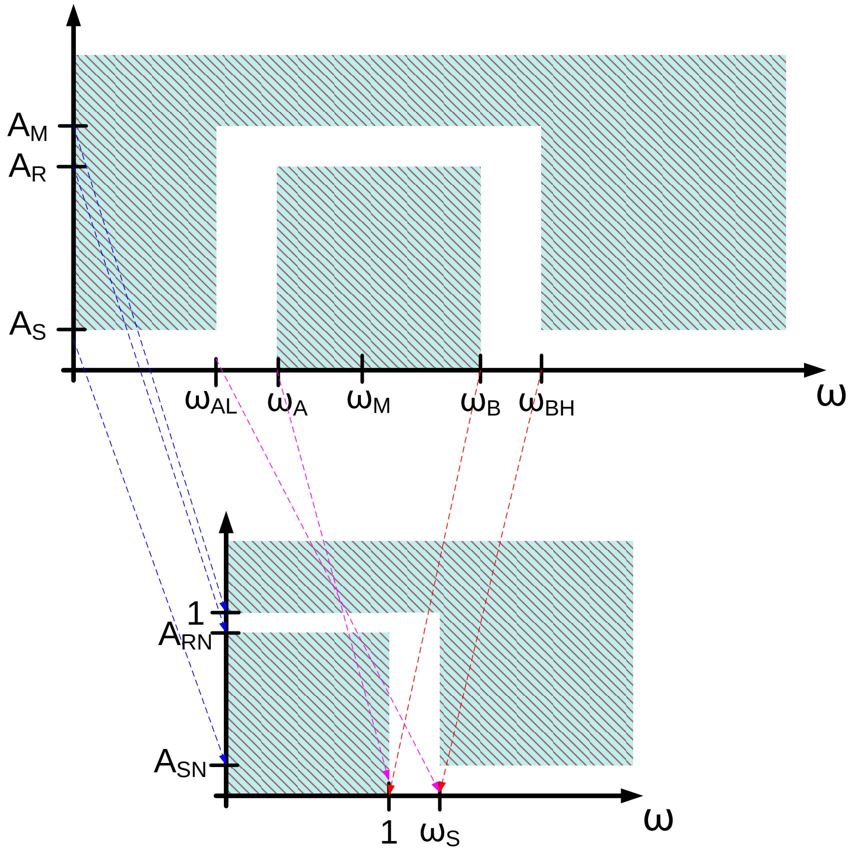
$$BW=\omega_B-\omega_A$$

$$\omega_M=\sqrt{\omega_B\bullet\omega_A}$$

Assume that ω_{AL}, ω_{BH} and ω_M satsify

$$\frac{\omega_M^2-\omega_{AL}^2}{\omega_{AL}\bullet BW}=\frac{\omega_{BH}^2-\omega_M^2}{\omega_{BH}\bullet BW}$$

Example 1: Obtain an approximation that meets the following specifications



$$A_{RN}=\frac{A_R}{A_M}$$

$$A_{SN}=\frac{A_S}{A_M}$$

$$\frac{1}{\sqrt{1+\varepsilon^2}}=\frac{A_R}{A_M}$$

$$\varepsilon=\sqrt{\left(\frac{A_M}{A_R}\right)^2-1}$$

$$\omega_S=\frac{\omega_M^2-\omega_{AL}^2}{\omega_{AL}\bullet BW}$$

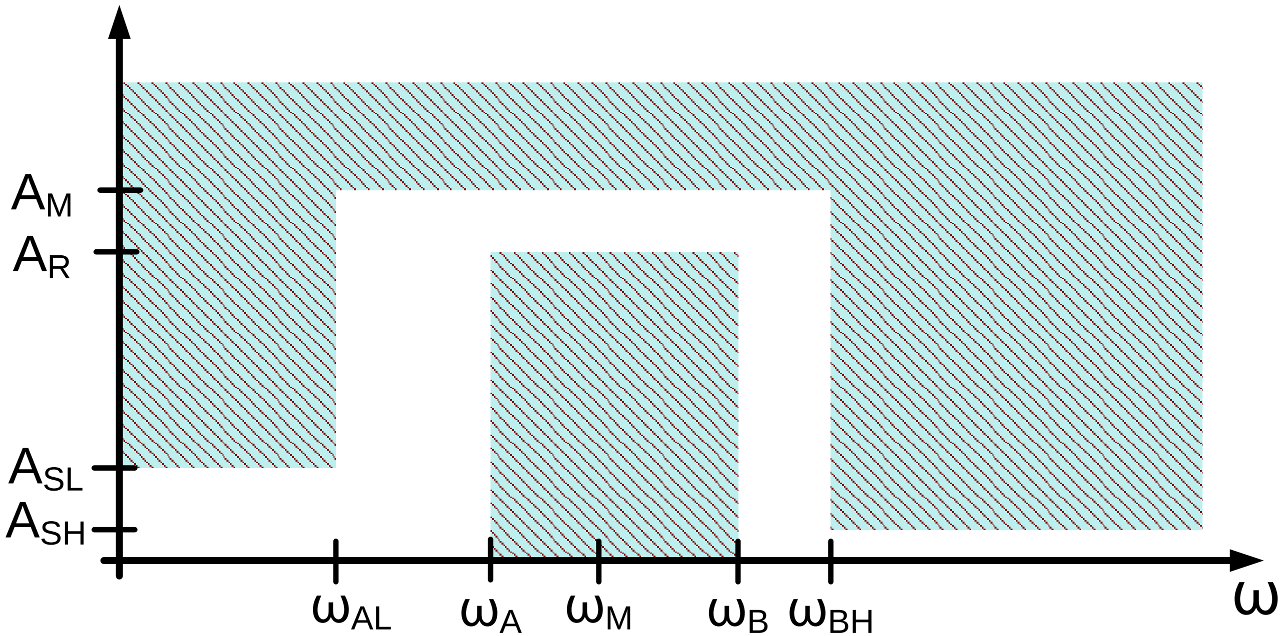
$$BW=\omega_B-\omega_A$$

$$\omega_M=\sqrt{\omega_B\bullet\omega_A}$$

$$\frac{\omega_M^2-\omega_{AL}^2}{\omega_{AL}\bullet BW}=\frac{\omega_{BH}^2-\omega_M^2}{\omega_{BH}\bullet BW}$$

(actually $-\omega_A$ and $-\omega_{AL}$ that map to 1 and ω_S respectively but show ω_A and ω_{AL} for convenience)

