EE 508
Lecture 23

Sensitivity Functions

- Transfer Function Sensitivity
- Examples

Review from last time

Corollary 3: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors and if p_k is any pole and z_h is any zero, then

$$\sum_{i=1}^{k_1} S_{R_i}^{p_k} = -1 \qquad \sum_{i=1}^{k_2} S_{C_i}^{p_k} = -1$$

and

$$\sum_{i=1}^{k_1} S_{R_i}^{z_h} = -1 \qquad \sum_{i=1}^{k_2} S_{C_i}^{z_h} = -1$$

Review from last time

Corollary 4: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors and if Z_{IN} is any input impedance of the network, then

$$\sum_{i=1}^{k_1} S_{R_i}^{Z_{IN}} - \sum_{i=1}^{k_2} S_{C_i}^{Z_{IN}} = 1$$

Review from last time

Bilinear Property of Electrical Networks

Theorem: Let x be any component or Op Amp time constant (1st order Op Amp model) of any linear active network employing a finite number of amplifiers and lumped passive components. Any transfer function of the network can be expressed in the form

$$T(s)=\frac{N_0(s)+xN_1(s)}{D_0(s)+xD_1(s)}$$

where N_0, N_1, D_0, and D_1 are polynomials in s that are not dependent upon x

A function that can be expressed as given above is said to be a bilinear function in the variable x and this is termed a bilateral property of electrical networks.

The bilinear relationship is useful for

1. Checking for possible errors in an analysis
2. Pole sensitivity analysis

Review from last time

Root Sensitivities

Consider expressing T(s) as a bilinear fraction in x

$$T(s)=\frac{N_0(s)+xN_1(s)}{D_0(s)+xD_1(s)}=\frac{N(s)}{D(s)}$$

Theorem: If z_i is any simple zero and/or p_i is any simple pole of T(s), then

$$S_x^{z_i}=\left(\frac{x}{z_i}\right)\left(\frac{-N_1(z_i)}{\frac{dN(z_i)}{dz_i}}\right) \quad \text{and} \quad S_x^{p_i}=\left(\frac{x}{p_i}\right)\left(\frac{-D_1(p_i)}{\frac{dD(p_i)}{dp_i}}\right)$$

Note: Do not need to find expressions for the poles or the zeros to fine the pole and zero sensitivities !

Review from last time

Root Sensitivities

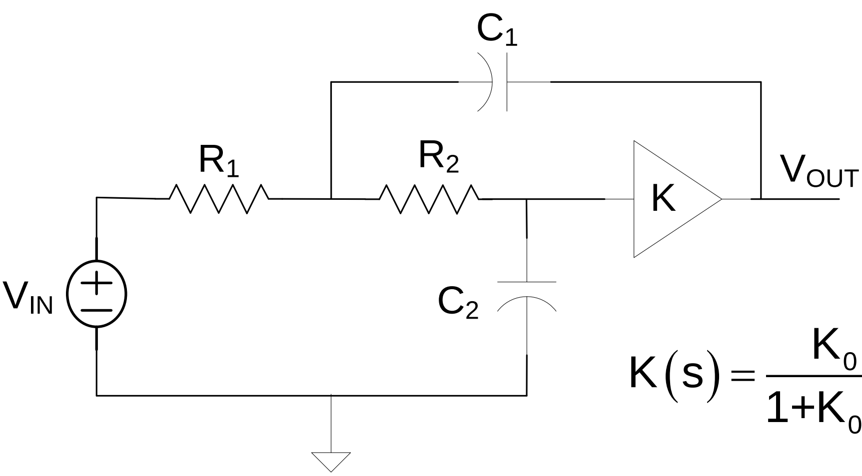
$$S_x^{p_i} = \frac{x}{p_i}\frac{\partial p_i}{\partial x} = -\left(\frac{x}{p_i}\right)\frac{D_1(p_i)}{\left(\frac{\partial D(p_i)}{\partial p_i}\right)}$$

Observation: Although the sensitivity expression is readily obtainable, direction information about the pole movement is obscured because the derivative is multiplied by the quantity p_i which is often complex. Usually will use either $\frac{\partial p_i}{\partial x}$ or

$$\tilde{S}_x^{p_i} = \frac{x}{|p_i|}\frac{\partial p_i}{\partial x} = -\left(\frac{x}{|p_i|}\right)\frac{D_1(p_i)}{\left(\frac{\partial D(p_i)}{\partial p_i}\right)}$$

which preserve direction information when working with pole or zero sensitivity analysis.

Example: Determine $\tilde{S}_{R_1}^{p_i}$ for the +KRC Lowpass Filter for equal R, equal C



$$T(s)=\frac{N_0(s)+xN_1(s)}{D_0(s)+xD_1(s)}$$

$$\tilde{S}_x^{p_i}=\frac{x}{|p_i|}\frac{\partial p_i}{\partial x}=-\left(\frac{x}{|p_i|}\right)\frac{D_1(p_i)}{\left(\frac{\partial D(p_i)}{\partial p_i}\right)}$$

$$K(s)=\frac{K_0}{1+K_0\tau s}$$

$$T(s)=\frac{\frac{K_0}{R_1R_2C_1C_2}}{s^2+s\left[\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{(1\text{-}K_0)}{R_2C_2}\right]+\frac{1}{R_1R_2C_1C_2}+K_0\tau s\left(s^2+s\left[\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1}{R_2C_2}\right]+\frac{1}{R_1R_2C_1C_2}\right)}$$

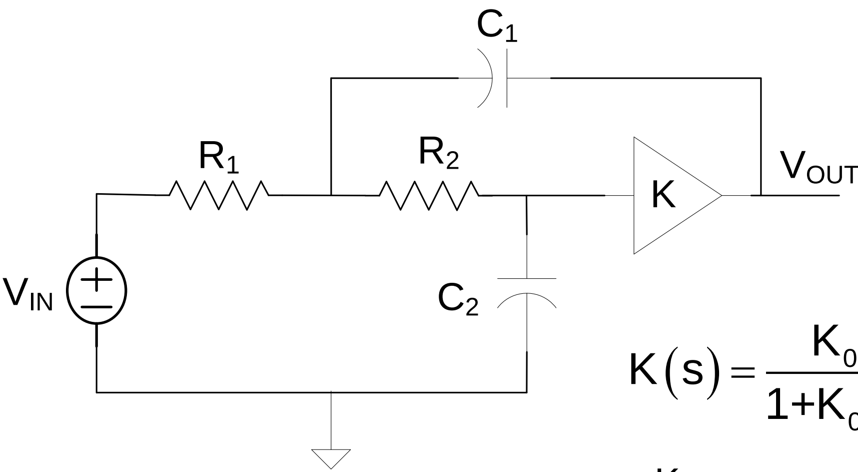
write in bilinear form

$$T(s)=\frac{\frac{K_0}{R_2C_1C_2}}{\left(s\frac{1}{C_1}+\frac{1}{R_2C_1C_2}+K_0\tau s\left(s\frac{1}{C_1}\right)+\frac{1}{R_2C_1C_2}\right)+R_1\left[s^2+s\left[\frac{1}{R_2C_1}+\frac{(1\text{-}K_0)}{R_2C_2}+K_0\tau s\left(s^2+s\left[\frac{1}{R_2C_1}+\frac{1}{R_2C_2}\right]\right)\right]\right]}$$

evaluate at τ=0

$$T(s)=\frac{\frac{K_0}{R_2C_1C_2}}{\left(s\frac{1}{C_1}+\frac{1}{R_2C_1C_2}+\frac{1}{R_2C_1C_2}\right)+R_1\left[s^2+s\left[\frac{1}{R_2C_1}+\frac{(1\text{-}K_0)}{R_2C_2}\right]\right]}$$

