EE 508
Lecture 21

Sensitivity Functions

- Comparison of Filter Structures
- Performance Prediction
- Design Characterization

Review from last time

Define the standard sensitivity function as

$$S_x^f = \frac{\partial f}{\partial x} \bullet \frac{x}{f}$$

S_x^f Is widely used except when x or f assume extreme values of 0 or ∞

Define the derivative sensitivity function as

$$\mathcal{S}_x^f = \frac{\partial f}{\partial x}$$

$\mathcal{S}_x^f$ Is more useful when x or f ideally assume extreme values of 0 or ∞

Review from last time

$$\frac{dF}{F} = \sum_{i=1}^{k} \left(S_{x_i}^{f} \Big|_{\vec{X}_N} \bullet \frac{dx_i}{x_{iN}} \right)$$

Dependent only on components (not circuit structure)

Dependent on circuit structure (for some circuits, also not dependent on components**)**

The sensitivity functions are thus useful for comparing different circuit structures

The variability which is the product of the sensitivity function and the normalized component differential is more important for predicting circuit performance

Variability Formulation

Review from last time

$$V_{x_i}^{f} = S_{x_i}^{f}\Big|_{\vec{X}_N} \bullet \frac{dx_i}{x_{iN}}$$

$$\frac{dF}{F} = \sum_{i=1}^{k} V_{x_i}^{f}\Big|_{\vec{X}_N}$$

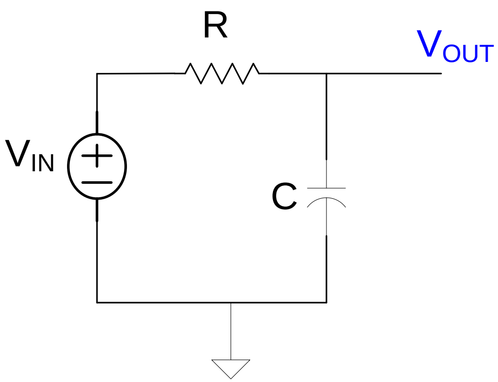
Variability includes effects of both circuit structure and components on performance

If component variations are small, high sensitivities are acceptable

If component variations are large, low sensitivities are critical

Observation:

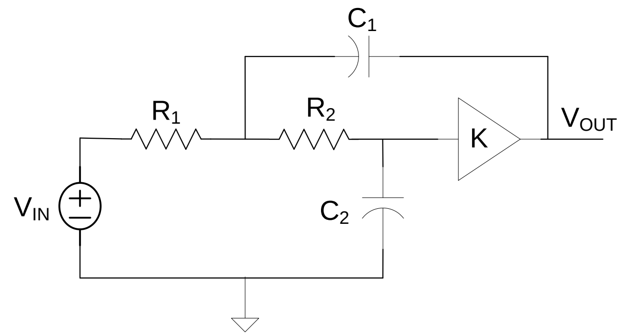
Review from last time



$$\omega_0 = \frac{1}{RC}$$

$$S_R^{\omega_0} = -1 \qquad S_C^{\omega_0} = -1$$

$$\sum_{All\ resistors} S_{R_i}^{\omega_0} = -1 \qquad \sum_{All\ capacitors} S_{C_i}^{\omega_0} = -1$$



$$\omega_0 = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}}$$

$$S_{R_1}^{\omega_0} = -\frac{1}{2} \qquad S_{C_1}^{\omega_0} = -\frac{1}{2}$$

$$S_{R_2}^{\omega_0} = -\frac{1}{2} \qquad S_{C_2}^{\omega_0} = -\frac{1}{2}$$

$$\sum_{All\ resistors} S_{R_i}^{\omega_0} = -1 \qquad \sum_{All\ capacitors} S_{C_i}^{\omega_0} = -1$$

At this stage, this is just an observation about summed sensitivities but later will establish some fundamental properties of summed sensitivities

Consider

$$\frac{dF}{F}=\left(\sum_{all\ resistors} S^{f}_{R_i}\bullet\frac{dR_i}{R_i}\right)+\left(\sum_{all\ capacitors} S^{f}_{C_i}\bullet\frac{dC_i}{C_i}\right)+\left(\sum_{all\ opamps} S^{f}_{\tau_i}\bullet\frac{d\tau_i}{\tau_i}\right)+\ldots$$

The nominal value of the time constant of the op amps is 0 so this expression can not be evaluated at the ideal (nominal) value of GB=∞

