EE 508
Lecture 22

Sensitivity Functions

- Root Sensitivity
- Bilinear Property of Filters
- Root Sensitivities

Review from last time

Homogeniety (defn)

A function f is homogeneous of order m in the n variables $\{x_1, x_2, \ldots x_n\}$ if

$$f(\lambda x_1, \lambda x_2, \ldots \lambda x_n) = \lambda^m f(x_1, x_2, \ldots x_n)$$

Note: f may be comprised of more than n variables

Review from last time

Theorem: If a function f is homogeneous of order m in the n variables $(x_1, x_2, \ldots x_n)$ then

$$\sum_{i=1}^{n} S_{x_i}^{f} = m$$

$$f(\lambda x_1, \lambda x_2, \ldots \lambda x_n) = \lambda^m f(x_1, x_2, \ldots x_n)$$

The concept of homogeneity and this theorem were somewhat late to appear

Are there really any useful applications of this rather odd observation?

Theorem: If all op amps in a filter are ideal, then ω_o, Q, BW, all band edges, and all poles and zeros are homogeneous of order 0 in the impedances.

Theorem: If all op amps in a filter are ideal and if T(s) is a dimensionless transfer function, T(s), $T(j\omega)$, $|T(j\omega)|$, $\angle T(j\omega)$, are homogeneous of order 0 in the impedances

Review from last time

Theorem 1: If all op amps in a filter are ideal and if T(s) is an impedance transfer function, T(s) and T(jω) are homogeneous of order 1 in the impedances

Theorem 2: If all op amps in a filter are ideal and if T(s) is a conductance transfer function, T(s) and T(jω) are homogeneous of order -1 in the impedances

Corollary 1: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors and if a function f is homogeneous of order 0 in the impedances, then

$$\sum_{i=1}^{k_1} S_{R_i}^{f} = \sum_{i=1}^{k_2} S_{C_i}^{f}$$

Corollary 2: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors then

$$\sum_{i=1}^{k_1} S_{R_i}^{Q} = 0$$

$$\sum_{i=1}^{k_2} S_{C_i}^{Q} = 0$$

Proof of Corollary 1:

Corollary 1: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors and if a function f is homogeneous of order 0 in the impedances, then

$$\sum_{i=1}^{k_1} S_{R_i}^{f} = \sum_{i=1}^{k_2} S_{C_i}^{f}$$

Since f is homogenous of order zero in the impedances, z_1, z_2, … z_{k1+k2},

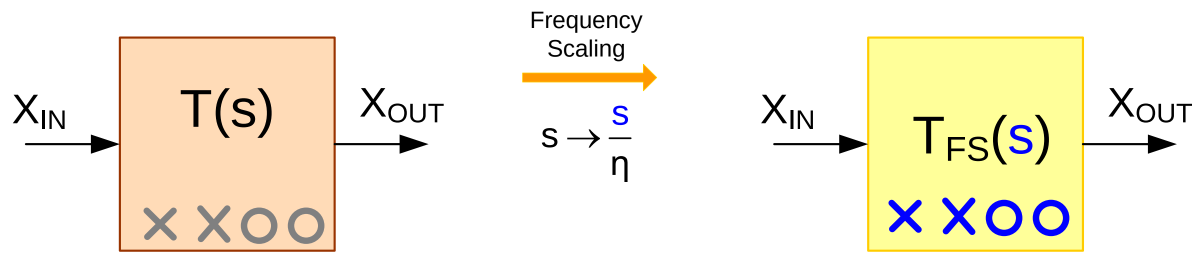
$$\sum_{i=1}^{k_1+k_2} S_{z_i}^{f} = 0$$

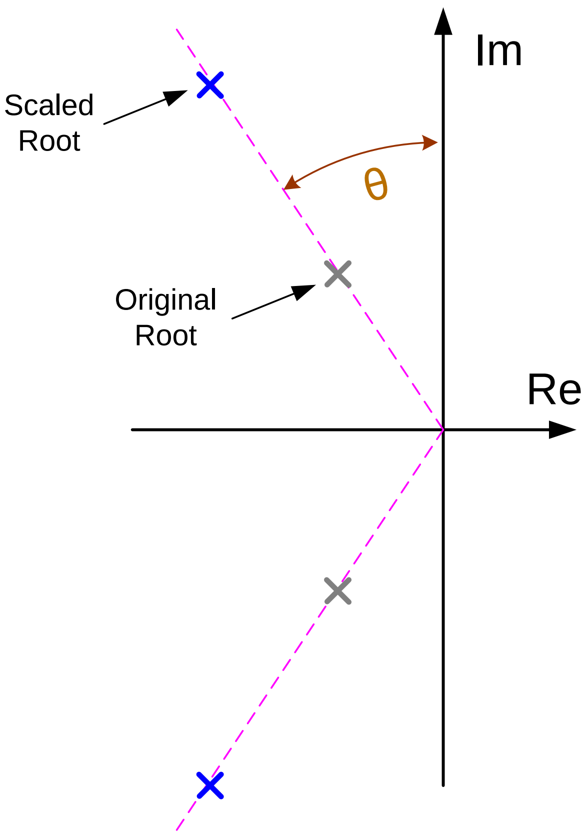
$$\therefore \quad \sum_{i=1}^{k_1} S_{R_i}^{f} + \sum_{i=1}^{k_2} S_{1/C_i}^{f} = 0$$

$$\therefore \quad \sum_{i=1}^{k_1} S_{R_i}^{f} - \sum_{i=1}^{k_2} S_{C_i}^{f} = 0$$