Example: Determine $\tilde{S}_{R_1}^{p_i}$ for the +KRC Lowpass Filter for equal R, equal C



$$T(s)=\frac{N_0(s)+xN_1(s)}{D_0(s)+xD_1(s)}$$

$$\tilde{S}_x^{p_i}=\frac{x}{|p_i|}\frac{\partial p_i}{\partial x}=-\left(\frac{x}{|p_i|}\right)\frac{D_1(p_i)}{\left(\frac{\partial D(p_i)}{\partial p_i}\right)}$$

$T(s)=$

$$K(s)=\frac{K_0}{1+K_0\tau s}$$

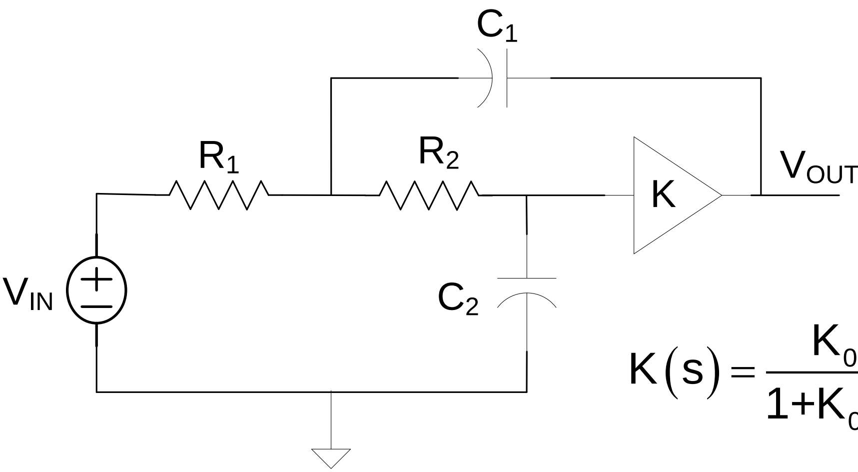
$$T(s)=\frac{\frac{K_0}{R_2C_1C_2}}{\left(s\frac{1}{C_1}+\frac{1}{R_2C_1C_2}+\frac{1}{R_2C_1C_2}\right)+R_1\left[s^2+s\left[\frac{1}{R_2C_1}+\frac{(1-K_0)}{R_2C_2}\right]\right]}$$

$$D_1(s)=s^2+s\left[\frac{1}{R_2C_1}+\frac{(1-K_0)}{R_2C_2}\right]$$

$$D(s)=\left(s\frac{1}{C_1}+\frac{1}{R_2C_1C_2}+\frac{1}{R_2C_1C_2}\right)+R_1\left[s^2+s\left[\frac{1}{R_2C_1}+\frac{(1-K_0)}{R_2C_2}\right]\right]=R_1\left(s^2+s\left[\frac{\omega_0}{Q}\right]+\omega_0^2\right)$$

$$\tilde{S}_{R_1}^{p}=\frac{x}{|p_i|}\frac{\partial p_i}{\partial x}=-\left(\frac{1}{|p_i|}\right)\frac{p^2+p\left[\frac{1}{R_2C_1}+\frac{(1-K_0)}{R_2C_2}\right]}{\left(2p_i+\frac{\omega_0}{Q}\right)}$$

Example: Determine $\tilde{S}_{R_1}^{p_i}$ for the +KRC Lowpass Filter for equal R, equal C



$$K(s)=\frac{K_0}{1+K_0\tau s}$$

$$T(s)=\frac{N_0(s)+xN_1(s)}{D_0(s)+xD_1(s)}$$

$$\tilde{S}_x^{p_i}=\frac{x}{|p_i|}\frac{\partial p_i}{\partial x}=-\left(\frac{x}{|p_i|}\right)\frac{D_1(p_i)}{\left(\frac{\partial D(p_i)}{\partial p_i}\right)}$$

$T(s)=$

$$T(s)=\frac{\frac{K_0}{R_1R_2C_1C_2}}{s^2+s\left[\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{(1-K_0)}{R_2C_2}\right]+\frac{1}{R_1R_2C_1C_2}}$$

$$p^2+p\left[\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{(1-K_0)}{R_2C_2}\right]+\frac{1}{R_1R_2C_1C_2}=0$$

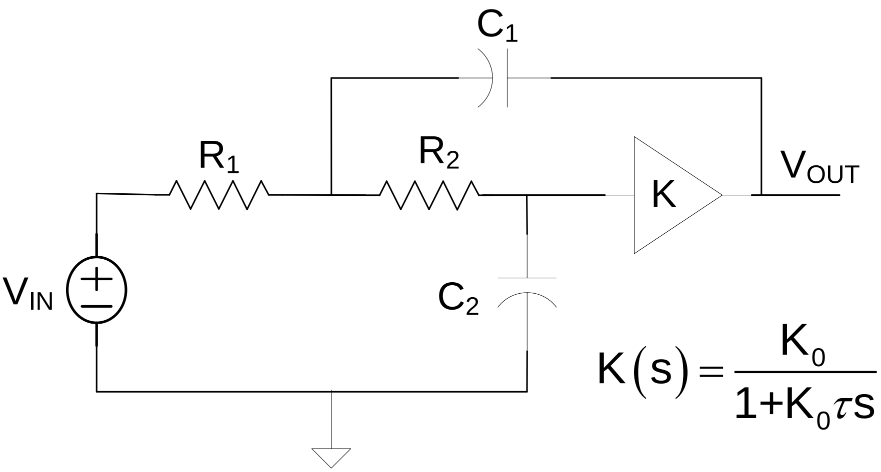
$$p^2+p\left[\frac{1}{R_2C_1}+\frac{(1-K_0)}{R_2C_2}\right]=-\frac{1}{R_1R_2C_1C_2}-p\frac{1}{R_1C_1}$$

$$\tilde{S}_{R_1}^{p}=\frac{x}{|p_i|}\frac{\partial p_i}{\partial x}=-\left(\frac{1}{|p_i|}\right)\frac{p^2+p\left[\frac{1}{R_2C_1}+\frac{(1-K_0)}{R_2C_2}\right]}{\left(2p_i+\frac{\omega_0}{Q}\right)}$$

$$\tilde{S}_{R_1}^{p}=\frac{x}{|p_i|}\frac{\partial p_i}{\partial x}=\left(\frac{1}{|p_i|}\right)\frac{\frac{1}{R_1R_2C_1C_2}+p\frac{1}{R_1C_1}}{\left(2p_i+\frac{\omega_0}{Q}\right)}$$

$$\tilde{S}_{R_1}^{p}=\frac{x}{|p_i|}\frac{\partial p_i}{\partial x}=\left(\frac{1}{\omega_0}\right)\frac{\omega_0^2+p\frac{1}{R_1C_1}}{\left(2p_i+\frac{\omega_0}{Q}\right)}$$

Example: Determine $\tilde{S}_{R_1}^{p_i}$ for the +KRC Lowpass Filter for equal R, equal C



$$K(s)=\frac{K_0}{1+K_0\tau s}$$

$$\tilde{S}_x^{p_i}=\frac{x}{|p_i|}\frac{\partial p_i}{\partial x}=\left(\frac{1}{\omega_0}\right)\frac{\omega_0^2+p\frac{1}{R_1C_1}}{\left(2p_i+\frac{\omega_0}{Q}\right)}$$

