EE 508
Lecture 20

Sensitivity Functions

- Comparison of Filter Structures
- Performance Prediction
- Design Characterization
Define the standard sensitivity function as

$$S^f_x = \frac{\partial f}{\partial x} \cdot \frac{x}{f}$$

$S^f_x$ is widely used except when $x$ or $f$ assume extreme values of 0 or $\infty$.

Define the derivative sensitivity function as

$$\mathcal{D}^f_x = \frac{\partial f}{\partial x}$$

$\mathcal{D}^f_x$ is more useful when $x$ or $f$ ideally assume extreme values of 0 or $\infty$.
Dependent on circuit structure (for some circuits, also not dependent on components)

The sensitivity functions are thus useful for comparing different circuit structures

The variability which is the product of the sensitivity function and the normalized component differential is more important for predicting circuit performance
Variability Formulation

\[ V_{x_i}^f = S_{x_i}^f \bigg|_{\bar{X}_N} \cdot \frac{dx_i}{X_{iN}} \]

\[ \frac{dF}{F} = \sum_{i=1}^{k} V_{x_i}^f \bigg|_{\bar{X}_N} \]

Variability includes effects of both circuit structure and components on performance

If component variations are small, high sensitivities are acceptable

If component variations are large, low sensitivities are critical
Observation:

\[ \omega_0 = \frac{1}{RC} \]

\[ S^{\omega_0}_R = -1 \quad S^{\omega_0}_C = -1 \]

\[ \sum S^{\omega_0}_{R_i} = -1 \quad \sum S^{\omega_0}_{C_i} = -1 \]

\[ \omega_0 = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}} \]

\[ S^{\omega_0}_{R_1} = -\frac{1}{2} \quad S^{\omega_0}_{C_1} = -\frac{1}{2} \]

\[ S^{\omega_0}_{R_2} = -\frac{1}{2} \quad S^{\omega_0}_{C_2} = -\frac{1}{2} \]

At this stage, this is just an observation about summed sensitivities but later will establish some fundamental properties of summed sensitivities.
Consider

$$\frac{dF}{F} = \left( \sum_{\text{all resistors}} S_{R_i}^f \cdot \frac{dR_i}{R_i} \right) + \left( \sum_{\text{all capacitors}} S_{C_i}^f \cdot \frac{dC_i}{C_i} \right) + \left( \sum_{\text{all opamps}} S_{\tau_i}^f \cdot \frac{d\tau_i}{\tau_i} \right) + \ldots$$

The nominal value of the time constant of the op amps is 0 so this expression cannot be evaluated at the ideal (nominal) value of $GB=\infty$.

Let \( \{x_i\} \) be the components in a circuit whose nominal value is not 0.

Let \( \{y_i\} \) be the components in a circuit whose nominal value is 0.

$$\frac{dF}{F} = \sum_{i=1}^{k_x} \frac{\partial F}{\partial x_i} \cdot \frac{dx_i}{F} + \sum_{i=1}^{k_y} \frac{\partial F}{\partial y_i} \cdot \frac{dy_i}{F} = \sum_{i=1}^{k} \left( \frac{\partial F}{\partial x_i} \cdot \frac{x_i}{F} \right) \cdot dx_i + \frac{1}{F} \sum_{i=1}^{k_y} \frac{\partial F}{\partial y_i} dy_i$$

$$\frac{dF}{F} = \sum_{i=1}^{k} \left( S_{x_i}^f \bigg|_{X_N,Y_N=0} \cdot \frac{dx_i}{x_i} \right) + \frac{1}{F} \sum_{i=1}^{k_y} \left( S_{y_i}^f \bigg|_{X_N,Y_N=0} \cdot y_i \right)$$

This expression can be used for predicting the effects of all components in a circuit.

Can set $Y_N=0$ before calculating $S_{x_i}^f$ functions.
Some useful sensitivity theorems

\[ S_{x}^{kf} = S_{x}^{f} \]

where \( k \) is a constant

\[ S_{x}^{f^n} = n \cdot S_{x}^{f} \]

\[ S_{x}^{1/f} = -S_{x}^{f} \]

\[ S_{x}^{\sqrt{f}} = \frac{1}{2} S_{x}^{f} \]

\[ S_{x}^{\prod_{i=1}^{k} f_{i}} = k \sum_{i=1}^{k} S_{x}^{f_{i}} \]
Example:

\[ A(s) = \frac{1}{\tau s} \]

Ideally

\[ I(s) = -\frac{1}{RCs} = -\frac{I_0}{s} \]

