Filter Transformations

Lowpass to Bandpass
Lowpass to Highpass
Lowpass to Band-reject
Review from Last Time

Flat Passband/Stopband Filters

\[ |T(j\omega)| \]

Lowpass

\[ |T(j\omega)| \]

Bandpass

\[ |T(j\omega)| \]

Highpass

\[ |T(j\omega)| \]

Bandreject
Review from Last Time

**Standard LP to BP Transformation**

s-domain

map $s=0$ to $s=j1$

map $s=j1$ to $s=j\omega_{BN}$

map $s=-j1$ to $s=j\omega_{AN}$

$\omega$-domain

map $\omega=0$ to $\omega=1$

map $\omega=1$ to $\omega=\omega_{BN}$

map $\omega=-1$ to $\omega=\omega_{AN}$

Verification of mapping Strategy:

$$T_{LPN}(f(s))$$

$$s \rightarrow \frac{s^2+1}{s\cdot BW_N}$$

Image of Im axis:

$$j\omega = \frac{s^2+1}{s\cdot BW_N}$$

solving for $s$, obtain

$$s = \frac{j\omega \cdot BW_N \pm \sqrt{(BW_N \cdot j\omega)^2 - 4}}{2} = j \left( \frac{\omega \cdot BW_N \pm \sqrt{(BW_N \cdot \omega)^2 + 4}}{2} \right)$$

this has no real part so the imaginary axis maps to the imaginary axis

Can readily show this mapping maps PB to PB and SB to SB

The mapping $s \rightarrow \frac{s^2+1}{s\cdot BW_N}$ is termed the standard LP to BP transformation
Standard LP to BP Transformation

\[ T_{LPN}(s) = \frac{s^2 + 1}{s \cdot BW_N} \]

\[ T_{BPN}(s) \]

Review from Last Time
Standard LP to BP Transformation

Pole Mappings

\[ p \leftarrow \frac{p_{x} \cdot BW_N \pm \sqrt{(BW_N \cdot p_{x})^2 - 4}}{2} \]

Image of the cc pole pair is the two pairs of poles

Review from Last Time
Can show that the upper hp pole maps to one upper hp pole and one lower hp pole as shown. Corresponding mapping of the lower hp pole is also shown.
Review from Last Time

Standard LP to BP Transformation

Pole Mappings

\[ p \leftarrow \frac{p_x \cdot BW_N \pm \sqrt{(BW_N \cdot p_x)^2 - 4}}{2} \]

Note doubling of poles, addition of zeros, and likely Q enhancement
LP to BP Transformation

Claim: Other variable mapping transforms exist that satisfy the imaginary axis mapping properties needed to obtain the LP to BP transformation but are seldom, if ever, discussed. The Standard LP to BP transform is by far the most popular and most authors treat it as if it is unique.
LP to BP Transformation

Pole Q of BP Approximations

Consider a pole in the LP approximation characterized by \( \{\omega_{0\text{LP}}, Q_{\text{LP}}\} \)

It can be shown that the corresponding BP poles have the same Q
(i.e. both bp poles lie on a common radial line)

\[
\omega_M = \sqrt{\omega_H \omega_L}
\]

\[
\text{BW} = \omega_H - \omega_L
\]
LP to BP Transformation

Pole Q of BP Approximations

Define:

\[ \delta = \left( \frac{BW}{\omega_M} \right) \omega_{0LP} \]

It can be shown that

\[ Q_{BPL} = Q_{BPH} = \frac{Q_{LP}}{\sqrt{2}} \sqrt{1 + \frac{4}{\delta^2} + \sqrt{\left(1 + \frac{4}{\delta^2}\right)^2 - \frac{4}{\delta^2}Q_{2LP}^2}} \]

For \( \delta \) small,

\[ Q_{BP} \approx \frac{2Q_{LP}}{\delta} \]