For equal R, equal C $\quad \omega_0 = \frac{1}{RC}$

$$\tilde{S}_{R_1}^{p}=\frac{x}{|p_i|}\frac{\partial p_i}{\partial x}=\left(\frac{1}{\omega_0}\right)\frac{\omega_0^2+p\omega_0}{\left(2p_i+\frac{\omega_0}{Q}\right)}$$

$$\tilde{S}_{R_1}^{p}=\frac{x}{|p_i|}\frac{\partial p_i}{\partial x}=\frac{\omega_0+p}{\left(2p+\frac{\omega_0}{Q}\right)}$$

$$\tilde{S}_{R_1}^{p}=\frac{\omega_0-\frac{\omega_0}{2Q}\pm\frac{\omega_0}{2Q}\sqrt{1\text{-}4Q^2}}{\pm\frac{\omega_0}{Q}\sqrt{1\text{-}4Q^2}}$$

$$\tilde{S}_{R_1}^{p}=\frac{Q\text{-}\frac{1}{2}\pm\frac{1}{2}\sqrt{1\text{-}4Q^2}}{\pm\sqrt{1\text{-}4Q^2}}$$

Transfer Function Sensitivities

$$\left. S_x^{T(s)} \right|_{s=j\omega} = S_x^{T(j\omega)}$$

$$S_x^{T(j\omega)} = S_x^{|T(j\omega)|} + j\theta S_x^{\theta}$$ where $\theta = \angle T(j\omega)$

$$S_x^{|T(j\omega)|} = \mathrm{Re}\left(S_x^{T(j\omega)}\right)$$

$$S_x^{\theta} = \frac{1}{\theta}\,\mathrm{Im}\left(S_x^{T(j\omega)}\right)$$

Transfer Function Sensitivities

If T(s) is expressed as $T(s) = \dfrac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i} = \dfrac{N(s)}{D(s)}$

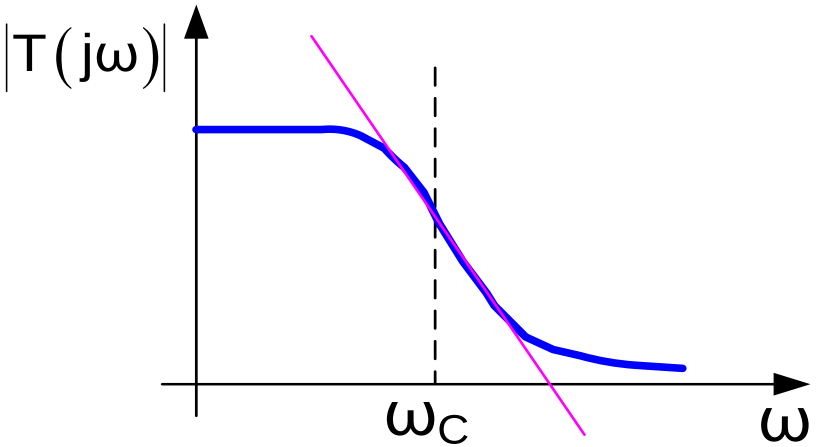
then $$S_x^{T(s)} = \frac{\sum_{i=0}^{m} a_i s^i S_x^{a_i}}{N(s)} - \frac{\sum_{i=0}^{n} b_i s^i S_x^{b_i}}{D(s)}$$

If T(s) is expressed as $T(s) = \dfrac{N_0(s) + xN_1(s)}{D_0(s) + xD_1(s)}$

$$S_x^{T(s)} = \frac{x\left[D_0(s)N_1(s) - N_0(s)D_1(s)\right]}{\left(N_0(s) + xN_1(s)\right)\left(D_0(s) + xD_1(s)\right)}$$

Band-edge Sensitivities

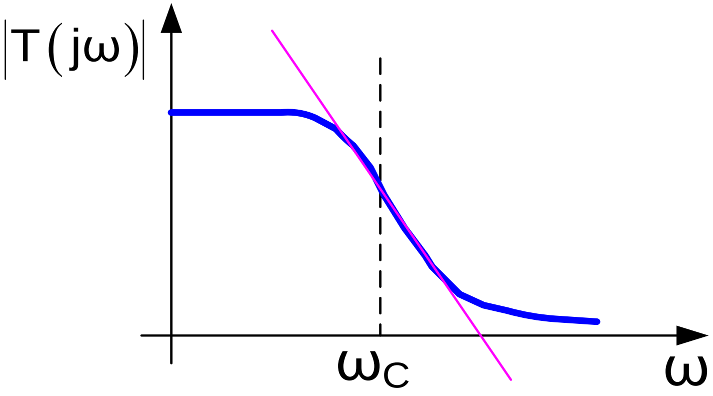
The band edge of a filter is often of interest. A closed-form expression for the band-edge of a filter may not be attainable and often the band-edges are distinct from the ω_0 of the poles. But the sensitivity of the band-edges to a parameter x is often of interest.



Want

$$S_x^{\omega_C} = \frac{\partial \omega_C}{\partial x} \bullet \frac{x}{\omega_C}$$

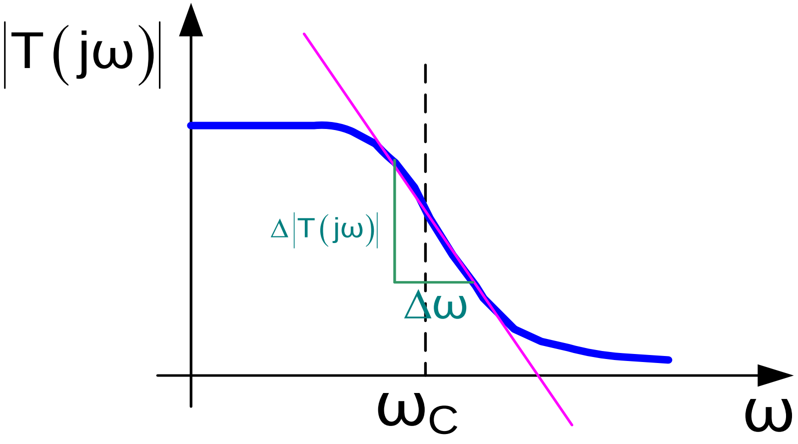
Band-edge Sensitivities



Theorem: The sensitivity of the band-edge of a filter is given by the expression

$$S_x^{\omega_C} = \frac{S_x^{|T(j\omega)|}\Big|_{\omega=\omega_C}}{S_\omega^{|T(j\omega)|}\Big|_{\omega=\omega_C}}$$

Band-edge Sensitivities



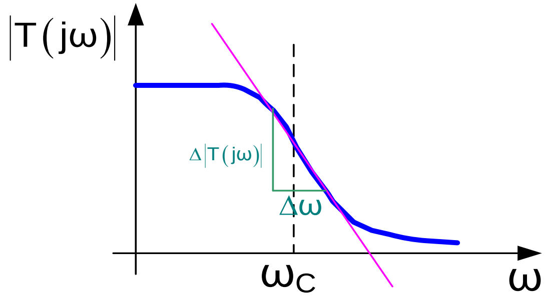
Proof:

Observe
$$\frac{\partial\left|T(j\omega)\right|}{\partial\omega} \simeq \frac{\Delta\left|T(j\omega)\right|}{\Delta\omega}$$

$$\frac{\partial\left|T(j\omega)\right|}{\partial\omega} \simeq \frac{\Delta\left|T(j\omega)\right|}{\Delta x} \bullet \frac{\Delta x}{\Delta\omega} \simeq \frac{\dfrac{\partial\left|T(j\omega)\right|}{\partial x}}{\dfrac{\partial\omega}{\partial x}}$$

Band-edge Sensitivities

$$\frac{\partial\left|T(j\omega)\right|}{\partial\omega} \simeq \frac{\Delta\left|T(j\omega)\right|}{\Delta x}\bullet\frac{\Delta x}{\Delta\omega} \simeq \frac{\dfrac{\partial\left|T(j\omega)\right|}{\partial x}}{\dfrac{\partial\omega}{\partial x}}$$



$$\frac{\partial\omega}{\partial x} \simeq \frac{\dfrac{\partial\left|T(j\omega)\right|}{\partial x}}{\dfrac{\partial\left|T(j\omega)\right|}{\partial\omega}}$$

$$\frac{\partial\omega}{\partial x} \simeq \frac{\dfrac{\partial\left|T(j\omega)\right|}{\partial x}\bullet\dfrac{x}{\left|T(j\omega)\right|}}{\dfrac{\partial\left|T(j\omega)\right|}{\partial\omega}\bullet\dfrac{\omega}{\left|T(j\omega)\right|}}\left(\frac{\omega}{x}\right)$$

$$\frac{\partial\omega}{\partial x}\bullet\left(\frac{x}{\omega}\right) \simeq \frac{\dfrac{\partial\left|T(j\omega)\right|}{\partial x}\bullet\dfrac{x}{\left|T(j\omega)\right|}}{\dfrac{\partial\left|T(j\omega)\right|}{\partial\omega}\bullet\dfrac{\omega}{\left|T(j\omega)\right|}}$$

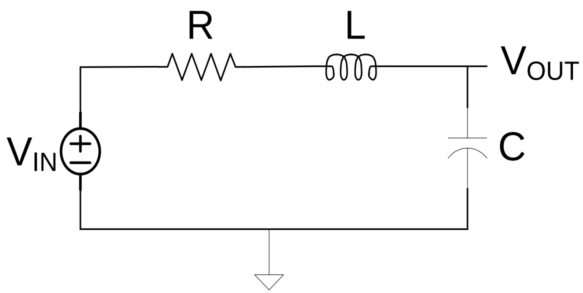
$$S_x^{\omega} = \frac{S_x^{\left|T(j\omega)\right|}}{S_\omega^{\left|T(j\omega)\right|}}$$

$$S_x^{\omega_C} = \frac{\left.S_x^{\left|T(j\omega)\right|}\right|_{\omega=\omega_C}}{\left.S_\omega^{\left|T(j\omega)\right|}\right|_{\omega=\omega_C}}$$

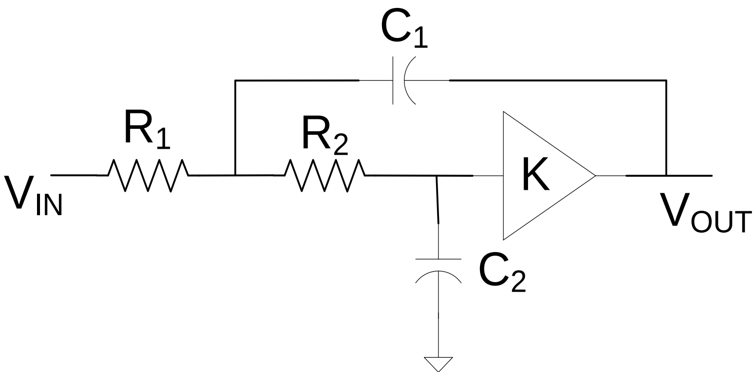
Sensitivity Comparisons

Consider 5 second-order lowpass filters

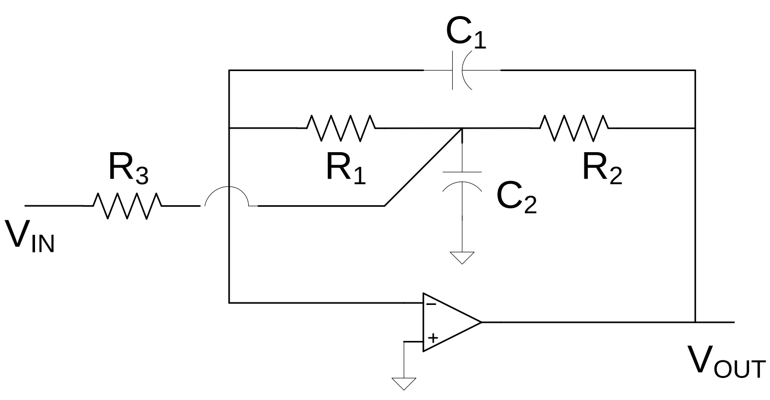
(all can realize same T(s) within a gain factor)