Proof of Corollary 2:

Recall:





Frequency Scaling: Scaling all frequency-dependent elements by a constant

$$L \rightarrow \eta L$$

$$C \rightarrow \eta C$$

Theorem: If all components are frequency scaled, roots (poles and zeros) will move along a constant Q locus

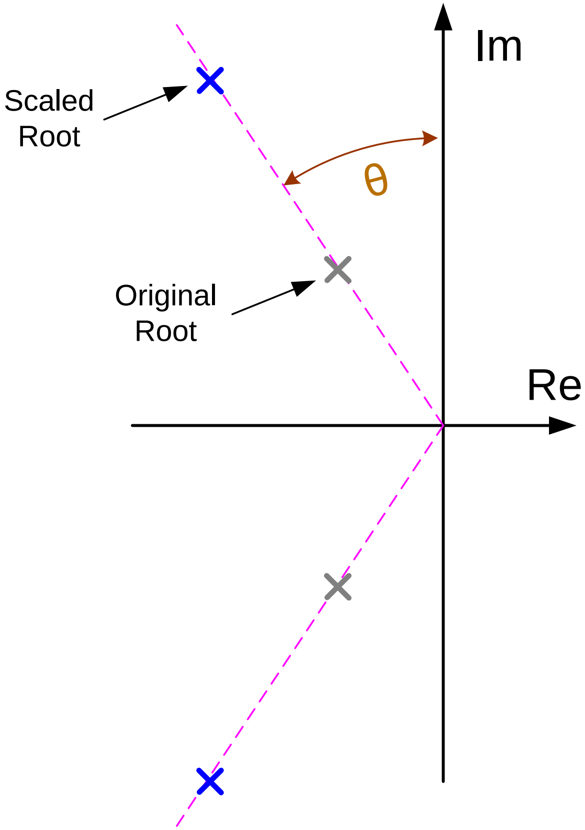
Proof:

$$T_{FS}(s) = T(s)\Big|_{s=\frac{s}{\eta}}$$

Proof of Corollary 2:

Recall:

Theorem: If all components are frequency scaled, roots (poles and zeros) will move along a constant Q locus



Proof:

$$T_{FS}(s) = T(s)\Big|_{s=\frac{s}{\eta}}$$

Let p be a pole (or zero) of T(s)

$$T(p)=0 \qquad \text{consider} \qquad p = \frac{p}{\eta}$$

$$T_{FS}(s) = T\left(\frac{s}{\eta}\right) = T(s)$$

Since true for any variable, substitute in p

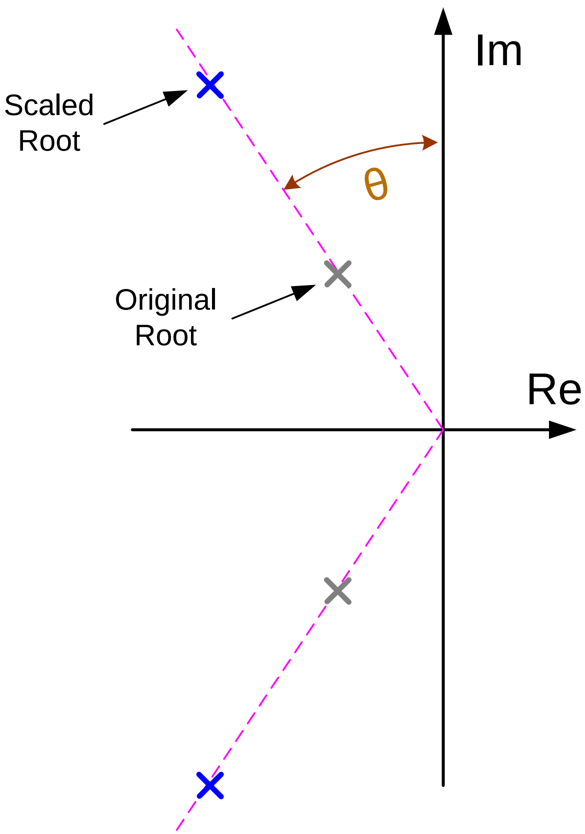
$$T_{FS}(p) = T\left(\frac{p}{\eta}\right) = T(p) = 0$$

Thus **p** is a pole (or zero) of $T_{FS}(s)$

Proof of Corollary 2:

Recall:

Theorem: If all components are frequency scaled, roots (poles and zeros) will move along a constant Q locus



Proof: Thus **p** is a pole (or zero) of $T_{FS}(s)$

$$p = \frac{p}{\eta}$$

$$p = p\eta$$

Express **p** in polar form

$$p = re^{j\beta}$$

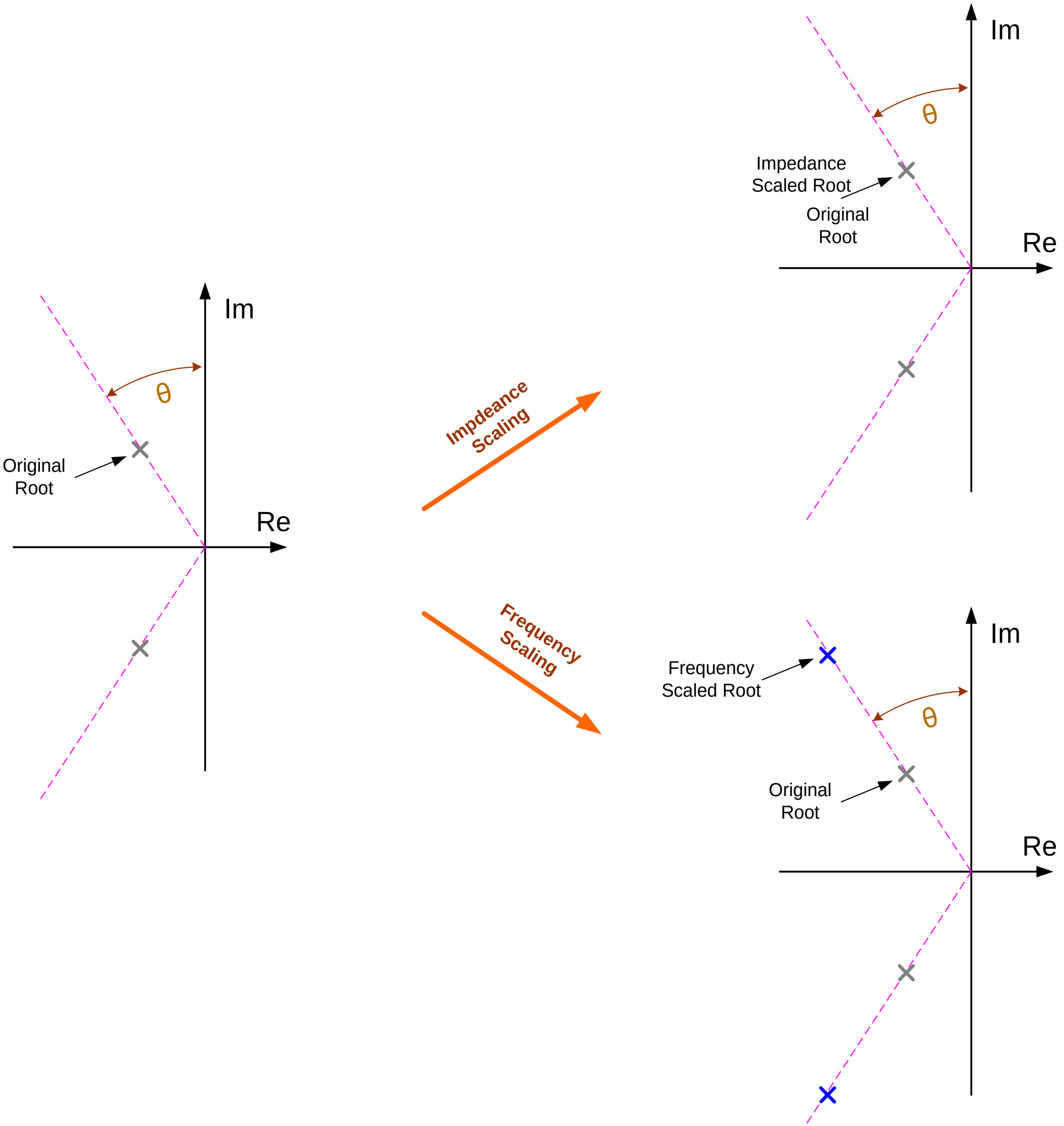
$$p = \eta p = \eta re^{j\beta}$$

Thus **p** and **p** have the same angle

Thus the scaled root has the same root Q

Proof of Corollary 2: Impedance and Frequency Scaling

Recall:



Proof of Corollary 2:

Corollary 2: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors then $\sum_{i=1}^{k_2} S_{C_i}^{Q} = 0$ and $\sum_{i=1}^{k_1} S_{R_i}^{Q} = 0$

Since impedance scaling does not change pole (or zero) Q, the pole (or zero) Q must be homogeneous of order 0 in the impedances

(For more generality, assume k_3 inductors)

$$\sum_{i=1}^{k_1} S_{R_i}^{Q} + \sum_{i=1}^{k_2} S_{1/C_i}^{Q} + \sum_{i=1}^{k_3} S_{L_i}^{Q} = 0 \qquad (1)$$