It can be shown that

\[ \omega_{0BP} = \frac{\omega_M}{2} \left[ \delta \frac{Q_{BP}}{Q_{LP}} \pm \sqrt{\left( \delta \frac{Q_{BP}}{Q_{LP}} \right)^2 - 4} \right] \]

Note for \( \delta \) small, \( Q_{BP} \) can get very large
LP to BP Transformation

Pole Q of BP Approximations

\[ \delta = \left( \frac{BW}{\omega_M} \right) \omega_{0LP} \]

\[ Q_{BPL} = Q_{BPH} = \frac{Q_{LP}}{\sqrt{2}} \sqrt{1 + \frac{4}{\delta^2} + \sqrt{\left(1 + \frac{4}{\delta^2}\right)^2 - \frac{4}{\delta^2}}^2} - \frac{4}{\delta^2 \omega_{2LP}} \]
LP to BP Transformation

Pole locations vs $Q_{LP}$ and $\delta$

$$\delta = \left( \frac{BW}{\omega_M} \right) \omega_{0LP}$$

Fig. 20-3  Upper-half-plane band-pass poles as a function of $Q_{LP}$ and $\delta$
LP to BP Transformation

Classical BP Approximations

- Butterworth
- Chebyschev
- Elliptic
- Bessel

Obtained by the LP to BP transformation of the corresponding LP approximations
Standard LP to BP Transformation

\[ s \rightarrow \frac{s^2 + 1}{s \cdot BW_N} \]

- Standard LP to BP transform is a variable mapping transform
- Maps \(j\omega\) axis to \(j\omega\) axis
- Maps LP poles to BP poles
- Preserves basic shape but warps frequency axis
- Doubles order
- Pole Q of resultant band-pass functions can be very large for narrow pass-band
- Sequencing of frequency scaling and transformation does not affect final function
Example 1: Obtain an approximation that meets the following specifications

$$BW = \omega_B - \omega_A$$

$$\omega_M = \sqrt{\omega_B \bullet \omega_A}$$

Assume that $\omega_{AL}$, $\omega_{BH}$ and $\omega_M$ satisfy

$$\frac{\omega_M^2 - \omega_{AL}^2}{\omega_{AL} \bullet BW} = \frac{\omega_{BH}^2 - \omega_M^2}{\omega_{BH} \bullet BW}$$
Recall from last lecture

Standard LP to BP Transformation

Frequency and s-domain Mappings - Denormalized

(subscript variable in LP approximation for notational convenience)

Consider the transformation:

\[ T_{LPN}(s_x) \]

\[ s_x \rightarrow \frac{s^2 + \omega_M^2}{s \cdot BW} \]

\[ \omega_x \rightarrow \frac{\omega^2 - \omega_M^2}{\omega \cdot BW} \]

\[ T_{BP}(s) \]

\[ s \leftarrow \frac{s_x \cdot BW \pm \sqrt{(BW \cdot s_x)^2 - 4\omega_M^2}}{2} \]

\[ \omega \leftarrow \frac{\omega_x \cdot BW \pm \sqrt{(BW \cdot \omega_x)^2 + 4\omega_M^2}}{2} \]

Exercise: Resolve the dimensional consistency in the last equation
Example 1: Obtain an approximation that meets the following specifications

\[ A_{RN} = \frac{A_R}{A_M} \]

\[ A_{SN} = \frac{A_S}{A_M} \]

\[ 1 = \frac{A_R}{A_M} \left( 1 + \varepsilon^2 \right) \]

\[ \varepsilon = \sqrt{\left( \frac{A_M}{A_R} \right)^2 - 1} \]

\[ \omega_S = \frac{\omega^2_M - \omega^2_{AL}}{\omega_{M} \cdot BW} \]

\[ BW = \omega_B - \omega_A \]

\[ \omega_M = \sqrt{\omega_B \cdot \omega_A} \]

\[ \frac{\omega^2_M - \omega^2_{AL}}{\omega_{AL} \cdot BW} = \frac{\omega^2_B - \omega^2_M}{\omega_{BH} \cdot BW} \]