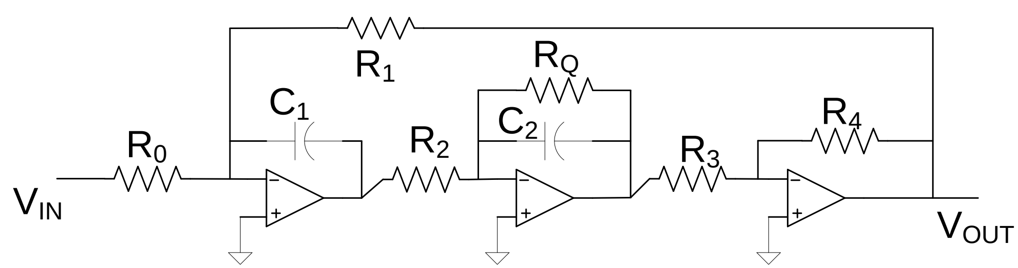
Passive RLC



+KRC



Bridged-T Feedback

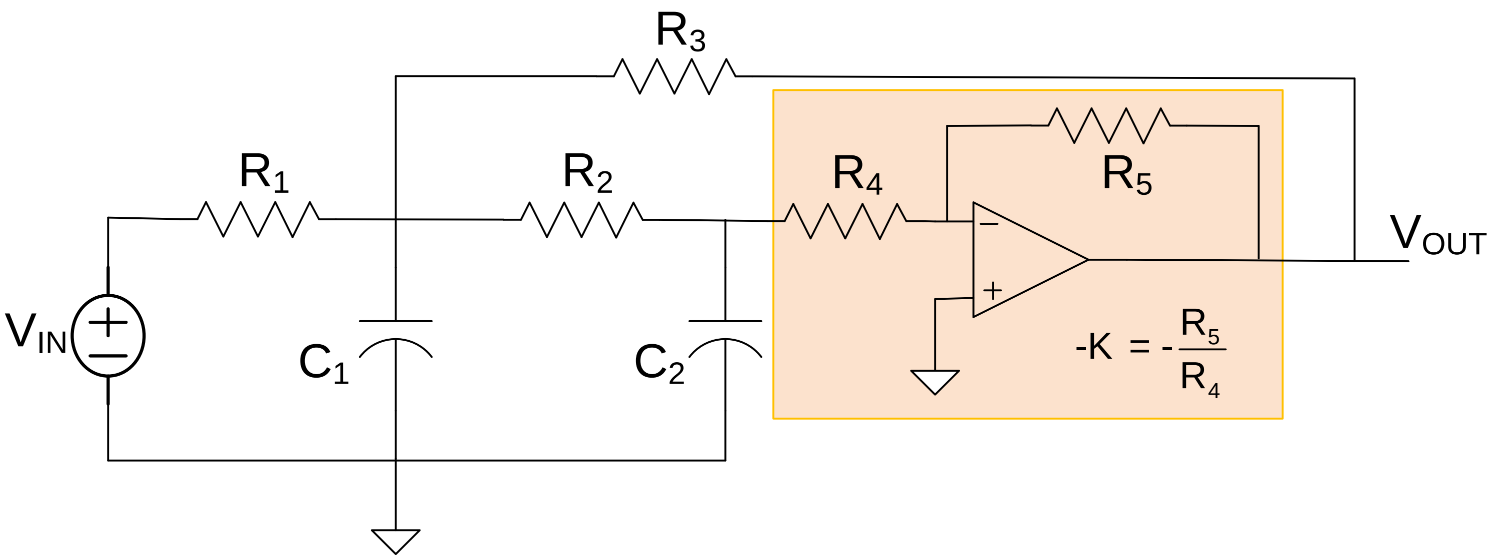


Two-Integrator Loop

Sensitivity Comparisons

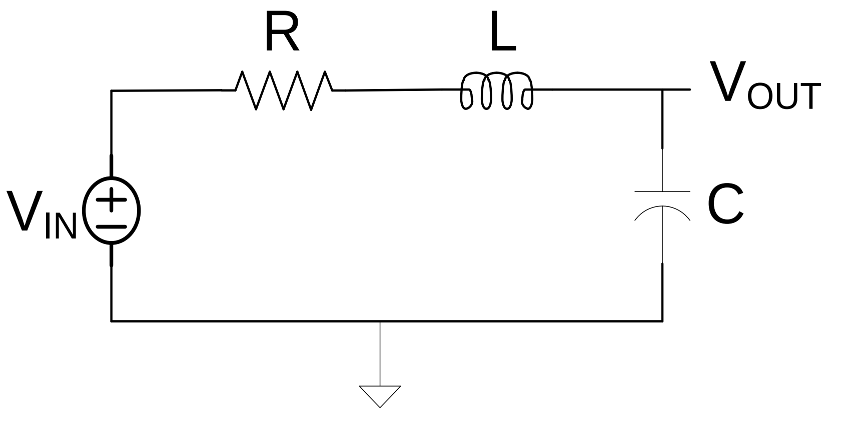
Consider 5 second-order lowpass filters

(all can realize same T(s) within a gain factor)



-KRC Lowpass

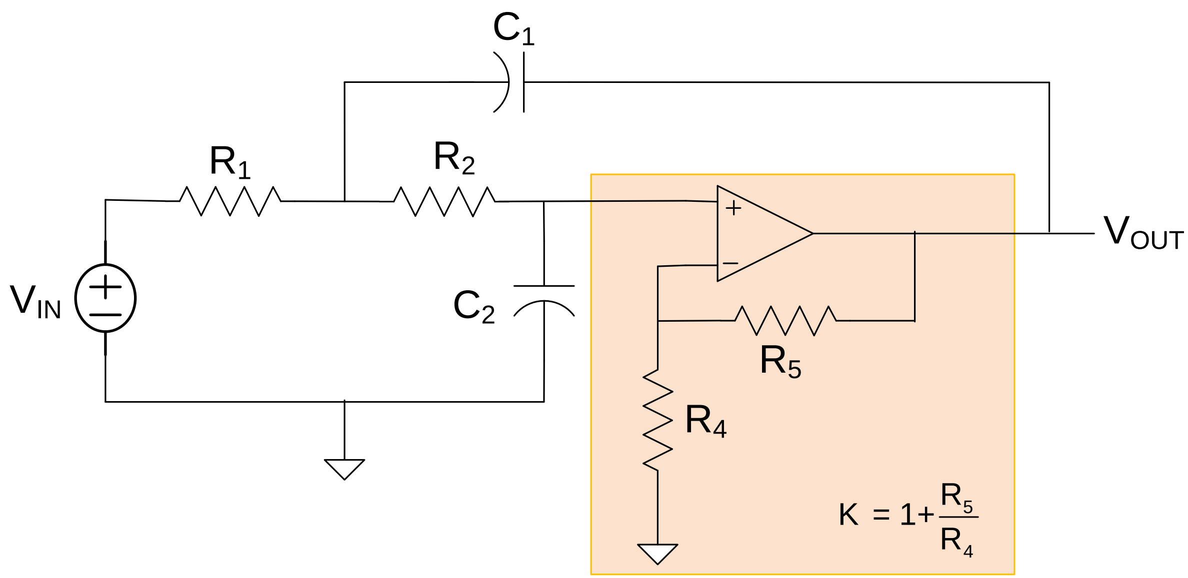
a) – Passive RLC



$$T(s) = \frac{V_{OUT}}{V_{IN}} = \frac{1/LC}{s^2 + s\frac{R}{L} + 1/LC}$$

$$\omega_0 = \sqrt{\frac{1}{LC}} \qquad Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

b) + KRC (a Sallen and Key filter)



$$T(s) = \frac{\dfrac{K}{R_1R_2C_1C_2}}{s^2+s\left[\left(\dfrac{1}{\sqrt{R_1R_2C_1C_2}}\right)\left(\sqrt{\dfrac{R_1C_1}{R_2C_2}}+\sqrt{\dfrac{R_2C_2}{R_1C_1}}+\sqrt{\dfrac{R_1C_2}{R_2C_1}}-K\sqrt{\dfrac{R_1C_1}{R_2C_2}}\right)\right]+\dfrac{1}{R_1R_2C_1C_2}}$$

$$\omega_0 = \sqrt{\frac{1}{R_1R_2C_1C_2}}$$

$$Q = \frac{1}{\left(\sqrt{\dfrac{R_1C_1}{R_2C_2}}+\sqrt{\dfrac{R_2C_2}{R_1C_1}}+\sqrt{\dfrac{R_1C_2}{R_2C_1}}-K\sqrt{\dfrac{R_1C_1}{R_2C_2}}\right)}$$

Case b1 : Equal R, Equal C

$$R_1 = R_2 = R \qquad C_1 = C_2 = C$$

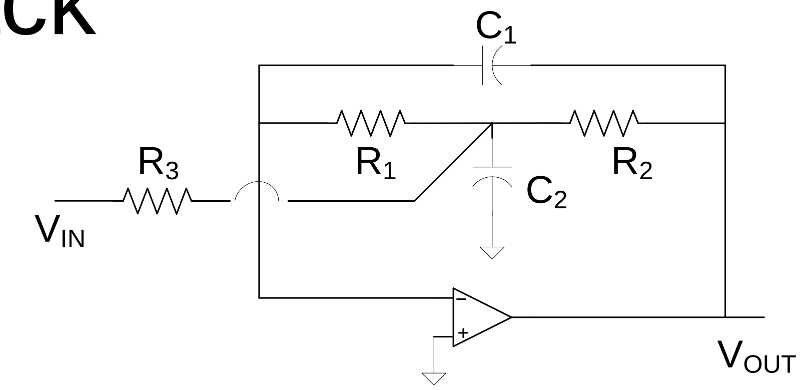
$$\omega_0 = \frac{1}{RC} \qquad K = 3 - \frac{1}{Q}$$