Since frequency scaling does not change pole (or zero) Q, the pole (or zero) Q must be homogeneous of order 0 in the frequency scaling elements

$$\sum_{i=1}^{k_2} S_{C_i}^{Q} + \sum_{i=1}^{k_3} S_{L_i}^{Q} = 0 \qquad (2)$$

Proof of Corollary 2:

$$\sum_{i=1}^{k_1} S_{R_i}^Q + \sum_{i=1}^{k_2} S_{1/C_i}^Q + \sum_{i=1}^{k_3} S_{L_i}^Q = 0 \qquad (1)$$

$$\sum_{i=1}^{k_2} S_{C_i}^Q + \sum_{i=1}^{k_3} S_{L_i}^Q = 0 \qquad (2)$$

From theorem about sensitivity of reciprocals, can write (1) as

$$\sum_{i=1}^{k_1} S_{R_i}^Q - \sum_{i=1}^{k_2} S_{C_i}^Q + \sum_{i=1}^{k_3} S_{L_i}^Q = 0 \qquad (3)$$

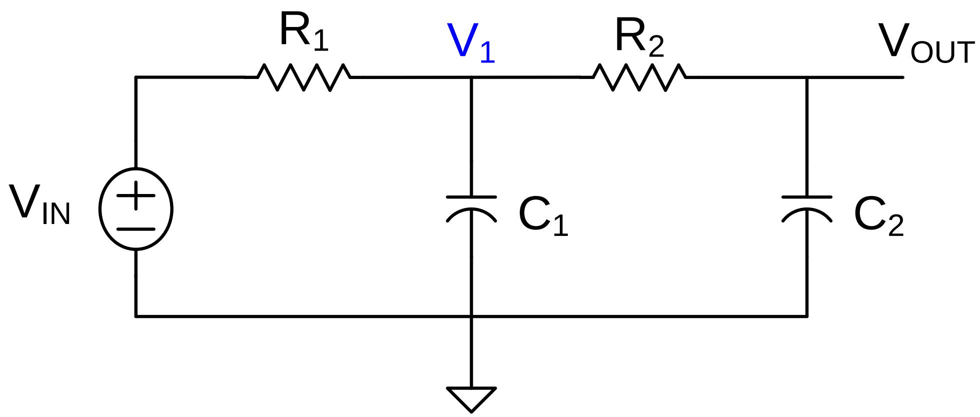
It follows from (2) and (3) that

$$\sum_{i=1}^{k_1} S_{R_i}^Q - 2\sum_{i=1}^{k_3} S_{L_i}^Q = 0 \qquad (4)$$

Since RC network, it follows from (4) and (2) that

$$\sum_{i=1}^{k_1} S_{R_i}^Q = 0 \qquad \sum_{i=1}^{k_2} S_{C_i}^Q = 0$$

Example



Determine the passive Q sensitivities

$$S^Q_{R_1} \quad S^Q_{R_2} \quad S^Q_{C_1} \quad S^Q_{C_2}$$

$$\left.\begin{array}{l} V_{OUT}\left(sC_1+G_2\right) = V_1 G_2 \\ V_1\left(sC_1+G_1+G_2\right)=V_{IN}G_1 + V_{OUT}G_2 \end{array}\right\}$$

$$T(s) = \frac{1}{s^2\left(R_1R_2C_1C_2\right)+s\left(R_1C_1+R_1C_2+R_2C_2\right)+1}$$

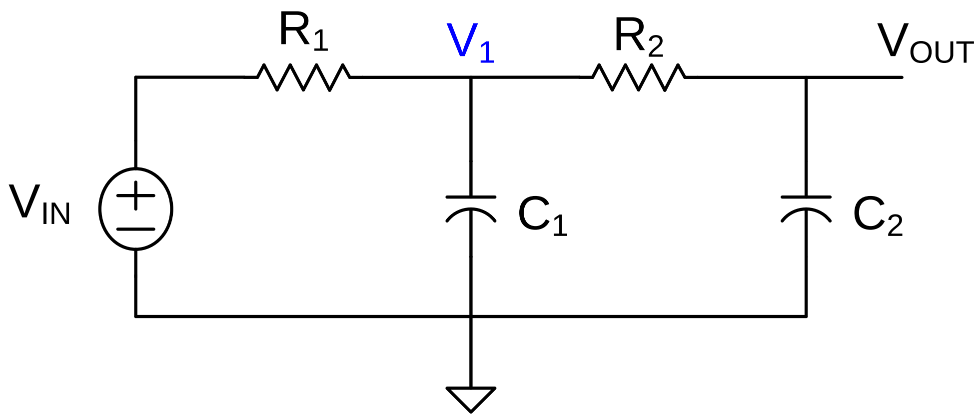
$$\omega_0 = \frac{1}{\sqrt{R_1R_2C_1C_2}}$$

$$Q = \frac{\sqrt{R_1R_2C_1C_2}}{R_1C_1+R_1C_2+R_2C_2}$$

By the definition of sensitivity, it follows that

$$S^Q_{R_1} = \frac{\left(R_1C_1+R_1C_2+R_2C_1\right)\frac{1}{2}\left(R_1R_2C_1C_2\right)^{-1/2}R_2C_1C_2-\left(C_1+C_2\right)\left(R_1R_2C_1C_2\right)^{1/2}}{\left(R_1C_1+R_1C_2+R_2C_2\right)^2} \bullet \frac{R_1}{Q}$$