Let {xi} be the components in a circuit whose nominal value is not 0

Let {yi} be the components in a circuit whose nominal value is 0

$$\frac{dF}{F}=\sum_{i=1}^{kx}\frac{\partial F}{\partial x_i}\bullet\frac{dx_i}{F}+\sum_{i=1}^{ky}\frac{\partial F}{\partial y_i}\bullet\frac{dy_i}{F}=\sum_{i=1}^{k}\left(\frac{\partial F}{\partial x_i}\bullet\frac{x_i}{F}\right)\bullet\frac{dx_i}{x_i}+\frac{1}{F}\sum_{i=1}^{ky}\frac{\partial F}{\partial y_i}dy_i$$

$$\frac{dF}{F}=\sum_{i=1}^{k}\left(S^{f}_{x_i}\Big|_{\vec{X}_N,\vec{Y}_N=0}\bullet\frac{dx_i}{x_i}\right)+\frac{1}{F_N}\sum_{i=1}^{ky}\left(\mathcal{S}^{f}_{y_i}\Big|_{\vec{X}_N,\vec{Y}_N=0}\bullet y_i\right)$$

This expression can be used for predicting the effects of all components in a circuit

Can set Y_N=0 before calculating $S^{f}_{x_i}$ functions

$$\frac{dF}{F} = \sum_{i=1}^{k}\left(S_{x_i}^{f}\Big|_{\vec{X}_N} \bullet \frac{dx_i}{x_i} \right) + \frac{1}{F_N}\sum_{i=1}^{ky}\left(\mathcal{S}_{y_i}^{f}\Big|_{\vec{Y}_N=0} \bullet y_i \right)$$

Low sensitivities in a circuit are often preferred but in some applications, low sensitivities would be totally unacceptable

Examples where low sensitivities are unacceptable are circuits where a charactristics F must be tunable or adjustable!

Some useful sensitivity theorems

$$S_x^{kf} = S_x^f$$

where k is a constant

$$S_x^{f^n} = n \bullet S_x^f$$

$$S_x^{1/f} = -S_x^f$$

$$S_x^{\sqrt{f}} = \frac{1}{2} S_x^f$$

$$S_x^{\prod_{i=1}^{k} f_i} = \sum_{i=1}^{k} S_x^{f_i}$$

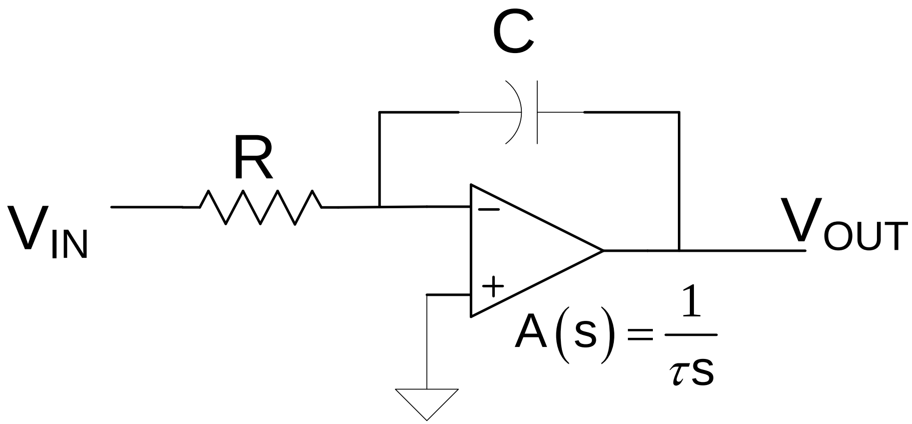
Some useful sensitivity theorems (cont)

$$S_x^{f/g} = S_x^f - S_x^g$$

$$S_x^{\sum_{i=1}^{k} f_i} = \frac{\sum_{i=1}^{k} f_i S_x^{f_i}}{\sum_{i=1}^{k} f_i}$$

$$S_{1/x}^{f} = -S_x^f$$

Example:



Ideally $$I(s) = -\frac{1}{RCs} = -\frac{I_0}{s}$$ I_0 termed the unity gain freq of integrator