Case b2 : Equal R, K=1

$$R_1 = R_2 = R \qquad Q = \frac{1}{2}\sqrt{\frac{C_1}{C_2}}$$

$$T(s) = \frac{K\omega_0^2}{s^2 + s\frac{\omega_0}{Q} + \omega_0^2}$$

c) Bridged T Feedback



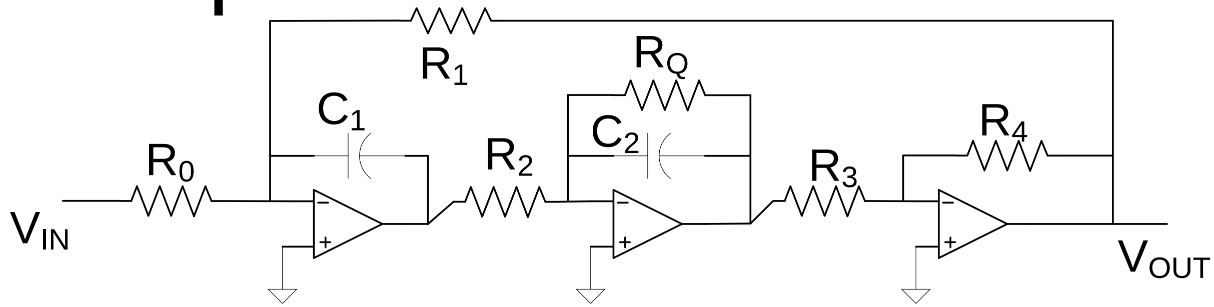
$$T(s) = \frac{\frac{1}{R_1R_3C_1C_2}}{s^2 + s\left[\left(\sqrt{\frac{C_2}{C_1}}\right)\left(\frac{1}{\sqrt{R_1R_2C_1C_2}}\right)\left(\sqrt{\frac{R_1}{R_3}} + \sqrt{\frac{R_2}{R_1}} + \frac{\sqrt{R_1R_2}}{R_3}\right)\right] + \frac{1}{R_1R_2C_1C_2}}$$

$$\omega_0 = \sqrt{\frac{1}{R_1R_2C_1C_2}} \qquad Q = \frac{1}{\left(\sqrt{\frac{C_2}{C_1}}\right)\left(\sqrt{\frac{R_1}{R_3}} + \sqrt{\frac{R_2}{R_1}} + \frac{\sqrt{R_1R_2}}{R_3}\right)}$$

If $R_1=R_2=R_3=R$ and $C_2=9Q^2C_1$

$$T(s) = \frac{\frac{1}{9Q^2R^2C_1^2}}{s^2 + s\left[\left(\frac{1}{3Q^2RC_1}\right)\right] + \frac{1}{9Q^2R^2C_1^2}}$$

d) 2 integrator loop



$$T(s) = -\frac{\frac{R_4}{R_3} \bullet \frac{1}{R_0 R_2 C_1 C_2}}{s^2 + s\left(\frac{1}{R_Q C_2}\right) + \frac{R_4}{R_3} \bullet \frac{1}{R_0 R_2 C_1 C_2}}$$

$$\omega_0 = \sqrt{\frac{R_4}{R_3} \bullet \frac{1}{R_0 R_2 C_1 C_2}}$$

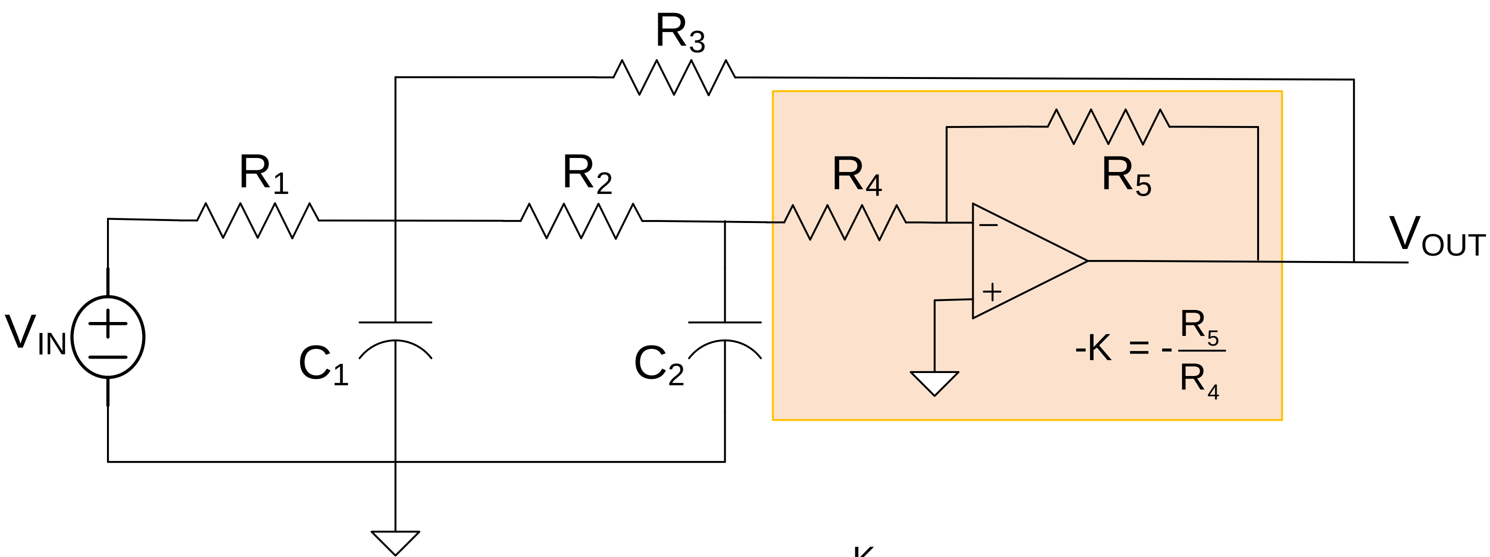
$$Q = \frac{R_Q}{\sqrt{R_0 R_2}} \sqrt{\frac{C_2}{C_1}}$$

For: $R_0 = R_1 = R_2 = R$ $\quad C_1 = C_2 = C$ $\quad R_3 = R_4$

$$T(s) = -\frac{\frac{1}{R^2 C^2}}{s^2 + s\left(\frac{1}{R_Q C}\right) + \frac{1}{R^2 C^2}}$$

$$R_Q = QR \qquad \omega_0 = \frac{1}{RC}$$

d) - KRC (a Sallen and Key filter)



$$T(s) = -\frac{\dfrac{K}{R_1R_2C_1C_2}}{s^2+s\left[\left(1+\dfrac{R_1}{R_3}\right)\left(\dfrac{1}{R_1C_1}\right)+\left(1+\dfrac{C_2}{C_1}\right)\left(\dfrac{1}{R_2C_2}\right)+\left(\dfrac{1}{R_4C_2}\right)\right]+\dfrac{1+(R_1/R_3)(1+K)+(R_1/R_4)(1+R_2/R_3+R_2/R_1)}{R_1R_2C_1C_2}}$$

$$\omega_0 = \sqrt{\frac{1+(R_1/R_3)(1+K)+(R_1/R_4)(1+R_2/R_3+R_2/R_1)}{R_1R_2C_1C_2}}$$

$$Q = \frac{\sqrt{\dfrac{1+(R_1/R_3)(1+K)+(R_1/R_4)(1+R_2/R_3+R_2/R_1)}{R_1R_2C_1C_2}}}{\left(1+\dfrac{R_1}{R_3}\right)\left(\dfrac{1}{R_1C_1}\right)+\left(1+\dfrac{C_2}{C_1}\right)\left(\dfrac{1}{R_2C_2}\right)+\left(\dfrac{1}{R_4C_2}\right)}$$