Example



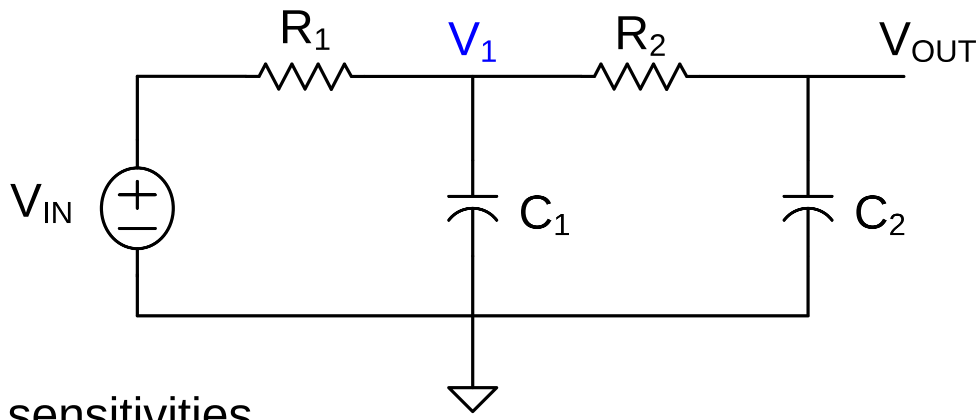
Determine the passive Q sensitivities $S^Q_{R_1}$ $S^Q_{R_2}$ $S^Q_{C_1}$ $S^Q_{C_2}$

$$S^Q_{R_1} = \frac{(R_1C_1+R_1C_2+R_2C_1)\frac{1}{2}(R_1R_2C_1C_2)^{-1/2}R_2C_1C_2-(C_1+C_2)(R_1R_2C_1C_2)^{1/2}}{(R_1C_1+R_1C_2+R_2C_2)^2} \bullet \frac{R_1}{Q}$$

Following some tedious manipulations, this simplifies to

$$S^Q_{R_1} = \frac{1}{2} - \frac{R_1(C_1+C_2)}{R_1C_1+R_1C_2+R_2C_2}$$

Example



Determine the passive Q sensitivities

Following the same type of calculations, can obtain

$$S_{R_1}^{Q} = \frac{1}{2} - \frac{R_1(C_1+C_2)}{R_1C_1+R_1C_2+R_2C_2} \qquad S_{R_2}^{Q} = \frac{1}{2} - \frac{R_2C_2}{R_1C_1+R_1C_2+R_2C_2}$$

$$S_{C_1}^{Q} = \frac{1}{2} - \frac{R_1C_1}{R_1C_1+R_1C_2+R_2C_2} \qquad S_{C_2}^{Q} = \frac{1}{2} - \frac{C_2(R_1+R_2)}{R_1C_1+R_1C_2+R_2C_2}$$

Verify

$$\sum_{i=1}^{k_2} S_{C_i}^{Q} = 0 \qquad \sum_{i=1}^{k_1} S_{R_i}^{Q} = 0$$

Could have saved considerable effort in calculations by using these theorems after $S_{R_1}^{Q}$ and $S_{C_1}^{Q}$ were calculated

Corollary 3: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors and if p_k is any pole and z_h is any zero, then

$$\sum_{i=1}^{k_1} S_{R_i}^{p_k} = -1 \qquad \sum_{i=1}^{k_2} S_{C_i}^{p_k} = -1$$

and

$$\sum_{i=1}^{k_1} S_{R_i}^{z_h} = -1 \qquad \sum_{i=1}^{k_2} S_{C_i}^{z_h} = -1$$

Corollary 3: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors and if p_k is any pole and z_h is any zero, then

$$\sum_{i=1}^{k_1} S_{R_i}^{p_k} = -1 \qquad \sum_{i=1}^{k_2} S_{C_i}^{p_k} = -1$$

and

$$\sum_{i=1}^{k_1} S_{R_i}^{z_h} = -1 \qquad \sum_{i=1}^{k_2} S_{C_i}^{z_h} = -1$$

Proof:

It was shown that scaling the frequency dependent elements by a factor η divides the pole (or zero) by η

Thus roots (poles and zeros) are homogeneous of order -1 in the frequency scaling elements

Proof:

Thus roots (poles and zeros) are homogeneous of order -1 in the frequency scaling elements