Example 2: Obtain an approximation that meets the following specifications



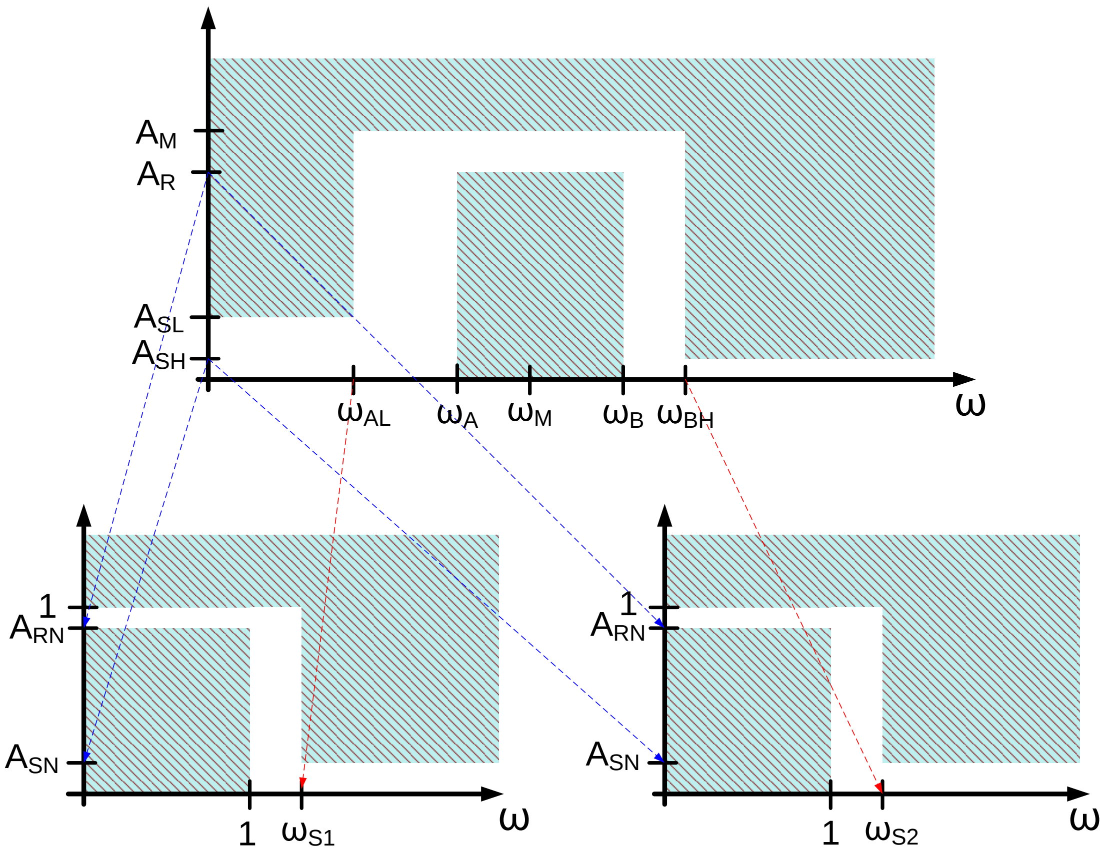
$$BW = \omega_B - \omega_A$$

$$\omega_M = \sqrt{\omega_B \bullet \omega_A}$$

In this example,

$$\frac{\omega_M^2 - \omega_{AL}^2}{\omega_{AL} \bullet BW} \neq \frac{\omega_{BH}^2 - \omega_M^2}{\omega_{BH} \bullet BW}$$

Example 2: Obtain an approximation that meets the following specifications



$$A_{RN}=\frac{A_R}{A_M}$$

$$\frac{1}{\sqrt{1+\varepsilon^2}}=\frac{A_R}{A_M}$$

$$A_{SN}=\min\left\{\frac{A_{SH}}{A_M},\frac{A_{SL}}{A_M}\right\}$$

$$\varepsilon=\sqrt{\left(\frac{A_M}{A_R}\right)^2-1}$$

$$\omega_{S1}=\frac{\omega_M^2-\omega_{AL}^2}{\omega_{AL}\bullet BW}$$

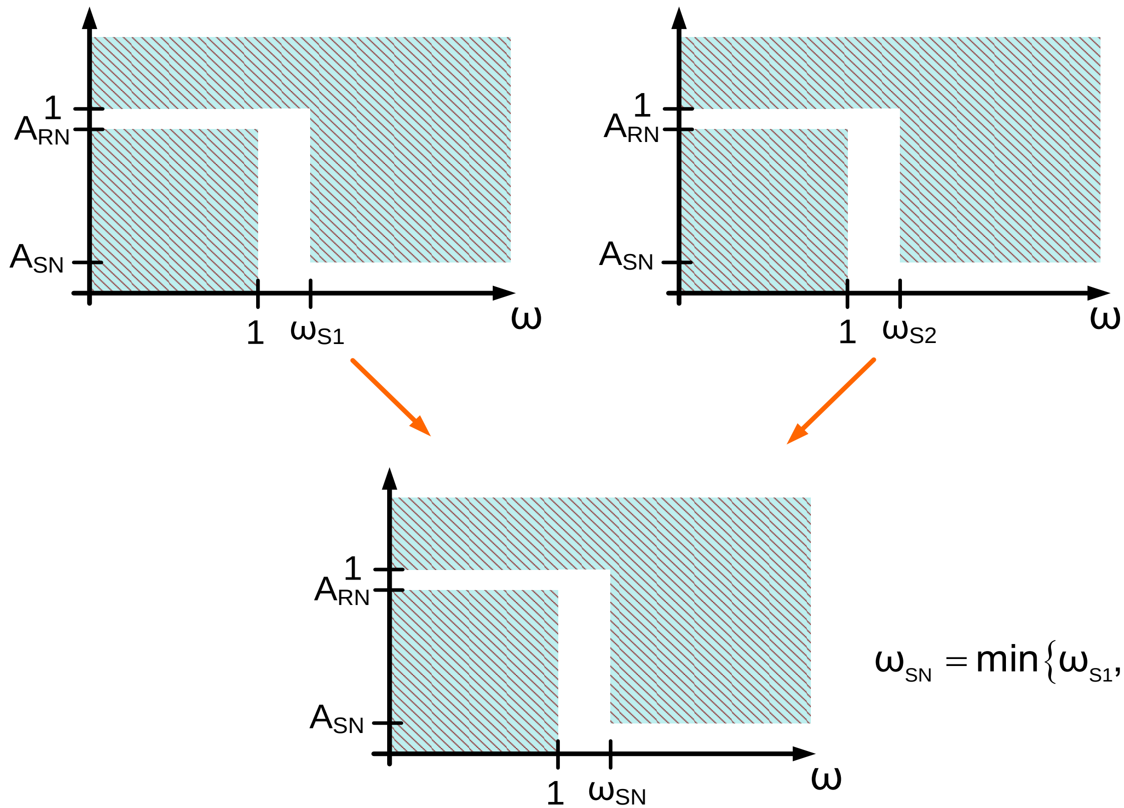
$$\omega_{S2}=\frac{\omega_{BH}^2-\omega_M^2}{\omega_{BH}\bullet BW}$$

$$BW=\omega_B-\omega_A$$

$$\omega_M=\sqrt{\omega_B\bullet\omega_A}$$

$$\omega_{SN}=\min\{\omega_{S1},\omega_{S2}\}$$

Example 2: Obtain an approximation that meets the following specifications



$$A_{RN}=\frac{A_R}{A_M}$$

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$$\varepsilon=\sqrt{\left(\frac{A_M}{A_R}\right)^2-1}$$

$$\omega_{SN}=\min\{\omega_{S1},\omega_{S2}\}$$

$$\omega_{S1}=\frac{\omega_M^2-\omega_{AL}^2}{\omega_{AL}\bullet BW}$$

$$\omega_{S2}=\frac{\omega_{BH}^2-\omega_M^2}{\omega_{BH}\bullet BW}$$

$$BW=\omega_B-\omega_A$$

$$\omega_M=\sqrt{\omega_B\bullet\omega_A}$$

$$\omega_{SN}=\min\{\omega_{S1},\omega_{S2}\}$$

Filter Transformations

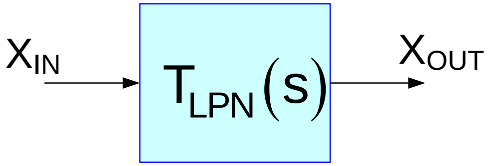
Lowpass to Bandpass **(LP to BP)**
Lowpass to Highpass **(LP to HP)**
→ Lowpass to Band-reject **(LP to BR)**

- Approach will be to take advantage of the results obtained for the standard LP approximations

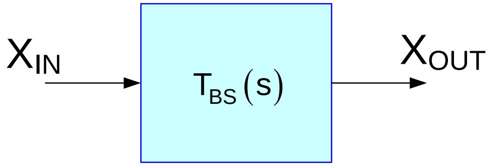
- Will focus on flat passband and zero-gain stop-band transformations

LP to BS Transformation

Strategy: As was done for the LP to BP approximations, will use a variable mapping strategy that maps the imaginary axis in the s-plane to the imaginary axis in the s-plane so the basic shape is preserved.