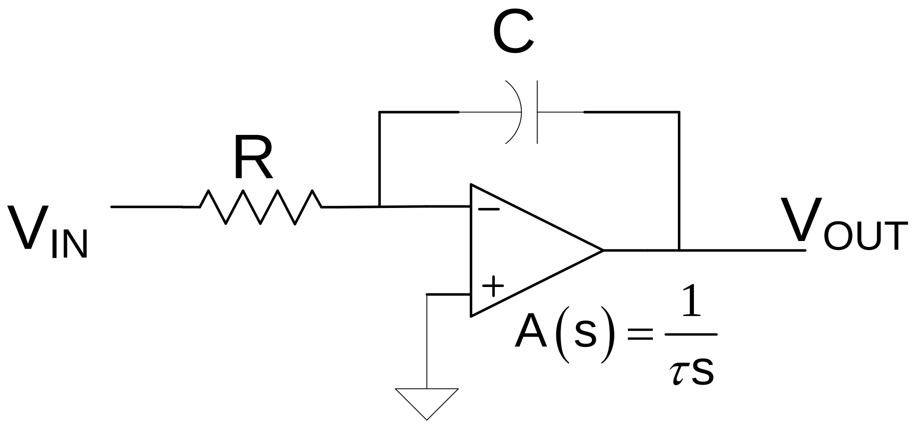
Assume ideally R=1K, C=3.18nF so that I_O=50KHz

Actually GB=600KHz, R=1.05K, and C=3.3nF

a) Determine an approximation to the actual unity gain frequency using a sensitivity analysis

b) Write an analytical expression for the actual unity gain frequency

Example:



Assume ideally R=1K, C=3.18nF so that I_O=50KHz

Actually GB=600KHz, R=1.05K, and C=3.3nF

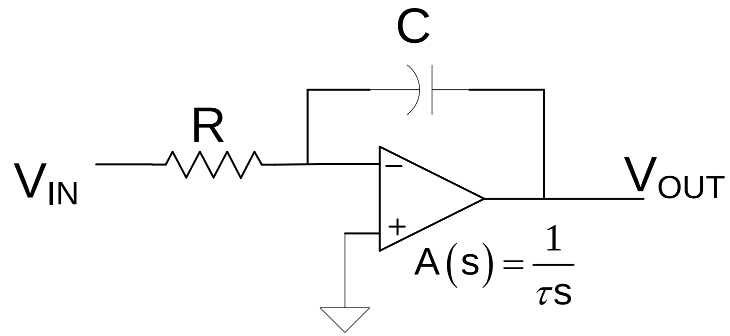
Observe

$$\frac{\Delta R}{R} = \frac{.05K}{1K} = .05$$

$$\frac{\Delta C}{C} = \frac{.12nF}{3.18nF} = .038$$

$$\frac{I_0}{GB} = \tau I_0 = \frac{50KHz}{600KHz} = .083$$

Example:



Ideally

$$I(s) = -\frac{1}{RCs} = -\frac{I_0}{s}$$

Solution:

Define I_{0A} to be the actual unity gain frequency

$$I_0 = \frac{1}{RC}$$

$$\frac{dF}{F} = \sum_{i=1}^{k}\left(S_{x_i}^{f}\Big|_{\vec{X}_N,\vec{Y}_N=0} \bullet \frac{dx_i}{x_i} \right) + \frac{1}{F_N}\sum_{i=1}^{ky}\left(\mathcal{S}_{y_i}^{f}\Big|_{\vec{X}_N,\vec{Y}_N=0} \bullet y_i \right)$$

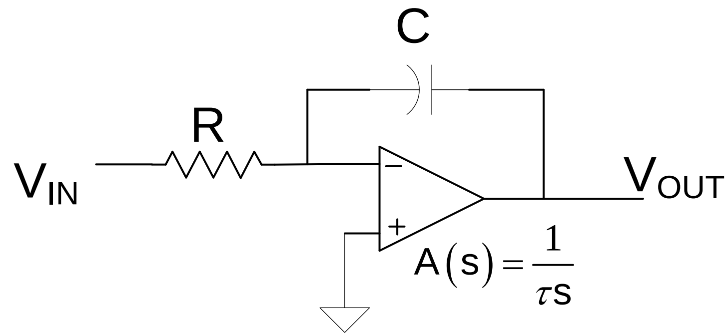
$$\frac{dI_{0A}}{I_{0A}} = \left[S_R^{I_{0A}}\Big|_{R_N,C_N,\tau=0} \right]\frac{dR}{R_N} + \left[S_C^{I_{0A}}\Big|_{R_N,C_N,\tau=0} \right]\frac{dC}{C_N} + \frac{1}{I_{0N}}\left(\mathcal{S}_{\tau}^{I_{0A}}\Big|_{\vec{X}_N,\vec{Y}_N=0} \bullet \tau \right)$$

$$S_R^{I_{0A}}\Big|_{R_N,C_N,\tau=0} = S_R^{I_0}\Big|_{R_N,C_N} \qquad S_C^{I_{0A}}\Big|_{R_N,C_N,\tau=0} = S_C^{I_0}\Big|_{R_N,C_N}$$

$$S_R^{I_0}\Big|_{R_N,C_N} = -1 \qquad S_C^{I_0}\Big|_{R_N,C_N} = -1$$

It remains to calculate $\mathcal{S}_{\tau}^{I_{0A}}\Big|_{\vec{X}_N,\vec{Y}_N=0}$