(For more generality, assume k_3 inductors)

$$\sum_{i=1}^{k_2} S_{C_i}^{p} + \sum_{i=1}^{k_3} S_{L_i}^{p} = -1 \qquad (1)$$

Since impedance scaling does not affects the poles, they are homogenous of order 0 in the impedances

$$\sum_{i=1}^{k_1} S_{R_i}^{p} + \sum_{i=1}^{k_2} S_{1/C_i}^{p} + \sum_{i=1}^{k_3} S_{L_i}^{p} = 0 \qquad (2)$$

Since there are no inductors in an active RC network, is follows from (1) that

$$\sum_{i=1}^{k_2} S_{C_i}^{p} = -1$$

And then from (2) and the theorem about sensitivity to reciprocals that

$$\sum_{i=1}^{k_1} S_{R_i}^{p} = -1$$

Corollary 4: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors and if Z_{IN} is any input impedance of the network, then

$$\sum_{i=1}^{k_1} S_{R_i}^{Z_{IN}} - \sum_{i=1}^{k_2} S_{C_i}^{Z_{IN}} = 1$$

Claim: If op amps in the filters considered previously are not ideal but are modeled by a gain $A(s)=1/(\tau s)$, then all previous summed sensitivities developed for ideal op amps hold provided they are evaluated a the nominal value of $\tau=0$

Sensitivity Analysis

If a closed-form expression for a function f is obtained, a straightforward but tedious analysis can be used to obtain the sensitivity of the function to any components

$$S_x^f = \frac{\partial f}{\partial x} \bullet \frac{x}{f}$$

Consider:

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i} = K \frac{\prod_{i=1}^{m} (s-z_i)}{\prod_{i=1}^{n} (s-p_i)}$$

Closed-form expressions for T(s), T(jω), |T(jω)|, $\angle T(j\omega)$, a_i, b_i, can be readily obtained

Sensitivity Analysis

If a closed-form expression for a function f is obtained, a straightforward but tedious analysis can be used to obtain the sensitivity of the function to any components

$$S_x^f = \frac{\partial f}{\partial x} \bullet \frac{x}{f}$$

Consider:

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i} = K \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)}$$

Closed-form expressions for p_i, z_i, pole or zero Q, pole or zero ω_0, peak gain, ω_{3dB}, BW, … (generally the most critical and useful circuit characteristics) are difficult or impossible to obtain !

Bilinear Property of Electrical Networks

Theorem: Let x be any component or Op Amp time constant (1st order Op Amp model) of any linear active network employing a finite number of amplifiers and lumped passive components. Any transfer function of the network can be expressed in the form

$$T(s)=\frac{N_0(s)+xN_1(s)}{D_0(s)+xD_1(s)}$$

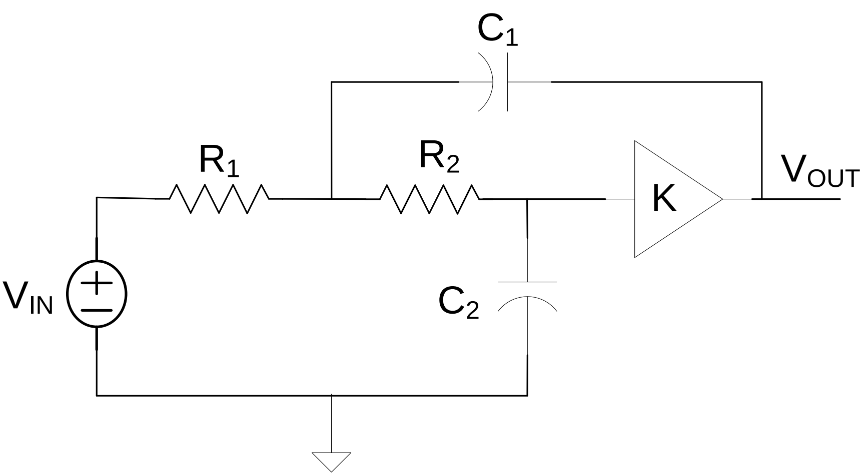
where N_0, N_1, D_0, and D_1 are polynomials in s that are not dependent upon x

A function that can be expressed as given above is said to be a bilinear function in the variable x and this is termed a bilateral property of electrical networks.