(actually \(-\omega_A\) and \(-\omega_{AL}\) that map to 1 and \(\omega_S\) respectively but show \(\omega_A\) and \(\omega_{AL}\) for convenience)
Example 2: Obtain an approximation that meets the following specifications

\[
\begin{align*}
BW &= \omega_B - \omega_A \\
\omega_M &= \sqrt{\omega_B \cdot \omega_A}
\end{align*}
\]

In this example,

\[
\frac{\omega_M^2 - \omega_{AL}^2}{\omega_{AL} \cdot BW} \neq \frac{\omega_B^2 - \omega_{M}^2}{\omega_{BH} \cdot BW}
\]
Example 2: Obtain an approximation that meets the following specifications

\[
A_{RN} = \frac{A_R}{A_M}
\]

\[
\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{A_R}{A_M}
\]

\[
A_{SN} = \min\left\{\frac{A_{SH}}{A_M}, \frac{A_{SL}}{A_M}\right\}
\]

\[
\varepsilon = \sqrt{\left(\frac{A_M}{A_R}\right)^2 - 1}
\]

\[
\omega_{S1} = \frac{\omega_B^2 - \omega_{AL}^2}{\omega_{AL} \cdot \text{BW}}
\]

\[
\omega_{S2} = \frac{\omega_{BH}^2 - \omega_M^2}{\omega_{BH} \cdot \text{BW}}
\]

\[
\omega_{SN} = \min\left\{\omega_{S1}, \omega_{S2}\right\}
\]

\[
\text{BW} = \omega_B - \omega_A
\]

\[
\omega_M = \sqrt{\omega_B \cdot \omega_A}
\]
Example 2: Obtain an approximation that meets the following specifications

\[ A_{RN} = \frac{A_R}{A_M} \]

\[ \frac{1}{\sqrt{1 + \varepsilon^2}} = \frac{A_R}{A_M} \]

\[ A_{SN} = \min \left\{ \frac{A_{SH}}{A_M}, \frac{A_{SL}}{A_M} \right\} \]

\[ \varepsilon = \sqrt{\left( \frac{A_M}{A_R} \right)^2} - 1 \]

\[ \omega_{s1} = \frac{\omega_M^2 - \omega_{AL}^2}{\omega_{AL} \cdot BW} \]

\[ \omega_{s2} = \frac{\omega_{BH}^2 - \omega_M^2}{\omega_{BH} \cdot BW} \]

\[ BW = \omega_B - \omega_A \]

\[ \omega_M = \sqrt{\omega_B \cdot \omega_A} \]

\[ \omega_{SN} = \min \{ \omega_{s1}, \omega_{s2} \} \]
Filter Transformations

- Lowpass to Bandpass (LP to BP)
- Lowpass to Highpass (LP to HP)
- Lowpass to Band-reject (LP to BR)

- Approach will be to take advantage of the results obtained for the standard LP approximations
- Will focus on flat passband and zero-gain stop-band transformations
LP to BS Transformation

Strategy: As was done for the LP to BP approximations, will use a variable mapping strategy that maps the imaginary axis in the s-plane to the imaginary axis in the s-plane so the basic shape is preserved.

\[ T_{BS}(s) = T_{LPN}(f(s)) \]

\[ f(s) = \sum_{i=0}^{m_T} a_{Ti} s^i \sum_{i=0}^{n_T} b_{Ti} s^i \]
LP to BS Transformation

\[ T(j\omega) \quad \rightarrow \quad T_{BS}(j\omega) \]

Normalized BW:

\[ BW_N = \omega_{BN} - \omega_{AN} \]

\[ \sqrt{\omega_{AN}\omega_{BN}} = 1 \]
Standard LP to BS Transformation

Mapping Strategy:

\[ |T(j\omega)| \]