Example:



Ideally

$$I(s) = -\frac{1}{RCs} = -\frac{I_0}{s}$$

Solution:

Still need $S_{\tau}^{I_{0A}}\Big|_{\vec{X}_N,\vec{Y}_N=0}$

Define I_{0A} to be the actual unity gain frequency

$$I_A(s) = -\frac{1}{RCs+\tau s(1+RCs)}$$

$$I_A(j\omega) = -\frac{1}{-\tau\omega^2 + j(\omega RC+\tau\omega)}$$

$$|I_A(j\omega)|^2 = \frac{1}{\tau^2\omega^4+\omega^2(RC+\tau)^2}$$

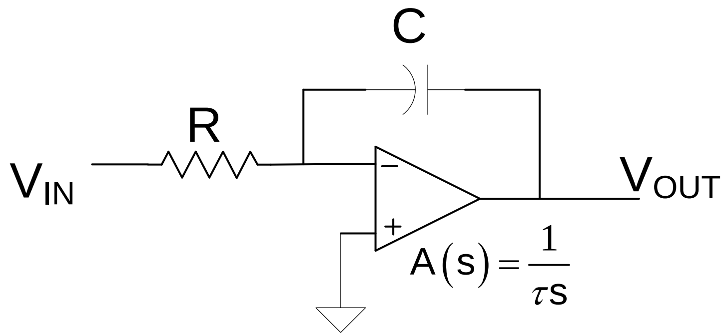
$$|I_A(j\omega)|^2 = \frac{1}{\tau^2\omega^4+\omega^2(RC+\tau)^2} = 1$$

$$\frac{1}{\tau^2 I_{0A}^4 + I_{OA}^2(RC+\tau)^2} = 1$$

$$\tau^2 I_{0A}^4 + I_{OA}^2(RC+\tau)^2 = 1$$

$S_{\tau}^{I_{0A}}\Big|_{\vec{X}_N,\vec{Y}_N=0}$ =?

Example:



Ideally

$$I(s) = -\frac{1}{RCs} = -\frac{I_0}{s}$$

Solution:

Still need $\mathcal{S}_{\tau}^{I_{0A}}\Big|_{\vec{X}_N,\vec{Y}_N=0}$

Define I_{0A} to be the actual unity gain frequency

$$\tau^2 I_{0A}^4 + I_{OA}^2 \left(RC+\tau\right)^2 = 1$$

$$\mathcal{S}_{\tau}^{I_{0A}}\Big|_{\vec{X}_N,\vec{Y}_N=0} = \left(\frac{\partial I_{0A}}{\partial \tau}\right)\Bigg|_{\vec{X}_N,\vec{Y}_N=0}$$

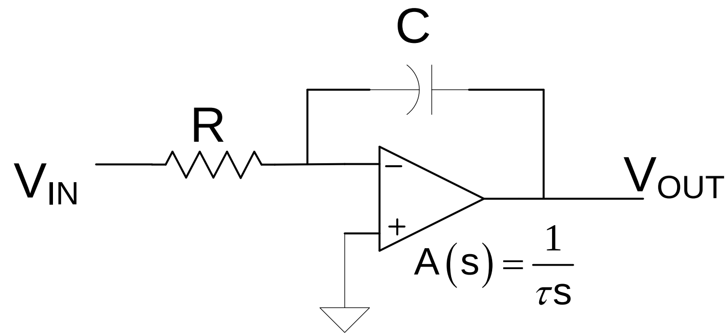
$$\tau^2 4 I_{0A}^3 \left(\frac{\partial I_{0A}}{\tau}\right) + 2\tau I_{0A}^4 + 2 I_{OA}^1 \left(\frac{\partial I_{0A}}{\tau}\right)\left(RC+\tau\right)^2 + 2\left(RC+\tau\right) I_{OA}^2 = 0$$