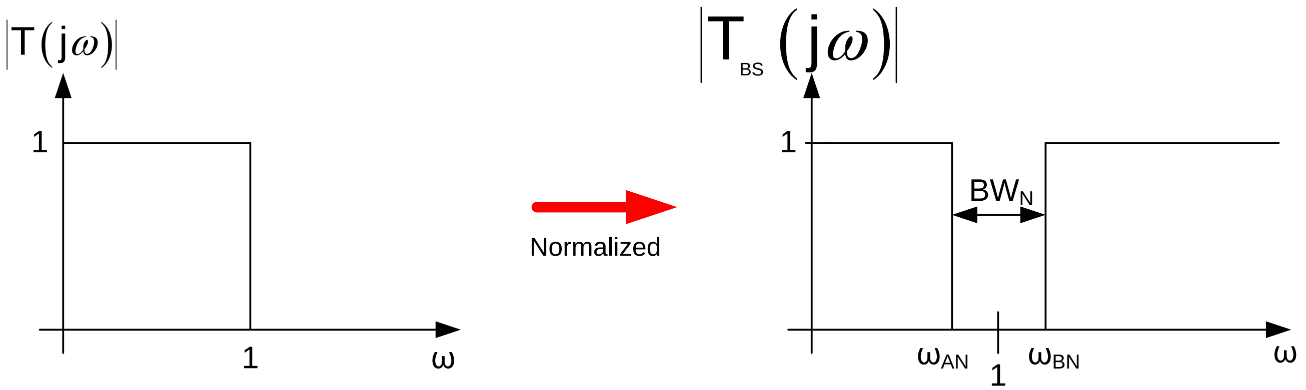
$$s \rightarrow f(s)$$



$$T_{BS}(s) = T_{LPN}(f(s))$$

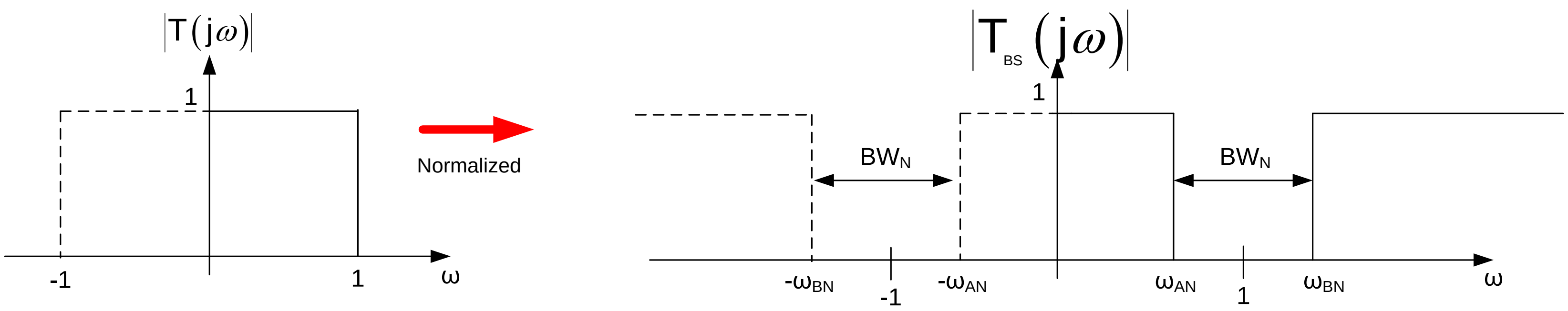
$$f(s) = \frac{\sum_{i=0}^{m_T} a_{Ti} s^i}{\sum_{i=0}^{n_T} b_{Ti} s^i}$$

LP to BS Transformation



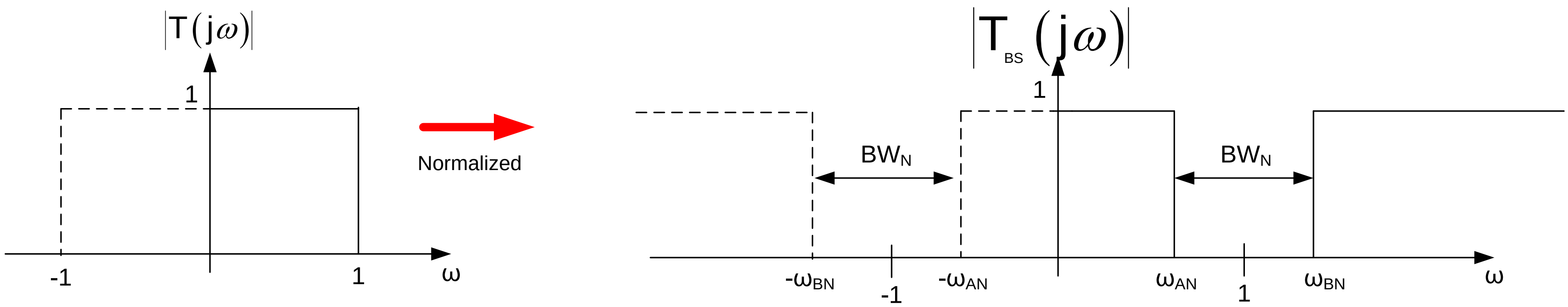
$$BW_N = \omega_{BN} - \omega_{AN}$$

$$\sqrt{\omega_{AN}\omega_{BN}} = 1$$



Standard LP to BS Transformation

Mapping Strategy:



Variable Mapping Strategy to Preserve Shape of LP function:

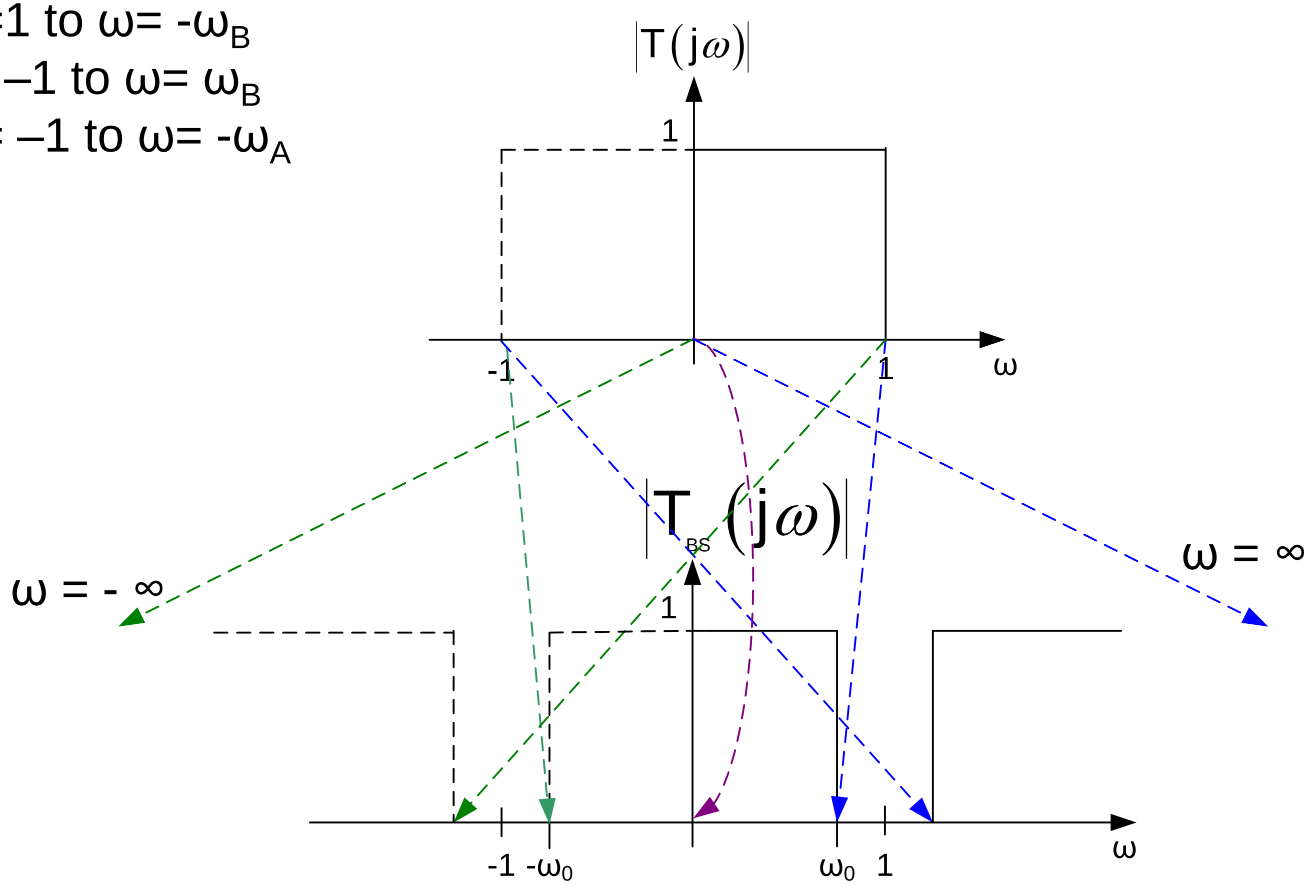
$F_N(s)$ should

map $s=0$ to $s=\pm j\infty$
map $s=0$ to $s=j0$
map $s=j1$ to $s=j\omega_A$
map $s=j1$ to $s=-j\omega_B$
map $s=-j1$ to $s=j\omega_B$
map $s=-j1$ to $s=-j\omega_A$

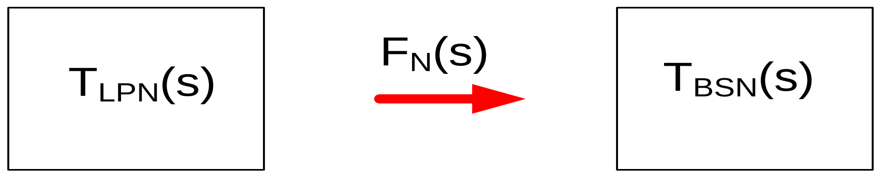
map $\omega=0$ to $\omega=\pm\infty$
map $\omega=0$ to $\omega=0$
map $\omega=1$ to $\omega=\omega_A$
map $\omega=1$ to $\omega=-\omega_B$
map $\omega=-1$ to $\omega=\omega_B$
map $\omega=-1$ to $\omega=-\omega_A$

Standard LP to BS Transformation

map ω=0 to ω = $\pm\infty$
map ω=0 to ω = 0
map ω=1 to ω = ω_A
map ω=1 to ω= $-\omega_B$
map ω= –1 to ω= ω_B
map ω= –1 to ω= $-\omega_A$



Standard LP to BS Transformation



Mapping Strategy: consider variable mapping transform

$F_N(s)$ should

map s=0 to s=± j∞	map ω=0 to ω = ±∞
map s=0 to s= j0	map ω=0 to ω = 0
map s=j1 to s=$j\omega_A$	map ω=1 to ω = ω_A
map s=j1 to s=-$j\omega_B$	map ω=1 to ω= -ω_B
map s=-j1 to s=$j\omega_B$	map ω= –1 to ω= ω_B
map s=-j1 to s=-$j\omega_A$	map ω= –1 to ω= -ω_A

Consider variable mapping

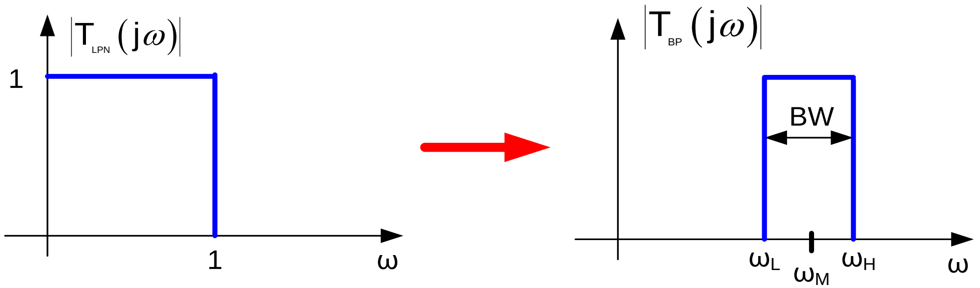
$$T_{LPN}\left(F_N(s)\right) = T_{BSN}(s)\Big|_{s=\frac{s\bullet BW_N}{s^2+1}}$$

$$s \rightarrow \frac{s \bullet BW_N}{s^2+1}$$

Comparison of Transforms

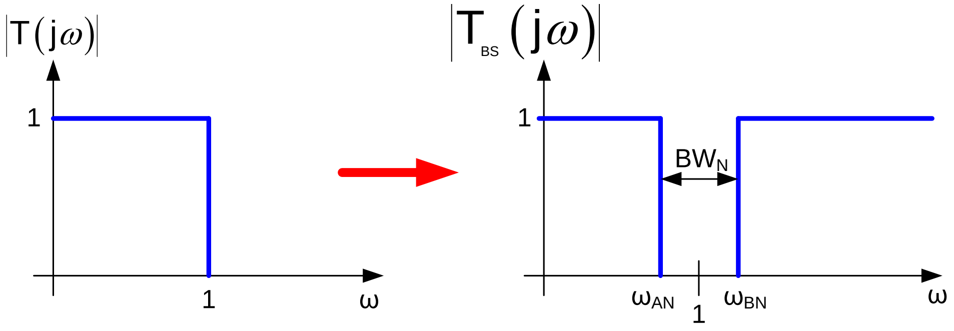
LP to BP

$$s \rightarrow \frac{s^2+1}{s \bullet BW_N}$$



LP to BS

$$s \rightarrow \frac{s \bullet BW_N}{s^2+1}$$



Standard LP to BS Transformation

Frequency and s-domain Mappings

(subscript variable in LP approximation for notational convenience)

$$T_{LPN}(s_X) \quad s_X \downarrow \frac{s \bullet BW_N}{s^2+1} \quad T_{BSN}(s)$$

$$s_X \rightarrow \frac{s \bullet BW_N}{s^2+1}$$

$$\omega_X \rightarrow \frac{\omega \bullet BW_N}{1-\omega^2}$$

$$s \leftarrow \frac{1}{2}\frac{BW_N}{s_X} \pm \frac{1}{2}\sqrt{\left(\frac{BW_N}{s_X}\right)^2 - 4}$$

$$\omega \leftarrow \frac{-1}{2}\frac{BW_N}{\omega_X} \pm \frac{1}{2}\sqrt{\left(\frac{BW_N}{\omega_X}\right) + 4}$$