Often $R_1=R_2=R_3=R_4=R$, $C_1=C_2=C$

$$Q = \frac{\sqrt{5+K_0}}{5}$$

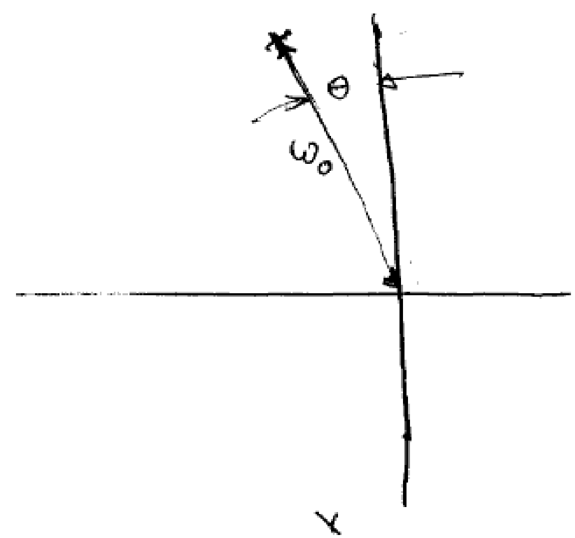
ω_0 and Q notation

$$s^2 + s\frac{\omega_0}{Q} + \omega_0^2$$

Poles $\omega_0 \left[-\frac{1}{2Q} \pm j\sqrt{1 - \frac{1}{4Q^2}} \right]$

$$|P| = \omega_0$$

$$\angle P \underset{Q > \frac{1}{2}}{=} \tan^{-1} \sqrt{4Q^2 - 1}$$



θ determined by Q

How do these five circuits compare?

a) From a passive sensitivity viewpoint?

- If Q is small
- If Q is large

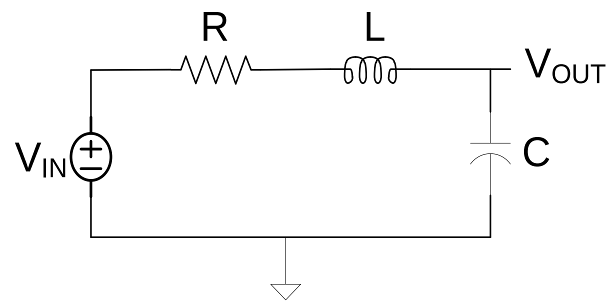
b) From an active sensitivity viewpoint?

- If Q is small
- If Q is large
- If $\tau\omega_0$ is large

Comparison: Calculate all ω_0 and Q sensitivities

Consider passive sensitivities first

a) – Passive RLC



$$S_R^{\omega_0} = 0$$

$$S_L^{\omega_0} = -\frac{1}{2}$$

$$S_C^{\omega_0} = -\frac{1}{2}$$

$$S_R^{Q} = -1$$

$$S_C^{Q} = -\frac{1}{2}$$

$$S_L^{Q} = \frac{1}{2}$$

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

$$\omega_0 = \sqrt{\frac{1}{LC}}$$

Case b1 : +KRC Equal R, Equal C

$$\omega_0 = \sqrt{\frac{1}{R_1R_2C_1C_2}}$$

$$Q = \frac{1}{\left(\sqrt{\frac{R_1C_1}{R_2C_2}}+\sqrt{\frac{R_2C_2}{R_1C_1}}+\sqrt{\frac{R_1C_2}{R_2C_1}}-K\sqrt{\frac{R_1C_1}{R_2C_2}}\right)}$$

$$S_{R_1}^{\omega_0} = S_{R_2}^{\omega_0} = S_{C_1}^{\omega_0} = S_{C_2}^{\omega_0} = -\frac{1}{2}$$

$$S_K^{\omega_0} = 0$$

$$S_{R_1}^{Q} = Q - \frac{1}{2}$$

$$S_{R_2}^{Q} = -Q + \frac{1}{2}$$

$$S_{C_1}^{Q} = 2Q - \frac{1}{2}$$

$$S_{C_2}^{Q} = -2Q + \frac{1}{2}$$

$$S_K^{Q} = 3Q - 1$$

$$Q = \frac{1}{3-K}$$

$$\omega_0 = \frac{1}{RC}$$

If $Q_N = 10$, what happens to Q if R_1 increases by 1% ?

10% ?

$\frac{\Delta R_1}{R_1} = .01$

$$\frac{\Delta Q}{Q} \approx S^{Q}_{R_1} \cdot \frac{\Delta R_1}{R_1} \approx (Q - 1/2)(.01) = .095$$

$\therefore$ Q changes by 9.5%

$\frac{\Delta R_1}{R_1} = 0.1$

$$\frac{\Delta Q}{Q} \approx S^{Q}_{R_1} \cdot \frac{\Delta R_1}{R_1} = (9.5)(.1) = .95$$

$\therefore$ Q changes by 95%

Actual: 10 → 11.04 for $\frac{\Delta R}{R} = .01$

10 → 105 for $\frac{\Delta R}{R} = 0.1$

Case b2 : +KRC Equal R, K=1

$$\omega_0 = \sqrt{\frac{1}{R_1R_2C_1C_2}}$$

$$Q = \frac{1}{\left(\sqrt{\frac{R_1C_1}{R_2C_2}} + \sqrt{\frac{R_2C_2}{R_1C_1}} + \sqrt{\frac{R_1C_2}{R_2C_1}} - K\sqrt{\frac{R_1C_1}{R_2C_2}}\right)}$$

$$S_{R_1}^{\omega_0} = S_{R_2}^{\omega_0} = S_{C_1}^{\omega_0} = S_{C_2}^{\omega_0} = -\frac{1}{2}$$

$$S_K^{\omega_0} = 0$$

$$S_{R_1}^{Q} = 0$$

$$S_{R_2}^{Q} = 0$$

$$S_{C_1}^{Q} = \frac{1}{2}$$

$$S_{C_2}^{Q} = -\frac{1}{2}$$

$$S_K^{Q} = 2Q^2$$

$$\omega_0 = \frac{1}{RC}$$

$$Q = \frac{1}{2}\sqrt{\frac{C_1}{C_2}}$$

c) Bridged T Feedback

$$\omega_0 = \sqrt{\frac{1}{R_1R_2C_1C_2}}$$

$$Q = \frac{1}{\left(\sqrt{\frac{C_2}{C_1}}\right)\left(\sqrt{\frac{R_1}{R_3}}+\sqrt{\frac{R_2}{R_1}}+\frac{\sqrt{R_1R_2}}{R_3}\right)}$$