\( I_0 \) termed the unity gain freq of integrator

Assume ideally \( R=1K, C=3.18nF \) so that \( I_0=50KHz \)

Actually \( GB=600KHz, R=1.05K, \) and \( C=3.3nF \)

a) Determine an approximation to the actual unity gain frequency using a sensitivity analysis

b) Write an analytical expression for the actual unity gain frequency
Example:

Assume ideally \( R=1\,\text{K}, \, C=3.18\,\text{nF} \) so that \( I_o=50\,\text{kHz} \)

Actually \( GB=600\,\text{kHz}, \, R=1.05\,\text{K}, \, \text{and} \, C=3.3\,\text{nF} \)

Observe

\[
\frac{\Delta R}{R} = \frac{.05\,\text{K}}{1\,\text{K}} = .05
\]

\[
\frac{\Delta C}{C} = \frac{.12\,\text{nF}}{3.18\,\text{nF}} = .038
\]

\[
\frac{I_o}{GB} = \frac{I_o}{50\,\text{kHz}} = \frac{600\,\text{kHz}}{600\,\text{kHz}} = .083
\]
Example:

\[
A(s) = \frac{1}{\tau s}
\]

Ideally

\[
I(s) = -\frac{1}{RCs} = -\frac{l_0}{s}
\]

Solution:

\[
l_0 = \frac{1}{RC}
\]

\[
\frac{dF}{F} = \sum_{i=1}^{k} \left( S_{x_i}^f \bigg|_{\bar{x}_N, \bar{y}_N=0} \cdot \frac{dx_i}{x_i} \right) + \frac{1}{F_N} \sum_{i=1}^{k_y} \left( \delta_{y_i}^f \bigg|_{\bar{x}_N, \bar{y}_N=0} \cdot y_i \right)
\]

\[
\frac{dl_{0A}}{l_{0A}} = \left[ S_{0A}^{l_{0A}} \bigg|_{R_N, C_N, \tau=0} \right] \frac{dR}{R_N} + \left[ S_{0A}^{l_{0A}} \bigg|_{R_N, C_N, \tau=0} \right] \frac{dC}{C_N} + \frac{1}{l_{0N}} \left( \delta_{\tau}^{l_{0A}} \bigg|_{\bar{x}_N, \bar{y}_N=0} \cdot \tau \right)
\]

\[
S_{0A}^{l_{0A}} \bigg|_{R_N, C_N, \tau=0} = S_{0A}^{l_{0}} \bigg|_{R_N, C_N}
\]

\[
S_{0}^{l_{0A}} \bigg|_{R_N, C_N} = -1
\]

\[
S_{0}^{l_{0}} \bigg|_{R_N, C_N} = -1
\]

It remains to calculate \( \delta_{\tau}^{l_{0A}} \bigg|_{\bar{x}_N, \bar{y}_N=0} \)
Example:

![Circuit Diagram](image)

Ideally

\[ I(s) = -\frac{1}{RCs} = -\frac{l_0}{s} \]

Solution:

Still need \( s |_{\tau, Y_N=0} \)

Define \( I_{0A} \) to be the actual unity gain frequency

\[ I_A(s) = -\frac{1}{RCs + \tau s (1 + RCs)} \]

\[ I_A(j\omega) = -\frac{1}{-\tau RC\omega^2 + j(\omega RC + \tau \omega)} \]

\[ |I_A(j\omega)|^2 = \frac{1}{(RC)^2 \tau^2 \omega^4 + \omega^2 (RC + \tau)^2} \]

\[ |I_A(j\omega)|^2 = \frac{1}{(RC)^2 \tau^2 \omega^4 + \omega^2 (RC + \tau)^2} = 1 \]

\[ \frac{1}{(RC)^2 \tau^2 I_{0A}^4 + I_{0A}^2 (RC + \tau)^2} = 1 \]
Example:

\[
\begin{align*}
V_{\text{IN}} & \quad R \quad C \quad \rightarrow \quad V_{\text{OUT}} \\
A(s) &= \frac{1}{\tau s}
\end{align*}
\]

Ideally

\[
I(s) = -\frac{1}{RCs} = -\frac{l_0}{s}
\]

Solution:

Still need

\[
\delta_{T_{0A}} \bigg|_{\bar{X}_N, \bar{Y}_N=0}
\]