Standard LP to BS Transformation

Un-normalized Frequency and s-domain Mappings

(subscript variable in LP approximation for notational convenience)

$$T_{LPN}(s_X)$$

$$s_X \downarrow \frac{s \bullet BW}{s^2 + \omega_M^2}$$

$$T_{BS}(s)$$

$$s_X \rightarrow \frac{s \bullet BW}{s^2 + \omega_M^2}$$

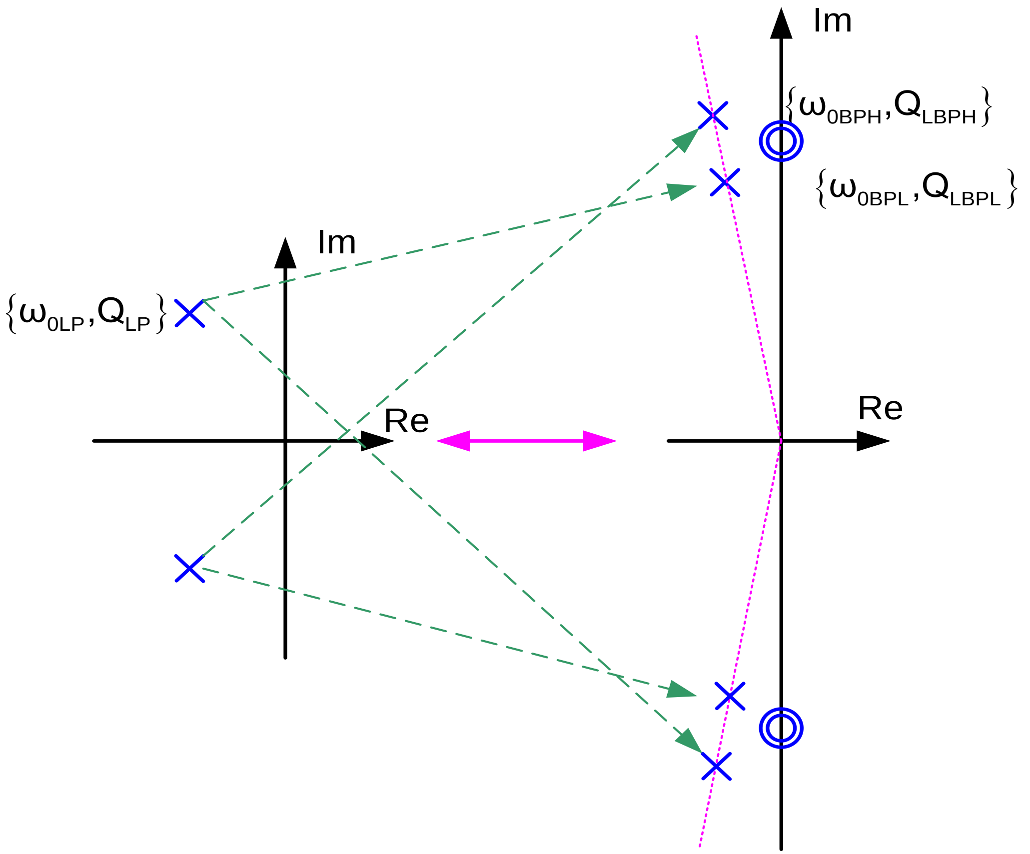
$$\omega_X \rightarrow \frac{\omega \bullet BW}{\omega_M^2 - \omega^2}$$

$$s \leftarrow \frac{1}{2}\frac{BW}{s_X} \pm \frac{1}{2}\sqrt{\left(\frac{BW}{s_X}\right)^2 - 4\omega_M^2}$$

$$\omega \leftarrow \frac{-1}{2}\frac{BW}{\omega_X} \pm \frac{1}{2}\sqrt{\left(\frac{BW}{\omega_X}\right) + 4\omega_M^2}$$

Standard LP to BS Transformation

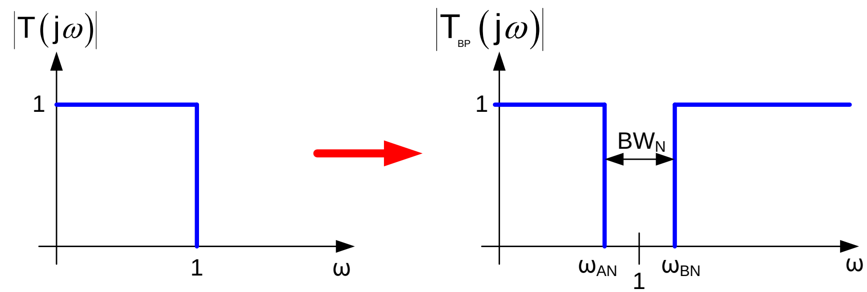
Pole Mappings



Can show that the upper hp pole maps to one upper hp pole and one lower hp pole as shown. Corresponding mapping of the lower hp pole is also shown

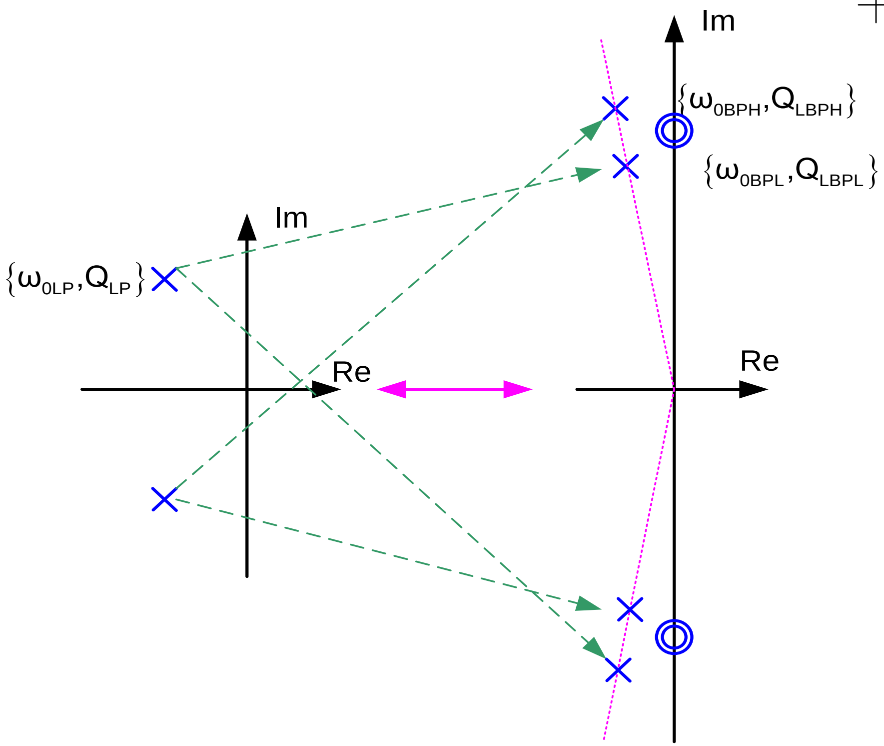
LP to BS Transformation

Pole Q of BS Approximations



$$BW = \omega_{BN} - \omega_{AN}$$

$$\omega_M = \sqrt{\omega_{AN}\omega_{BN}}$$



Define: $$\gamma = \left(\frac{BW}{\omega_M \omega_{0LP}}\right)$$

It can be shown that

$$Q_{BSL} = Q_{BSH} = \frac{Q_{LP}}{\sqrt{2}}\sqrt{1+\frac{4}{\gamma^2}+\sqrt{\left(1+\frac{4}{\gamma^2}\right)^2-\frac{4}{\gamma^2 Q_{LP}^2}}}$$