The bilinear relationship is useful for

1. Checking for possible errors in an analysis
2. Pole sensitivity analysis

Example of Bilinear Property : +KRC Lowpass Filter



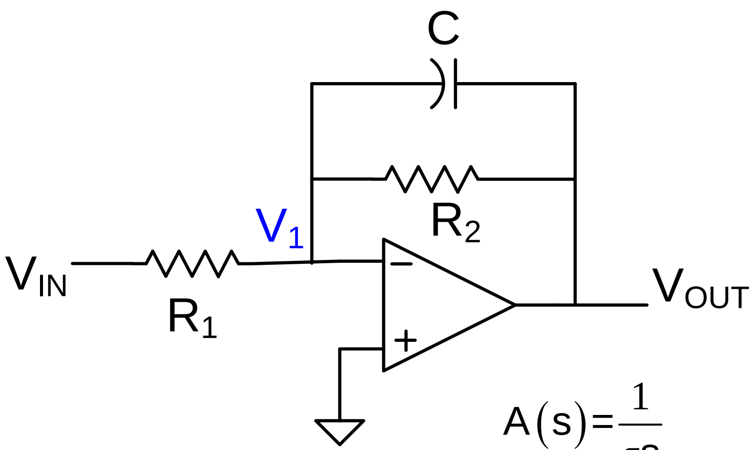
$$T(s)=\frac{\dfrac{K_0}{R_1R_2C_1C_2}}{s^2+s\left[\dfrac{1}{R_1C_1}+\dfrac{1}{R_2C_1}+\dfrac{(1-K_0)}{R_2C_2}\right]+\dfrac{1}{R_1R_2C_1C_2}+K_0\tau s\left(s^2+s\left[\dfrac{1}{R_1C_1}+\dfrac{1}{R_2C_1}+\dfrac{1}{R_2C_2}\right]+\dfrac{1}{R_1R_2C_1C_2}\right)}$$

Consider R_1

$$T(s)=\frac{\dfrac{K_0}{R_2C_1C_2}}{R_1s^2+s\left[\dfrac{1}{C_1}+R_1\dfrac{1}{R_2C_1}+R_1\dfrac{(1-K_0)}{R_2C_2}\right]+\dfrac{1}{R_2C_1C_2}+K_0\tau s\left(R_1s^2+s\left[\dfrac{1}{C_1}+R_1\dfrac{1}{R_2C_1}+R_1\dfrac{1}{R_2C_2}\right]+\dfrac{1}{R_2C_1C_2}\right)}$$

$$T(s)=\frac{\left[\dfrac{K_0}{R_2C_1C_2}\right]+R_1\bullet[0]}{\left[s\dfrac{1}{C_1}+\dfrac{1}{R_2C_1C_2}+K_0\tau s\left(s\dfrac{1}{C_1}\right)+\dfrac{1}{R_2C_1C_2}\right]+R_1\left[s^2+s\left[\dfrac{1}{R_2C_1}+\dfrac{(1-K_0)}{R_2C_2}\right]+K_0\tau s\left(s^2+s\left[\dfrac{1}{R_2C_1}+\dfrac{1}{R_2C_2}\right]\right)\right]}$$

Example of Bilinear Property



$$A(s)=\frac{1}{\tau s}$$

$$\left.\begin{aligned} &V_1\left(G_1+G_2+sC\right) = V_{IN}G_1+V_{OUT}\left(sC+G_2\right) \\ &V_{OUT}= -V_1\left(\frac{1}{\tau s}\right) \end{aligned}\right\}$$

$$T(s) = \frac{-R_2}{R_1+R_1R_2Cs+\tau s\left(sCR_1R_2+R_1+R_2\right)}$$

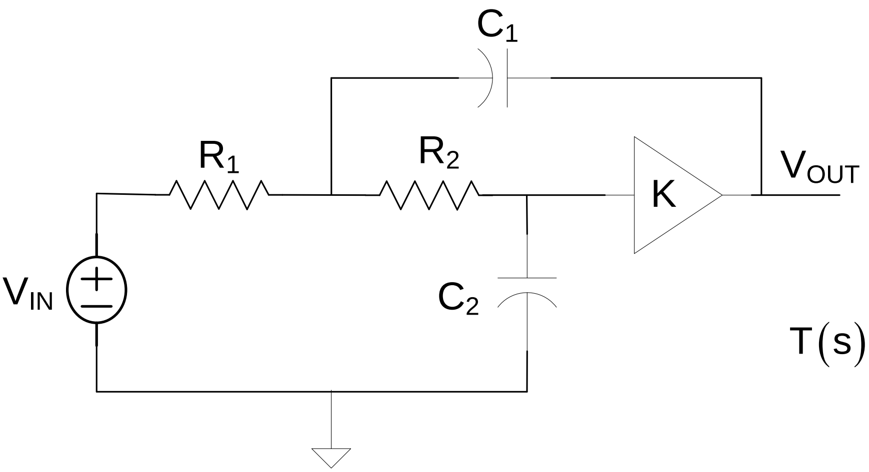
Consider R_1

$$T(s) = \frac{-R_2+0\bullet R_1}{\left[\tau sR_2\right]+R_1\left[1+R_2Cs+\tau s\left(sCR_2+1\right)\right]}$$

Consider τ

$$T(s) = \frac{-R_2+0\bullet \tau}{\left[R_1\left(1+R_2Cs\right)\right]+\tau\left[sR_2+sR_1\left(sCR_2+1\right)\right]}$$