Define \( I_{0A} \) to be the actual unity gain frequency

\[
(\text{RC})^2 \tau^2 l_{0A}^4 + l_{OA}^2 (\text{RC} + \tau)^2 = 1
\]

\[
\delta_{T_{0A}} \bigg|_{\bar{X}_N, \bar{Y}_N=0} = \left( \frac{\partial l_{0A}}{\partial \tau} \right)_{\bar{X}_N, \bar{Y}_N=0}
\]

\[
(\text{RC})^2 \tau^2 4l_{0A}^3 \left( \frac{\partial l_{0A}}{\tau} \right) + 2 \tau (\text{RC})^2 l_{0A}^4 + 2l_{0A}^1 \left( \frac{\partial l_{0A}}{\tau} \right)(\text{RC} + \tau)^2 + 2(\text{RC} + \tau)l_{OA}^2 = 0
\]

Evaluating at \( \bar{X}_N, \bar{Y}_N = 0 \)

\[
2l_{O}^1 \left( \frac{\partial l_{0A}}{\tau} \bigg|_{\bar{X}_N, \bar{Y}_N=0} \right)(\text{RC})^2 + 2(\text{RC})l_{O}^2 = 0
\]

\[
\left( \frac{\partial l_{0A}}{\tau} \bigg|_{\bar{X}_N, \bar{Y}_N=0} \right) = \frac{-l_0}{\text{RC}} = \delta_{T_{0A}} \bigg|_{X_N, Y_N=0} = -l_{O}^2
\]
Example:

\[ A(s) = \frac{1}{\tau s} \]

Ideally

\[ I(s) = -\frac{1}{RCs} = -\frac{l_{0N}}{s} \]

Solution:

\[
\begin{align*}
\frac{dl_{0A}}{l_{0A}} &= \left[ S_R^{l_{0A}} \bigg|_{R_N, C_N, \tau=0} \right] \frac{dR}{R_N} + \left[ S_C^{l_{0A}} \bigg|_{R_N, C_N, \tau=0} \right] \frac{dC}{C_N} + \frac{1}{l_{0N}} \left( \delta^{l_{0A}}_{\tau} \bigg|_{X_N, Y_N=0} \cdot \tau \right) \\
S_R^{l_{0A}} \bigg|_{R_N, C_N} &= S_C^{l_{0A}} \bigg|_{R_N, C_N} = -1 \\
\delta^{l_{0A}}_{\tau} \bigg|_{X_N, Y_N=0} &= -l_{0N}^2 \\
\frac{\Delta R}{R} &= .05 \quad \frac{\Delta C}{C} = .038 \quad \tau l_0 = .083
\end{align*}
\]

\[
\frac{dl_{0A}}{l_{0A}} = \left[ -1 \right] .05 + \left[ -1 \right] .038 + \frac{1}{l_{0N}} \left( -l_{0N}^2 \cdot \tau \right)
\]

Due to passives

\[
\frac{dl_{0A}}{l_{0A}} = \left[ -1 \right] .05 + \left[ -1 \right] .038 + (-.083)
\]

Due to actives

\[
\frac{dl_{0A}}{l_{0A}} = -.088 - .083
\]
Example:

Solution:

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\[ \frac{\text{d}I_{0A}}{I_{0A}} = -0.171 \quad I_{\text{ON}} = 50\text{KHz} \]

\[ I_{0A} \approx 0.829I_{\text{ON}} = 41.45\text{KHz} \]

Note that with the sensitivity analysis, it was not necessary to ever determine \( I_{0A} \)

a) Determine an approximation to the actual unity gain frequency using a sensitivity analysis

b) Write an analytical expression for the actual unity gain frequency

\[ \left( RC \right)^2 \tau^2 I_{0A}^4 + I_{OA}^2 (RC+\tau)^2 = 1 \]

Must solve this quadratic for \( I_{OA} \)  

Solving, obtain \( I_{OA} = 42.6\text{KHz} \)

Note this is close to the value obtained with the sensitivity analysis

Although in this simple example, it may have been easier to go directly to this expression, in more complicated circuits sensitivity analysis is much easier.
How can sensitivity analysis be used to compare the performance of different circuits?

Circuits have many sensitivity functions

If two circuits have exactly the same number of sensitivity functions and all sensitivity functions in one circuit are lower than those in the other circuit, then the one with the lower sensitivities is a less sensitive circuit

But usually this does not happen!