For γ small, $$Q_{BS} \simeq \frac{2Q_{LP}}{\gamma}$$

It can be shown that

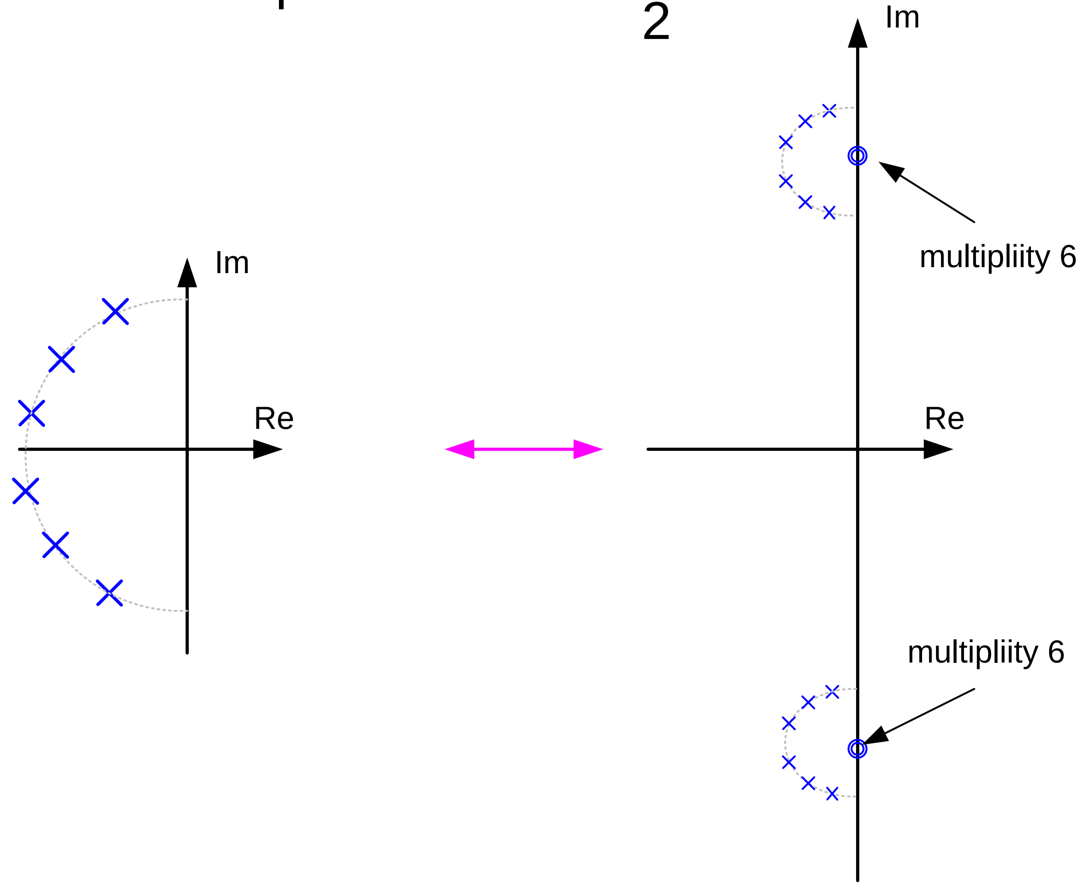
$$\omega_{0BS} = \frac{\omega_M}{2}\left[\gamma\frac{Q_{BS}}{Q_{LP}} \pm \sqrt{\left(\gamma\frac{Q_{BS}}{Q_{LP}}\right)^2-4}\right]$$

Note for γ **small, Q_{BS} can get very large**

Standard LP to BS Transformation

Pole Mappings

$$p \leftarrow \frac{\frac{BW_N}{p_X} \pm \sqrt{\left(\frac{BW_N}{p_X}\right)^2 - 4}}{2}$$



Note doubling of poles, addition of zeros, and likely Q enhancement

Standard LP to BS Transformation

$$s_X \rightarrow \frac{s \bullet BW}{s^2 + \omega_M^2}$$

- Standard LP to BS transformation is a variable mapping transform
- Maps jω axis to jω axis in the s-plane
- Preserves basic shape of an approximation but warps frequency axis
- Order of BS approximation is double that of the LP Approximation
- Pole Q and ω_0 expressions are identical to those of the LP to BP transformation
- Pole Q of BS approximation can get very large for narrow BW
- Other variable transforms exist but the standard is by far the most popular

Filter Transformations

Lowpass to Bandpass **(LP to BP)**

→ Lowpass to Highpass **(LP to HP)**

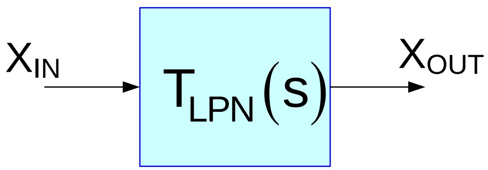
Lowpass to Band-reject **(LP to BR)**

- Approach will be to take advantage of the results obtained for the standard LP approximations

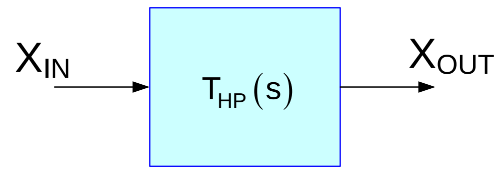
- Will focus on flat passband and zero-gain stop-band transformations

LP to HP Transformation

Strategy: As was done for the LP to BP approximations, will use a variable mapping strategy that maps the imaginary axis in the s-plane to the imaginary axis in the s-plane so the basic shape is preserved.