Variable Mapping Strategy to Preserve Shape of LP function:

\[ F_N(s) \text{ should} \]

map \( s=0 \) to \( s=\pm j\infty \)
map \( s=0 \) to \( s=j0 \)
map \( s=j1 \) to \( s=j\omega_A \)
map \( s=j1 \) to \( s=-j\omega_B \)
map \( s=-j1 \) to \( s=j\omega_B \)
map \( s=-j1 \) to \( s=-j\omega_A \)

map \( \omega=0 \) to \( \omega=\pm\infty \)
map \( \omega=0 \) to \( \omega=0 \)
map \( \omega=1 \) to \( \omega=\omega_A \)
map \( \omega=1 \) to \( \omega=-\omega_B \)
map \( \omega=-1 \) to \( \omega=\omega_B \)
map \( \omega=-1 \) to \( \omega=-\omega_A \)
Standard LP to BS Transformation

map $\omega = 0$ to $\omega = \pm \infty$
map $\omega = 0$ to $\omega = 0$
map $\omega = 1$ to $\omega = \omega_A$
map $\omega = 1$ to $\omega = -\omega_B$
map $\omega = -1$ to $\omega = \omega_B$
map $\omega = -1$ to $\omega = -\omega_A$
Standard LP to BS Transformation

\[ T_{LPN}(s) \quad F_N(s) \quad T_{BSN}(s) \]

Mapping Strategy: consider variable mapping transform

\[ F_N(s) \text{ should} \]

map \( s=0 \) to \( s=\pm j\infty \)
map \( s=0 \) to \( s=j0 \)
map \( s=j1 \) to \( s=j\omega_A \)
map \( s=j1 \) to \( s=-j\omega_B \)
map \( s=-j1 \) to \( s=j\omega_B \)
map \( s=-j1 \) to \( s=-j\omega_A \)

map \( \omega=0 \) to \( \omega = \pm \infty \)
map \( \omega=0 \) to \( \omega = 0 \)
map \( \omega=1 \) to \( \omega = \omega_A \)
map \( \omega=1 \) to \( \omega = -\omega_B \)
map \( \omega=-1 \) to \( \omega = \omega_B \)
map \( \omega=-1 \) to \( \omega = -\omega_A \)

Consider variable mapping

\[ T_{LPN} \left( F_N(s) \right) = T_{BSN} \left( s \right) \bigg|_{s = \frac{s \cdot BW_N}{s^2 + 1}} \]

\[ s \rightarrow \frac{s \cdot BW_N}{s^2 + 1} \]
Comparison of Transforms

LP to BP
\[ s \rightarrow \frac{s^2 + 1}{s \cdot BW_N} \]

LP to BS
\[ s \rightarrow \frac{s \cdot BW_N}{s^2 + 1} \]
Standard LP to BS Transformation

Frequency and s-domain Mappings

(subscript variable in LP approximation for notational convenience)

\[ T_{LPN}(s_x) \]

\[ T_{BSN}(s) \]

\[ s_x \rightarrow \frac{s \cdot BW_N}{s^2 + 1} \]

\[ \omega_x \rightarrow \frac{\omega \cdot BW_N}{1 - \omega^2} \]

\[ s \leftarrow \frac{1}{2} \frac{BW_N}{s_x} \pm \frac{1}{2} \sqrt{\left( \frac{BW_N}{s_x} \right)^2 - 4} \]

\[ \omega \leftarrow -\frac{1}{2} \frac{BW_N}{\omega_x} \pm \frac{1}{2} \sqrt{\left( \frac{BW_N}{\omega_x} \right)^2 + 4} \]
Standard LP to BS Transformation