Designers would like a single metric for comparing two circuits!
\[
\frac{dF}{F} = \sum_{i=1}^{k} S_{f_i}^{f} \left| X_{i} \right| X_{iN} N \cdot \frac{dx_i}{X_{iN}}
\]

Dependent on circuit structure
(for some circuits, also not dependent on components)

Dependent only on components
(not circuit structure)

Consider:

\[
T(s) = \frac{1}{1+RCs}
\]

\[
T(s) = \frac{\omega_0}{s + \omega_0}
\]

\[
\omega_0 = \frac{1}{RC}
\]
\( \omega_0 = \frac{1}{RC} \)

\( S_{R}^{\omega_0} = -1 \)

\( S_{C}^{\omega_0} = -1 \)

\[
\frac{d\omega_0}{\omega_0} = \sum_{i=1}^{2} \left( S_{x_i}^{\omega_0} \bigg|_{X_i} \cdot \frac{dx_i}{X_i} \right)
\]

\[
\frac{d\omega_0}{\omega_0} = \begin{bmatrix} -1 \end{bmatrix} \cdot \frac{dR}{R_N} + \begin{bmatrix} -1 \end{bmatrix} \cdot \frac{dC}{C_N}
\]

Dependent only on components (not circuit structure)

Dependent only on circuit structure
Dependent on circuit structure
(for some circuits, also not dependent on components)

Consider now:

\[
T(s) = \frac{\frac{R_2}{R_1+R_2}}{1 + \frac{R_1R_2}{(R_1+R_2)C}s}
\]

\[
\omega_0 = \frac{R_1+R_2}{R_1R_2C}
\]
\[ S_{R_1}^{\omega_0} = ? \]

\[ \omega_0 = \frac{G_1 + G_2}{C} \]

\[ S_{R_1}^{\omega_0} = -S_{G_1}^{\omega_0} \]

\[ S_{G_1}^{\omega_0} = S_{G_1}^{G_1 + G_2} \]

\[ S_{G_1}^{G_1 + G_2} = \left( \frac{\partial (G_1 + G_2)}{\partial G_1} \right) \frac{G_1}{G_1 + G_2} = \frac{G_1}{G_1 + G_2} \]

\[ S_{R_1}^{\omega_0} = -\frac{R_2}{R_1 + R_2} \]

Note this is dependent upon the components as well!
Actually dependent upon component ratio!
Theorem: If \( f(x_1, ..x_m) \) can be expressed as
\[
f = x_1^{\alpha_1} x_2^{\alpha_2} ... x_m^{\alpha_m}
\]

where \( \{\alpha_1, \alpha_2, \ldots \alpha_m\} \) are real numbers, then \( S_{x_i}^f \) is not dependent upon any of the variables in the set \( \{x_1, ..x_m\} \)

Proof:
\[
S_{x_i}^f = S_{x_i}^{x_i^{\alpha_i}}
\]
\[
S_{x_i}^{x_i^{\alpha_i}} = \frac{\partial x_i^{\alpha_i}}{\partial x_i} \cdot \frac{x_i}{x_i^{\alpha_i}}
\]
\[
S_{x_i}^{x_i^{\alpha_i}} = \alpha_i X_i^{\alpha_i-1} \cdot \frac{x_i}{X_i^{\alpha_i}}
\]

It is often the case that functions of interest are of the form expressed in the hypothesis of the theorem, and in these cases the previous claim is correct.

\[
S_{x_i}^{x_i^{\alpha_i}} = \alpha_i
\]
Theorem: If \( f(x_1, \ldots, x_m) \) can be expressed as
\[
f = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_m^{\alpha_m}
\]
where \( \{\alpha_1, \alpha_2, \ldots, \alpha_m\} \) are real numbers, then the sensitivity terms in
\[
\frac{df}{f} = \sum_{i=1}^{k} \left( S_{x_i}^f \frac{dx_i}{x_i^{N}} \right)
\]
are dependent only upon the circuit architecture and not dependent upon the components and and the right terms are dependent only upon the components and not dependent upon the architecture.

This observation is useful for comparing the performance of two or more circuits where the function \( f \) shares this property.
Metrics for Comparing Circuits

Summed Sensitivity

\[ \rho_S = \sum_{i=1}^{m} S_{f_i} \]

Not very useful because sum can be small even when individual sensitivities are large

Schoeffler Sensitivity

\[ \rho = \sum_{i=1}^{m} \left| S_{f_i} \right| \]

Strictly heuristic but does differentiate circuits with low sensitivities from those with high sensitivities
Metrics for Comparing Circuits

\[
\rho = \sum_{i=1}^{m} \left| S_{x_i}^f \right|
\]

Often will consider several distinct sensitivity functions to consider effects of different components

\[
\rho_R = \sum \left| S_{R_i}^f \right|
\]

All resistors

\[
\rho_C = \sum \left| S_{C_i}^f \right|
\]

All capacitors

\[
\rho_{OA} = \sum \left| \delta_{\tau_i}^f \right|
\]

All op amps
End of Lecture 20