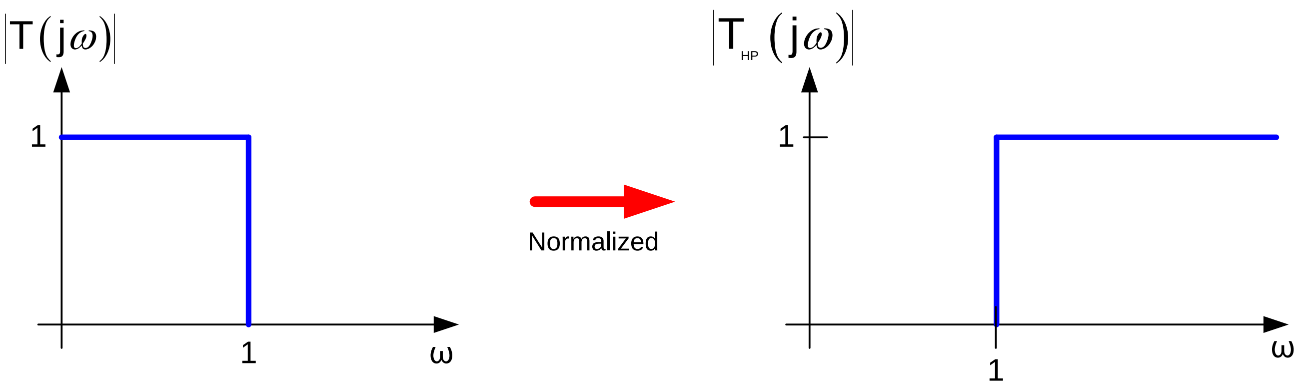
$$s \rightarrow f(s)$$

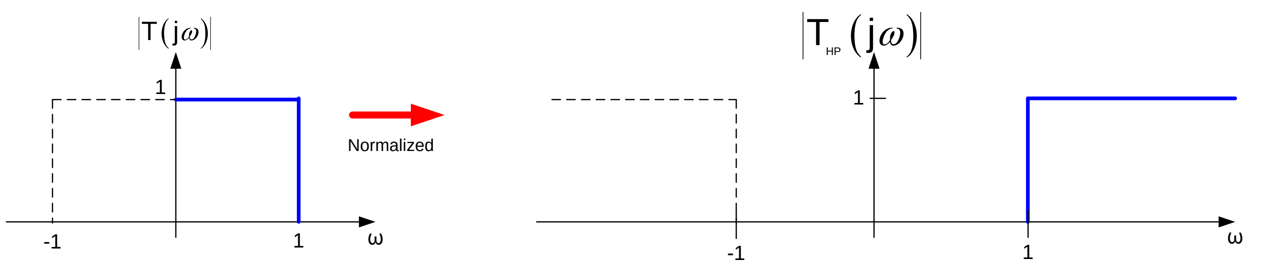


$$T_{HP}(s) = T_{LPN}(f(s))$$

$$f(s) = \frac{\sum_{i=0}^{m_T} a_{Ti} s^i}{\sum_{i=0}^{n_T} b_{Ti} s^i}$$

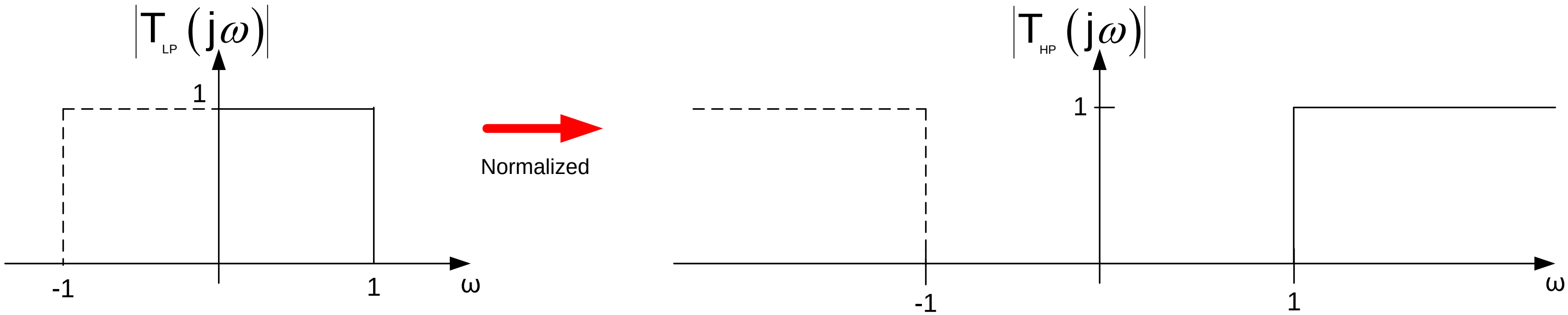
LP to HP Transformation





Standard LP to HP Transformation

Mapping Strategy:



Variable Mapping Strategy to Preserve Shape of LP function:

$F_N(s)$ should

map s=0 to s=± j∞
map s=j1 to s=-j1
map s= –j1 to s=j1

map ω=0 to ω=∞
map ω=1 to ω=-1
map ω= –1 to ω=1

Standard LP to HP Transformation

$T_{LPN}(s)$ $\xrightarrow{F_N(s)}$ $T_{HPN}(s)$

Mapping Strategy: consider variable mapping transform

$F_N(s)$ should

map s=0 to s=± j∞
map s=j1 to s=-j1
map s= –j1 to s=j1

→

map ω=0 to ω=∞
map ω=1 to ω=-1
map ω= –1 to ω=1

Consider variable mapping

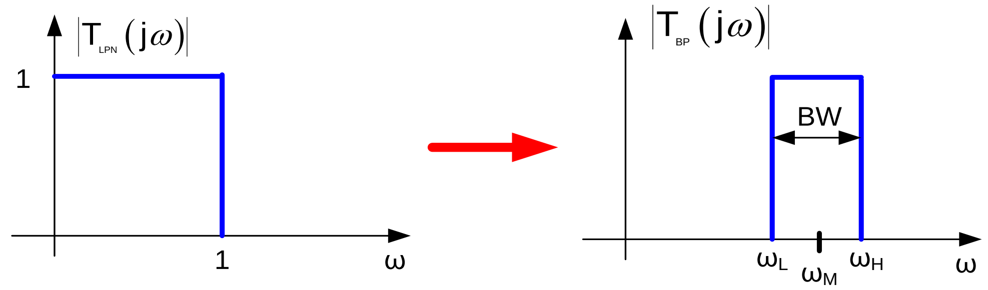
$$T_{LPN}(F(s)) = T_{LPN}(s)\Big|_{s=\frac{1}{s}}$$