Un-normalized Frequency and s-domain Mappings

(subscript variable in LP approximation for notational convenience)

\[ T_{LPN}(s_x) \] \[ T_{BS}(s) \]

\[ s_x \rightarrow \frac{s \cdot BW}{s^2 + \omega_m^2} \]

\[ \omega_x \rightarrow \frac{\omega \cdot BW}{\omega_m^2 - \omega^2} \]

\[ s \leftarrow \frac{1}{2} \frac{BW}{s_x} \pm \frac{1}{2} \sqrt{\left(\frac{BW}{s_x}\right)^2 - 4\omega_m^2} \]

\[ \omega \leftarrow \frac{-1}{2} \frac{BW}{\omega_x} \pm \frac{1}{2} \sqrt{\left(\frac{BW}{\omega_x}\right)^2 + 4\omega_m^2} \]
Can show that the upper hp pole maps to one upper hp pole and one lower hp pole as shown. Corresponding mapping of the lower hp pole is also shown.
LP to BS Transformation

Pole Q of BS Approximations

Define:

\[\gamma = \left(\frac{BW}{\omega_m \omega_{0LP}}\right)\]

It can be shown that

\[Q_{BSL} = Q_{BSH} = \frac{Q_{LP}}{\sqrt{2}} \sqrt{1 + \frac{4}{\gamma^2} + \sqrt{\left(1 + \frac{4}{\gamma^2}\right)^2 - \frac{4}{\gamma^2}Q_{LP}^2}}\]

For \(\gamma\) small,

\[Q_{BS} \approx \frac{2Q_{LP}}{\gamma}\]

It can be shown that

\[\omega_{0BS} = \frac{\omega_M}{2} \left[\gamma \frac{Q_{BS}}{Q_{LP}} \pm \sqrt{\left(\gamma \frac{Q_{BS}}{Q_{LP}}\right)^2 - 4}\right]\]

Note for \(\gamma\) small, \(Q_{BS}\) can get very large
Standard LP to BS Transformation

Pole Mappings

\[
\frac{BW_N}{p_x} \pm \sqrt{\left(\frac{BW_N}{p_x}\right)^2 - 4}
\]

Note doubling of poles, addition of zeros, and likely Q enhancement
Standard LP to BS Transformation

\[ s_x \rightarrow \frac{s \cdot BW}{s^2 + \omega^2_M} \]

- Standard LP to BS transformation is a variable mapping transform
- Maps j\(\omega\) axis to j\(\omega\) axis in the s-plane
- Preserves basic shape of an approximation but warps frequency axis
- Order of BS approximation is double that of the LP Approximation
- Pole Q and \(\omega_0\) expressions are identical to those of the LP to BP transformation
- Pole Q of BS approximation can get very large for narrow BW
- Other variable transforms exist but the standard is by far the most popular
Filter Transformations

- Lowpass to Bandpass \((\text{LP to BP})\)
- Lowpass to Highpass \((\text{LP to HP})\)
- Lowpass to Band-reject \((\text{LP to BR})\)

- Approach will be to take advantage of the results obtained for the standard LP approximations
- Will focus on flat passband and zero-gain stop-band transformations
LP to HP Transformation

Strategy: As was done for the LP to BP approximations, will use a variable mapping strategy that maps the imaginary axis in the s-plane to the imaginary axis in the s-plane so the basic shape is preserved.

\[ T_{HP}(s) = T_{LPN}(f(s)) \]

\[ f(s) = \sum_{i=0}^{m_T} a_{Ti}s^i \]

\[ \frac{\sum_{i=0}^{n_T} b_{Ti}s^i}{\sum_{i=0}^{n_T} b_{Ti}s^i} \]
LP to HP Transformation

\[ |T(j\omega)| \]

\[ |T_{\text{hp}}(j\omega)| \]

Normalized

\[ \omega \]

-1

1

1

Normalized

\[ \omega \]

-1

1

1
Standard LP to HP Transformation

Mapping Strategy:

\[ |T_{LP}(j\omega)| \quad \text{Normalized} \quad |T_{HP}(j\omega)| \]

Variable Mapping Strategy to Preserve Shape of LP function:

\[ F_N(s) \text{ should} \]

map \( s=0 \) to \( s=\pm j\infty \)
map \( s=j1 \) to \( s=-j1 \)
map \( s=-j1 \) to \( s=j1 \)

map \( \omega=0 \) to \( \omega=\infty \)
map \( \omega=1 \) to \( \omega=-1 \)
map \( \omega=-1 \) to \( \omega=1 \)
Standard LP to HP Transformation

\[ T_{LPN}(s) \rightarrow F_N(s) \rightarrow T_{HPN}(s) \]