Evaluating at $\vec{X}_N, \vec{Y}_N = 0$

$$2 I_O^1 \left(\frac{\partial I_{0A}}{\tau}\Bigg|_{\vec{X}_N,\vec{Y}_N=0}\right)\left(RC\right)^2 + 2\left(RC\right) I_O^2 = 0$$

$$\left(\frac{\partial I_{0A}}{\tau}\Bigg|_{\vec{X}_N,\vec{Y}_N=0}\right) = \frac{-I_O}{RC} = \mathcal{S}_{\tau}^{I_{0A}}\Big|_{\vec{X}_N,\vec{Y}_N=0} = -I_O^2$$

Example:



Ideally

$$I(s) = -\frac{1}{RCs} = -\frac{I_0}{s}$$

Solution:

$$\frac{dI_{0A}}{I_{0A}} = \left[S_R^{I_{0A}}\Big|_{R_N, C_N, \tau=0} \right]\frac{dR}{R_N} + \left[S_C^{I_{0A}}\Big|_{R_N, C_N, \tau=0} \right]\frac{dC}{C_N} + \frac{1}{I_{0N}}\left(\mathcal{S}_\tau^{I_{0A}}\Big|_{\vec{X}_N, \vec{Y}_N=0} \bullet \tau \right)$$

$$S_R^{I_0}\Big|_{R_N, C_N} = S_C^{I_0}\Big|_{R_N, C_N} = -1 \qquad \mathcal{S}_\tau^{I_{0A}}\Big|_{\vec{X}_N, \vec{Y}_N=0} = -I_O^2$$

$$\frac{\Delta R}{R} = .05 \qquad \frac{\Delta C}{C} = .038 \qquad \tau I_0 = .083$$

$$\frac{dI_{0A}}{I_{0A}} = [-1].05 + [-1].038 + \frac{1}{I_{0N}}\left(-I_{0N}^2 \bullet \tau \right)$$

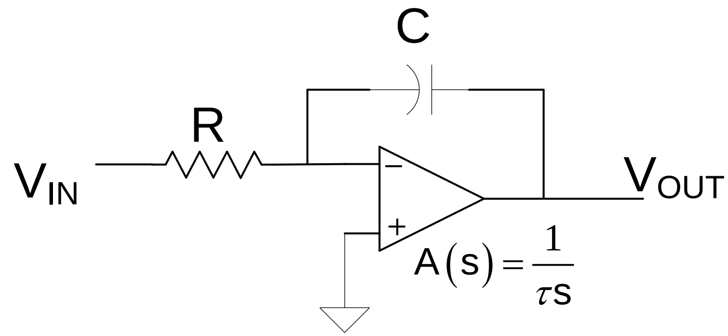
$$\frac{dI_{0A}}{I_{0A}} = [-1].05 + [-1].038 + (-.083)$$

Due to passives

Due to actives

$$\frac{dI_{0A}}{I_{0A}} = -.088 - .083$$

Example:



Ideally

$$I(s) = -\frac{1}{RCs} = -\frac{I_0}{s}$$

Solution:

$$\frac{dI_{0A}}{I_{0A}} = -.171$$

I_O=50KHz

$$I_{0A} \simeq 0.829 I_0 = 41.45\text{KHz}$$

Note that with the sensitivity analysis, it was not necessary to ever determine I_{0A}

a) Determine an approximation to the actual unity gain frequency using a sensitivity analysis

b) Write an analytical expression for the actual unity gain frequency

$$\tau^2 I_{0A}^4 + I_{OA}^2 (RC+\tau)^2 = 1$$

Must solve this quadratic for I_{0A}

Although in this simple example, it may have been easier to go directly to this expression, in more complicated circuits sensitivity analysis is much easier

How can sensitivity analysis be used to compare the performance of different circuits?

Circuits have many sensitivity functions

If two circuits have exactly the same number of sensitivity functions and all sensitivity functions in one circuit are lower than those in the other circuit, then the one with the lower sensitivities is a less sensitive circuit

But usually this does not happen !