Example of Bilinear Property : +KRC Lowpass Filter



Equal R Equal C

$$T(s)=\frac{\frac{K_0}{R^2C^2}}{s^2+s\left[\frac{(3\text{-}K_0)}{RC}\right]+\frac{1}{R^2C^2}+K_0\tau s\left(s^2+s\left[\frac{3}{RC}\right]+\frac{1}{R^2C^2}\right)}$$

$$T(s)=\frac{K_0}{R^2\left(C^2s^2+K_0\tau sC^2\right)+R\left(sC(3\text{-}K_0)+3K_0C\tau s^2\right)+(1+K_0\tau s)}$$

Can not eliminate the R^2 term

- Bilinear property only applies to individual components
- Bilinear property was established only for T(s)

Root Sensitivities

Consider expressing T(s) as a bilinear fraction in x

$$T(s)=\frac{N_0(s)+xN_1(s)}{D_0(s)+xD_1(s)}=\frac{N(s)}{D(s)}$$

Theorem: If z_i is any simple zero and/or p_i is any simple pole of T(s), then

$$S_x^{z_i}=\left(\frac{x}{z_i}\right)\left(\frac{-N_1(z_i)}{\frac{dN(z_i)}{dz_i}}\right) \quad \text{and} \quad S_x^{p_i}=\left(\frac{x}{p_i}\right)\left(\frac{-D_1(p_i)}{\frac{dD(p_i)}{dp_i}}\right)$$

Note: Do not need to find expressions for the poles or the zeros to fine the pole and zero sensitivities !

Root Sensitivities

Theorem: If p_i is any simple pole of T(s), then

$$S_x^{p_i} = \left(\frac{x}{p_i}\right)\left(\frac{-D_1(p_i)}{\frac{dD(p_i)}{dp_i}}\right)$$

Proof (similar argument for the zeros)

$$D(s) = D_0(s) + xD_1(s)$$

By definition of a pole,

$$D(p_i) = 0$$

$$\therefore \qquad D(p_i) = D_0(p_i) + xD_1(p_i)$$

Root Sensitivities

$$\therefore \qquad D(p_i) = D_0(p_i) + xD_1(p_i)$$

Differentiating this expression explicitly WRT x, we obtain

$$\frac{\partial D_0(p_i)}{\partial p_i}\frac{\partial p_i}{\partial x} + \left[x\frac{\partial D_1(p_i)}{\partial p_i}\frac{\partial p_i}{\partial x} + D_1(p_i) \right] = 0$$

Re-grouping, obtain

$$\frac{\partial p_i}{\partial x}\left[\frac{\partial D_0(p_i)}{\partial p_i} + x\frac{\partial D_1(p_i)}{\partial p_i} \right] = -D_1(p_i)$$

But term in brackets is derivative of $D(p_i)$, thus

$$\frac{\partial p_i}{\partial x} = -\frac{D_1(p_i)}{\left(\dfrac{\partial D(p_i)}{\partial p_i} \right)}$$

Root Sensitivities

$$\frac{\partial p_i}{\partial x} = -\frac{D_1(p_i)}{\left(\frac{\partial D(p_i)}{\partial p_i}\right)}$$

Finally, from the definition of sensitivity,

$$S_x^{p_i} = \frac{x}{p_i}\frac{\partial p_i}{\partial x} = -\left(\frac{x}{p_i}\right)\frac{D_1(p_i)}{\left(\frac{\partial D(p_i)}{\partial p_i}\right)}$$

Root Sensitivities

$$S_x^{p_i} = \frac{x}{p_i}\frac{\partial p_i}{\partial x} = -\left(\frac{x}{p_i}\right)\frac{D_1(p_i)}{\left(\frac{\partial D(p_i)}{\partial p_i}\right)}$$

Observation: Although the sensitivity expression is readily obtainable, direction information about the pole movement is obscured because the derivative is multiplied by the quantity p_i which is often complex. Usually will use either $\mathcal{S}_x^{p_i} = \frac{\partial p_i}{\partial x}$ or

$$\tilde{S}_x^{p_i} = \frac{x}{|p_i|}\frac{\partial p_i}{\partial x} = -\left(\frac{x}{|p_i|}\right)\frac{D_1(p_i)}{\left(\frac{\partial D(p_i)}{\partial p_i}\right)}$$

which preserve direction information when working with pole or zero sensitivity analysis.

Root Sensitivities

Summary: Pole (or zero) locations due to component variations can be approximated with simple analytical calculations without obtaining parametric expressions for the poles (or zeros).

$$p_i \simeq p_i\Big|_{\substack{\text{Ideal}\\ \text{Components}}} + \Delta p_i$$

where

$$\Delta p_i \simeq \Delta x \bullet \mathcal{S}_x^{p_i}$$

$$\mathcal{S}_x^{p_i} = -\frac{D_1(p_i)}{\left.\left(\dfrac{\partial D(p_i)}{\partial p_i}\right)\right|_{p_{iN}}}$$

and

$$D(s) = D_0(s) + x \bullet D_1(s)$$

Alternately,

$$\Delta p_i \simeq \left(|p_i| \frac{\Delta x}{x}\right) \tilde{S}_x^{p_i}$$

End of Lecture 22