Mapping Strategy: consider variable mapping transform

\[ F_N(s) \text{ should} \]

map 0 to \( \pm j^\infty \)
map \( j1 \) to \(- j1 \)
map \(- j1 \) to \( j1 \)
map \( \omega = 0 \) to \( \omega = \infty \)
map \( \omega = 1 \) to \( \omega = -1 \)
map \( \omega = -1 \) to \( \omega = 1 \)

Consider variable mapping

\[ T_{LPN}(F(s)) = T_{LPN}(s) \bigg|_{s=\frac{1}{s}} \]

\[ s \rightarrow \frac{1}{s} \]
Comparison of Transforms

LP to BP
\[
S \rightarrow \frac{s^2 + 1}{s \cdot \text{BW}_N}
\]

LP to BS
\[
S \rightarrow \frac{s \cdot \text{BW}_N}{s^2 + 1}
\]

LP to HP
\[
S \rightarrow \frac{1}{s}
\]
LP to HP Transformation

(Normalized Transform)

\[ |T(j\omega)| \]

\[ |T_{HP}(j\omega)| \]

\[ \omega = -\infty \]

\[ \omega = \infty \]
Standard LP to HP Transformation

Frequency and s-domain Mappings

(subscript variable in LP approximation for notational convenience)
Claim: With a variable mapping transform, the variable mapping naturally defines the mapping of the poles of the transformed function.

\[ T_{LPN}(s_x) \rightarrow T_{HPN}(s) \]

\[ p_x \rightarrow \frac{1}{p} \]

\[ p \rightarrow \frac{1}{p_x} \]
Standard LP to HP Transformation

Pole Mappings

\[ T_{LPN}(s_x) \]

\[ \downarrow \]

\[ 1 \]

\[ \downarrow \]

\[ s \]

\[ T_{HPN}(s) \]

\[ p \leftarrow \frac{1}{p_x} \]

If \( p_x = \alpha + j\beta \)

\[ p = \frac{1}{\alpha + j\beta} = \frac{\alpha - j\beta}{\alpha^2 + \beta^2} \]

and \( p_x = \alpha - j\beta \)

\[ p = \frac{1}{\alpha + j\beta} = \frac{\alpha + j\beta}{\alpha^2 + \beta^2} \]
Standard LP to HP Transformation

Pole Mappings

\[ T_{LPN}(s_X) \]

\[ s_X \]

\[ \downarrow \]

\[ \frac{1}{s} \]

\[ T_{HPN}(s) \]

If \( p_X = \alpha + j\beta \) and \( p_X = \alpha - j\beta \)

\[ p = \frac{1}{\alpha+j\beta} = \frac{\alpha-j\beta}{\alpha^2 + \beta^2} \]

\[ p = \frac{1}{\alpha+j\beta} = \frac{\alpha+j\beta}{\alpha^2 + \beta^2} \]

Highpass poles are scaled in magnitude but make identical angles with imaginary axis

HP pole Q is same as LP pole Q

Order is preserved
Standard LP to HP Transformation

(Un-normalized variable mapping transform)

\[ S \rightarrow \frac{\omega_0}{S} \]

\[ |T(j\omega)| \]

\[ T_{HP}(j\omega) \]
Filter Design Process

Establish Specifications
- possibly $T_D(s)$ or $H_D(z)$
- magnitude and phase characteristics or restrictions
- time domain requirements

Approximation
- obtain acceptable transfer functions $T_A(s)$ or $H_A(z)$
- possibly acceptable realizable time-domain responses

Synthesis
- build circuit or implement algorithm that has response close to $T_A(s)$ or $H_A(z)$
- actually realize $T_R(s)$ or $H_R(z)$
End of Lecture 17