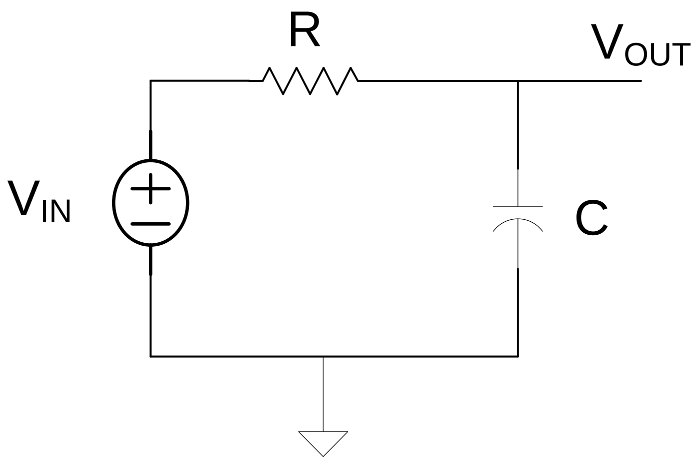
Designers would like a single metric for comparing two circuits !

$$\frac{dF}{F} = \sum_{i=1}^{k} \left(S_{x_i}^{f} \Big|_{\vec{X}_N} \bullet \frac{dx_i}{x_{iN}} \right)$$

Dependent on circuit structure
(for some circuits, also not dependent on components)

Dependent only on components
(not circuit structure)

Consider:

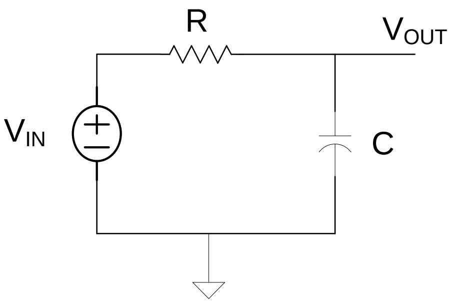


$$T(s) = \frac{1}{1+RCs}$$

$$T(s) = \frac{\omega_0}{s+\omega_0}$$

$$\omega_0 = \frac{1}{RC}$$

$$\omega_0 = \frac{1}{RC}$$



$$S_R^{\omega_0} = -1$$

$$S_C^{\omega_0} = -1$$

$$\frac{d\omega_0}{\omega_0} = \sum_{i=1}^{2} \left(S_{x_i}^{\omega_0} \Big|_{\vec{X}_N} \bullet \frac{dx_i}{x_{iN}} \right)$$

Dependent only on components
(not circuit structure)

$$\frac{d\omega_0}{\omega_0} = [-1] \bullet \frac{dR}{R_N} + [-1] \bullet \frac{dC}{C_N}$$

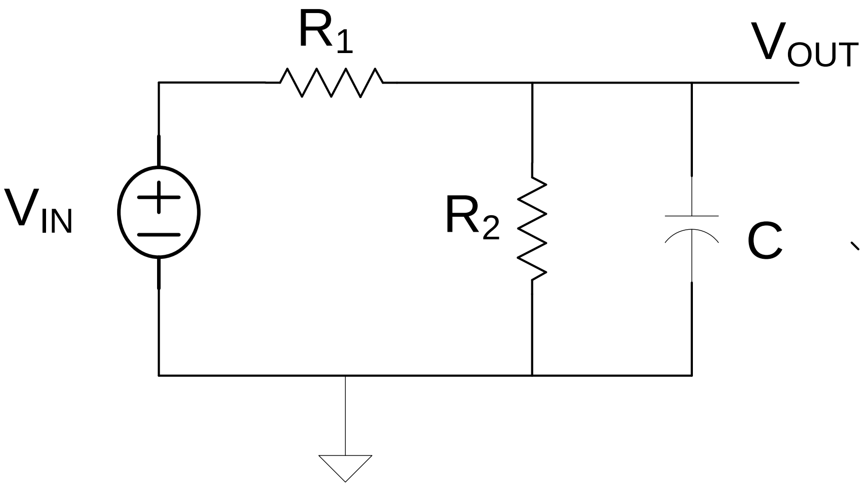
Dependent only on circuit structure

$$\frac{dF}{F} = \sum_{i=1}^{k} \left(S_{x_i}^{f} \Big|_{\vec{X}_N} \bullet \frac{dx_i}{x_{iN}} \right)$$

Dependent on circuit structure
(for some circuits, also not dependent on components)

Dependent only on components
(not circuit structure)

Consider now:



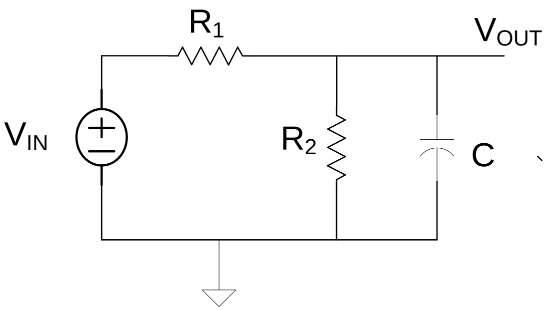
$$T(s) = \frac{\frac{R_2}{R_1+R_2}}{1+\left(\frac{R_1R_2}{R_1+R_2}C\right)s}$$

$$T(s) = \frac{R_2}{R_1+R_2} \bullet \frac{\omega_0}{s+\omega_0}$$

$$\omega_0 = \frac{R_1+R_2}{R_1R_2C}$$

$$S_{R_1}^{\omega_0} = ?$$

$$\omega_0 = \frac{R_1+R_2}{R_1 R_2 C}$$



$$\omega_0 = \frac{G_1+G_2}{C}$$

$$S_{R_1}^{\omega_0} = -S_{G_1}^{\omega_0}$$

$$S_{G_1}^{\omega_0} = S_{G_1}^{G_1+G_2}$$

$$S_{G_1}^{G_1+G_2} = \left(\frac{\partial (G_1+G_2)}{\partial G_1}\right)\frac{G_1}{G_1+G_2} = \frac{G_1}{G_1+G_2}$$

$$S_{R_1}^{\omega_0} = -\frac{R_2}{R_1+R_2}$$

Note this is dependent upon the components as well !
Actually dependent upon component ratio!

Theorem: If $f(x_1, ..x_m)$ can be expressed as $$f = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_m^{\alpha_m}$$

where $\{\alpha_1, \alpha_2, \ldots \alpha_m\}$ are real numbers, then $S_{x_i}^{f}$ is not dependent upon any of the variables in the set $\{x_1, ..x_m\}$

Proof:

$$S_{x_i}^{f} = S_{x_i}^{X_i^{\alpha_i}}$$

$$S_{x_i}^{X_i^{\alpha_i}} = \frac{\partial X_i^{\alpha_i}}{\partial x_i} \bullet \frac{x_i}{X_i^{\alpha_i}}$$

$$S_{x_i}^{X_i^{\alpha_i}} = \alpha_i X_i^{\alpha_i - 1} \bullet \frac{x_i}{X_i^{\alpha_i}}$$

$$S_{x_i}^{X_i^{\alpha_i}} = \alpha_i$$

$$S_{x_i}^{f} = \alpha_i$$

It is often the case that functions of interest are of the form expressed in the hypothesis of the theorem, and in these cases the previous claim is correct

Theorem: If $f(x_1, ..x_m)$ can be expressed as $f = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_m^{\alpha_m}$

where $\{\alpha_1, \alpha_2, \ldots \alpha_m\}$ are real numbers, then the sensitivity terms in

$$\frac{df}{f} = \sum_{i=1}^{k} \left(S_{x_i}^{f} \Big|_{\vec{X}_N} \bullet \frac{dx_i}{x_{iN}} \right)$$

are dependent only upon the circuit architecture and not dependent upon the components and and the right terms are dependent only upon the components and not dependent upon the architecture

This observation is useful for comparing the performance of two or more circuits where the function f shares this property

Metrics for Comparing Circuits

Summed Sensitivity

$$\rho_S = \sum_{i=1}^{m} S_{x_i}^{f}$$

Not very useful because sum can be small even when individual sensitivities are large

Schoeffler Sensitivity

$$\rho = \sum_{i=1}^{m} \left| S_{x_i}^{f} \right|$$

Strictly heuristic but does differentiate circuits with low sensitivities from those with high sensitivities

Metrics for Comparing Circuits

$$\rho = \sum_{i=1}^{m} \left| \mathrm{S}_{x_i}^{f} \right|$$

Often will consider several distinct sensitivity functions to consider effects of different components

$$\rho_R = \sum_{All\ resistors} \left| \mathrm{S}_{R_i}^{f} \right|$$

$$\rho_C = \sum_{All\ capacitors} \left| \mathrm{S}_{C_i}^{f} \right|$$

$$\rho_{OA} = \sum_{All\ op\ amps} \left| \mathcal{S}_{\tau_i}^{f} \right|$$

Homogeniety (defn)

A function f is homogeneous of order m in the n variables $\{x_1, x_2, \ldots x_n\}$ if

$$f(\lambda x_1, \lambda x_2, \ldots \lambda x_n) = \lambda^m f(x_1, x_2, \ldots x_n)$$