$$s \rightarrow \frac{1}{s}$$

Comparison of Transforms

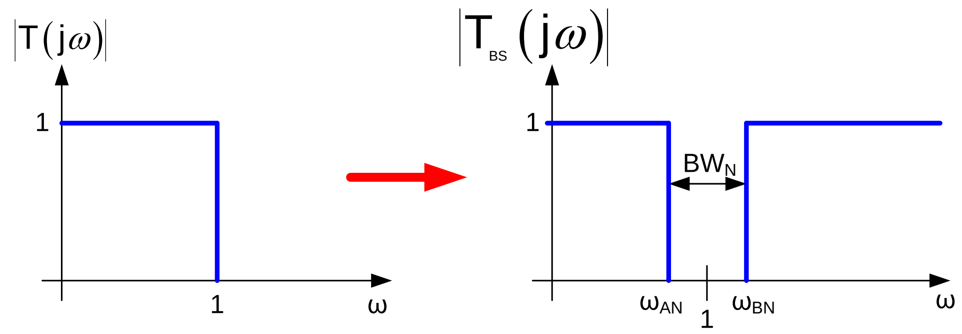
LP to BP

$$s \rightarrow \frac{s^2+1}{s \bullet BW_N}$$



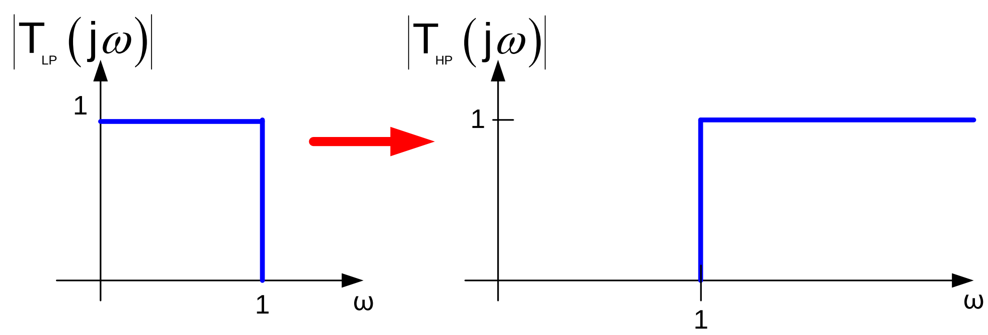
LP to BS

$$s \rightarrow \frac{s \bullet BW_N}{s^2+1}$$



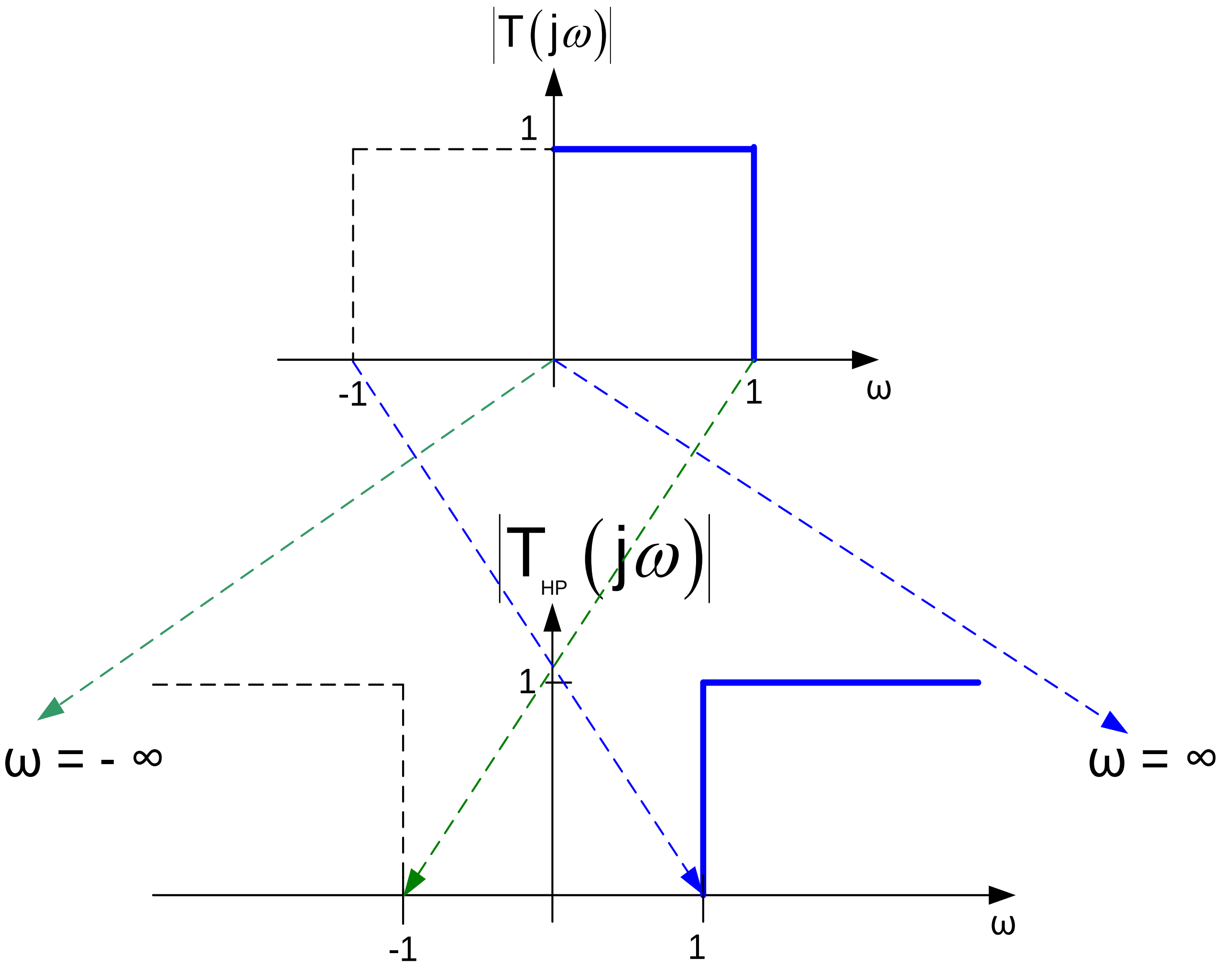
LP to HP

$$s \rightarrow \frac{1}{s}$$



LP to HP Transformation

(Normalized Transform)



Standard LP to HP Transformation

Frequency and s-domain Mappings

(subscript variable in LP approximation for notational convenience)

$$T_{LPN}(s_X)$$

$$s_X \downarrow \frac{1}{s}$$

$$T_{HPN}(s)$$

$$s_X \rightarrow \frac{1}{s}$$

$$\omega_X \rightarrow \frac{-1}{\omega}$$

$$s \leftarrow \frac{1}{s_X}$$

$$\omega \leftarrow \frac{-1}{\omega_X}$$

Standard LP to HP Transformation

Pole Mappings

Claim: With a variable mapping transform, the variable mapping naturally defines the mapping of the poles of the transformed function

$T_{LPN}(s_X)$

$$s_X \rightarrow \frac{1}{s}$$

$T_{HPN}(s)$

$$p_X \rightarrow \frac{1}{p}$$

$$p \leftarrow \frac{1}{p_X}$$

Standard LP to HP Transformation

Pole Mappings

$T_{LPN}(s_X)$

$$s_X \rightarrow \frac{1}{s}$$

$T_{HPN}(s)$

$$p \leftarrow \frac{1}{p_X}$$

If $p_X=\alpha+j\beta$

$$p=\frac{1}{\alpha+j\beta}=\frac{\alpha-j\beta}{\alpha^2+\beta^2}$$

and $p_X=\alpha-j\beta$

$$p=\frac{1}{\alpha+j\beta}=\frac{\alpha+j\beta}{\alpha^2+\beta^2}$$

Standard LP to HP Transformation

Pole Mappings

$T_{LPN}(s_X)$

$s_X \rightarrow \frac{1}{s}$

$T_{HPN}(s)$

$$p \leftarrow \frac{1}{p_X}$$

If $p_X = \alpha + j\beta$ and $p_X = \alpha - j\beta$

$$p = \frac{1}{\alpha + j\beta} = \frac{\alpha - j\beta}{\alpha^2 + \beta^2} \qquad p = \frac{1}{\alpha + j\beta} = \frac{\alpha + j\beta}{\alpha^2 + \beta^2}$$

Highpass poles are scaled in magnitude but make identical angles with imaginary axis

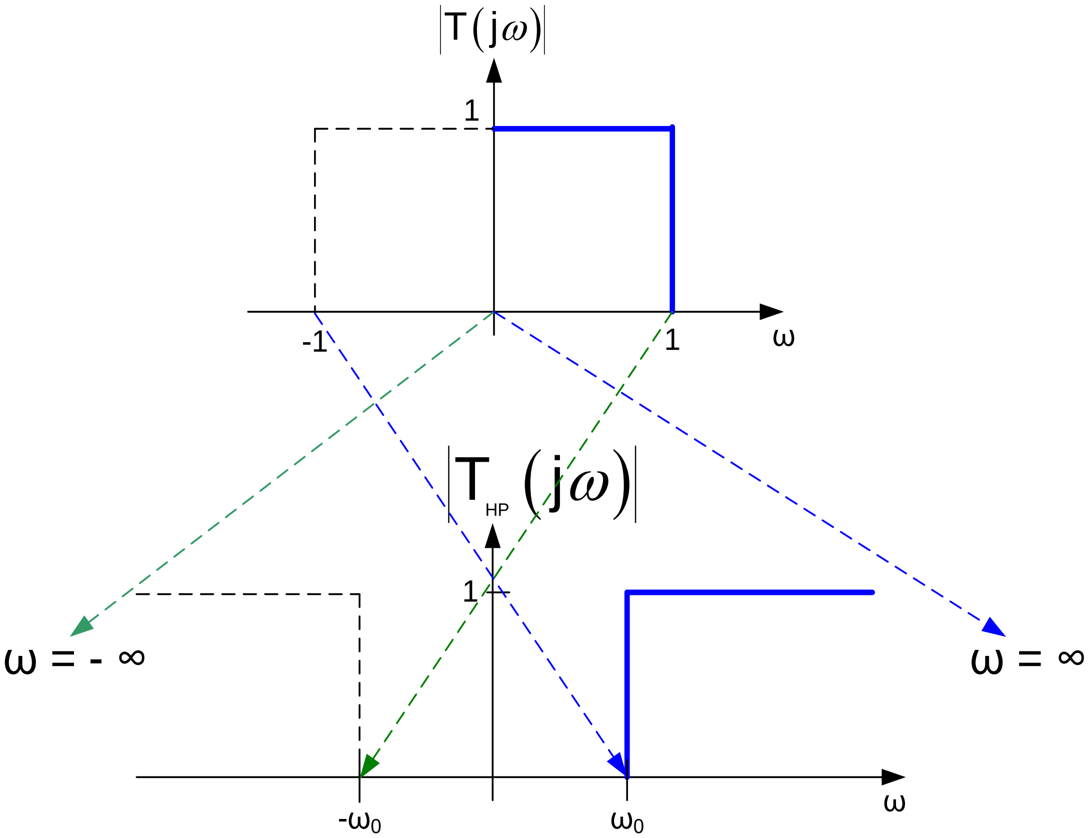
HP pole Q is same as LP pole Q

Order is preserved

Standard LP to HP Transformation

(Un-normalized variable mapping transform)

$$s \rightarrow \frac{\omega_0}{s}$$



Filter Design Process

Establish Specifications

- possibly $T_D(s)$ or $H_D(z)$
- magnitude and phase characteristics or restrictions
- time domain requirements

↓

Approximation

- obtain acceptable transfer functions $T_A(s)$ or $H_A(z)$
- possibly acceptable realizable time-domain responses

↓

Synthesis

- build circuit or implement algorithm that has response close to $T_A(s)$ or $H_A(z)$
- actually realize $T_R(s)$ or $H_R(z)$

↓

Filter

End of Lecture 17