Note: f may be comprised of more than n variables

Theorem: If a function f is homogeneous of order m in the n variables $\{x_1, x_2, \ldots x_n\}$ then

$$\sum_{i=1}^{n} S_{x_i}^{f} = m$$

Proof:

$$f(\lambda x_1, \lambda x_2, \ldots \lambda x_n) = \lambda^m f(x_1, x_2, \ldots x_n)$$

Differentiate WRT λ

$$\frac{\partial\left(f(\lambda x_1, \lambda x_2, \ldots \lambda x_n)\right)}{\partial \lambda} = m\lambda^{m-1} f(x_1, x_2, \ldots x_n)$$

$$\frac{\partial f}{\partial \lambda x_1} x_1 + \frac{\partial f}{\partial \lambda x_2} x_2 + \ldots + \frac{\partial f}{\partial \lambda x_n} x_n = m\lambda^{m-1} f(x_1, x_2, \ldots x_n)$$

$$\frac{\partial f}{\partial \lambda x_1} x_1 + \frac{\partial f}{\wp \lambda x_2} x_2 + \ldots + \frac{\partial f}{\partial \lambda x_n} x_n = m\lambda^{m-1} f\left(x_1, x_2, \ldots x_n\right)$$

Simplify notation

$$\frac{\partial f}{\partial \lambda x_1} x_1 + \frac{\partial f}{\partial \lambda x_2} x_2 + \ldots + \frac{\partial f}{\partial \lambda x_n} x_n = m\lambda^{m} f$$

Divide by f

$$\frac{\partial f}{\lambda x_1} \frac{x_1}{f} + \frac{\partial f}{\lambda x_2} \frac{x_2}{f} + \ldots + \frac{\partial f}{\lambda x_n} \frac{x_n}{f} = m\lambda^{m}$$

Since true for all λ, also true for λ=1, thus

$$\frac{\partial f}{x_1} \frac{x_1}{f} + \frac{\partial f}{x_2} \frac{x_2}{f} + \ldots + \frac{\partial f}{x_n} \frac{x_n}{f} = m$$

This can be expressed as

$$\sum_{i=1}^{n} S_{x_i}^{f} = m$$

Theorem: If a function f is homogeneous of order m in the n variables $\{x_1, x_2, \ldots x_n\}$ then

$$\sum_{i=1}^{n} S_{x_i}^{f} = m$$

$$f(\lambda x_1, \lambda x_2, \ldots \lambda x_n) = \lambda^m f(x_1, x_2, \ldots x_n)$$

The concept of homogeneity and this theorem were somewhat late to appear

Are there really any useful applications of this rather odd observation?

Let T(s) be a voltage or current transfer function

Observation: Impedance scaling does not change any of the following, provided Op Amps are ideal:

$T(s)$, $T(j\omega)$, $|T(j\omega)|$, ω_0, Q, p_k, z_k

So, consider impedance scaling by a parameter λ

$$R \to \lambda R$$
$$L \to \lambda L$$
$$C \to C/\lambda$$

For these <u>impedance</u> functions

$$f(\lambda x_1, \lambda x_2, \ldots \lambda x_n) = \lambda^0 f(x_1, x_2, \ldots x_n)$$

Thus, all of these functions are homogeneous of order m=0 in the impedances

Theorem: If all op amps in a filter are ideal, then ω_o, Q, BW, all band edges, and all poles and zeros are homogeneous of order 0 in the impedances.

Theorem: If all op amps in a filter are ideal and if T(s) is a dimensionless transfer function, T(s), T(jω), $|T(j\omega)|$, $\angle T(j\omega)$, are homogeneous of order 0 in the impedances

Theorem 1: If all op amps in a filter are ideal and if $T(s)$ is an impedance transfer function, $T(s)$ and $T(j\omega)$ are homogeneous of order 1 in the impedances

Theorem 2: If all op amps in a filter are ideal and if $T(s)$ is a conductance transfer function, $T(s)$ and $T(j\omega)$ are homogeneous of order -1 in the impedances

Corollary 1: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors and if a function f is homogeneous of order 0 in the impedances, then

$$\sum_{i=1}^{k_1} S_{R_i}^{f} = \sum_{i=1}^{k_2} S_{C_i}^{f}$$

Corollary 2: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors then

$$\sum_{i=1}^{k_1} S_{R_i}^{Q} = 0$$

$$\sum_{i=1}^{k_2} S_{C_i}^{Q} = 0$$

End of Lecture 21