For $R_1=R_2=R_3=R$

$$S_{R_1}^{\omega_0} = S_{R_2}^{\omega_0} = S_{C_1}^{\omega_0} = S_{C_2}^{\omega_0} = -\frac{1}{2}$$

$$S_{R_3}^{\omega_0} = 0$$

$$S_{R_1}^{Q} = -\frac{1}{6}$$

$$S_{R_2}^{Q} = -\frac{1}{6}$$

$$S_{R_3}^{Q} = \frac{1}{3}$$

$$S_{C_1}^{Q} = -\frac{1}{2}$$

$$S_{C_2}^{Q} = \frac{1}{2}$$

$$\omega_0 = \frac{3Q}{RC_1}$$

$$Q = \frac{1}{3}\sqrt{\frac{C_1}{C_2}}$$

d) 2 integrator loop

$$\omega_0 = \sqrt{\frac{R_4}{R_3} \bullet \frac{1}{R_0 R_2 C_1 C_2}}$$

$$Q = \frac{R_Q}{\sqrt{R_0 R_2}} \sqrt{\frac{C_2}{C_1}}$$

For: $R_0 = R_1 = R_2 = R$ $\quad C_1 = C_2 = C$ $\quad R_3 = R_4$

$$S_{R_1}^{\omega_0} = S_{R_2}^{\omega_0} = S_{R_3}^{\omega_0} = S_{C_1}^{\omega_0} = S_{C_2}^{\omega_0} - \frac{1}{2}$$

$$S_{R_4}^{\omega_0} = \frac{1}{2}$$

$$S_{R_1}^{Q} = S_{R_2}^{Q} = S_{R_3}^{Q} = S_{C_1}^{Q} = -\frac{1}{2}$$

$$S_{R_4}^{Q} = S_{C_2}^{Q} = \frac{1}{2}$$

$$S_{R_Q}^{Q} = 1$$

$$S_{R_0}^{Q} = 0$$

$$\omega_0 = \frac{1}{RC}$$

$$Q = \frac{R_Q}{R}$$

d) -KRC passive sensitivities

$$\omega_0 = \sqrt{\frac{1+(R_1/R_3)(1+K)+(R_1/R_4)(1+R_2/R_3+R_2/R_1)}{R_1R_2C_1C_2}}$$

$$Q = \frac{\sqrt{\frac{1+(R_1/R_3)(1+K)+(R_1/R_4)(1+R_2/R_3+R_2/R_1)}{R_1R_2C_1C_2}}}{\left(1+\frac{R_1}{R_3}\right)\left(\frac{1}{R_1C_1}\right)+\left(1+\frac{C_2}{C_1}\right)\left(\frac{1}{R_2C_2}\right)+\left(\frac{1}{R_4C_2}\right)}$$

For $R_1=R_2=R_3=R_4=R$, $C_1=C_2=C$

$$Q = \frac{\sqrt{5+K_0}}{5} \qquad \omega_0 = \frac{\sqrt{5+K}}{R\ C}$$

$$S_{R_1}^{\omega_0} = -\frac{1}{25Q^2} \qquad S_{R_2}^{\omega_0} = -\frac{1}{2}+\frac{1}{25Q^2} \qquad S_{R_3}^{\omega_0} = -\frac{1}{2}+\frac{3}{50Q^2}$$

$$S_{C_1}^{\omega_0} = S_{C_2}^{\omega_0} - \frac{1}{2} \qquad S_{R_4}^{\omega_0} = -\frac{3}{50Q^2} \qquad S_{K}^{\omega_0} = \frac{1}{2}+\frac{1}{10Q^2}$$

$$S_{R_1}^{Q} = \frac{1}{5} - \frac{1}{25Q^2} \qquad S_{R_2}^{Q} = -\frac{1}{10}+\frac{1}{25Q^2} \qquad S_{R_3}^{Q} = -\frac{3}{10}+\frac{3}{50Q^2}$$

$$S_{R_4}^{Q} = \frac{1}{5} - \frac{3}{50Q^2} \qquad S_{C_2}^{Q} = -\frac{1}{10} \qquad S_{C_1}^{Q} = \frac{1}{10} \qquad S_{K}^{Q} = \frac{1}{2} - \frac{1}{10Q^2}$$

Passive Sensitivity Comparisons

	$\left\|S_x^{\omega_0}\right\|$	$\left\|S_x^{Q}\right\|$
Passive RLC	$\leq \frac{1}{2}$	**1,1/2**
+KRC		
Equal R, Equal C (K=3-1/Q)	**0,1/2**	**Q, 2Q, 3Q**
Equal R, K=1 ($C_1=4Q^2C_2$)	**0,1/2**	**0,1/2, $2Q^2$**
Bridged-T Feedback	**0,1/2**	**1/3,1/2, 1/6**
Two-Integrator Loop	**0,1/2**	**1,1/2, 0**
-KRC	**less than or equal to 1/2**	**less than or equal to 1/2**

Substantial Differences Between (or in) Architectures

How do active sensitivities compare ?

$$S_{t}^{\omega_0} = ? \qquad S_{t}^{Q} = ?$$

Recall $S_x^f = \frac{\partial f}{\partial x} \frac{x}{f}$

so $\frac{\Delta f}{f} \simeq \frac{\Delta x}{x} S_x^f$

but if x is ideally 0, not useful

$$\mathcal{S}_x^f = \frac{\partial f}{\partial x}$$

$$\frac{\Delta f}{f} \simeq \mathcal{S}_x^f \frac{\Delta x}{f}$$

Where we are at with sensitivity analysis:

Considered a group of five second-order filters

Passive Sensitivity Analysis

- Closed form expressions were obtained for ω_0 and Q
- Tedious but straightforward calculations provided passive sensitivities directly from the closed form expressions ???

Active Sensitivity Analysis

- Closed form expressions for ω_0 and Q are very difficult or impossible to obtain

If we consider higher-order filters

Passive Sensitivity Analysis

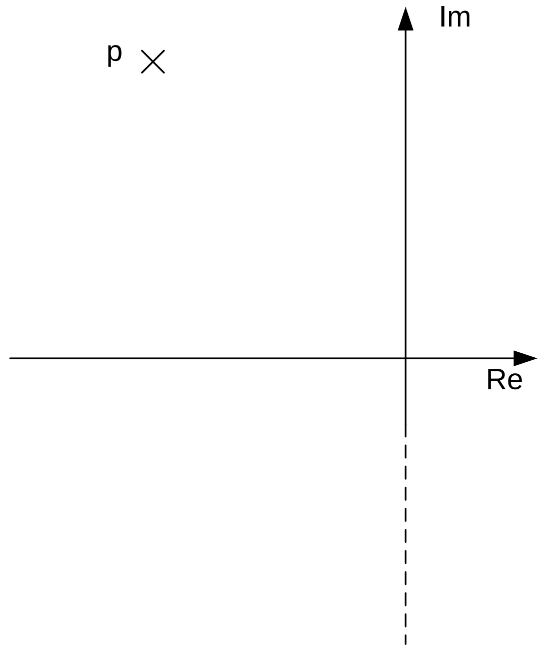
- Closed form expressions for ω_0 and Q are very difficult or impossible to obtain for many useful structures

Active Sensitivity Analysis

- Closed form expressions for ω_0 and Q are very difficult or impossible to obtain

Need some better method for obtaining sensitivities when closed-form expressions are difficult or impractical to obtain or manipulate !!

Relationship between pole sensitivities and ω_0 and Q sensitivities



$p = -\alpha + j\beta$

$D_2(s) = (s-p)(s-p^*)$

$D_2(s) = (s+\alpha-j\beta)(s+\alpha+j\beta)$

$D_2(s) = s^2 + s(2\alpha) + (\alpha^2+\beta^2)$

Relationship between active pole sensitivities and ω_0 and Q sensitivities

Define $D(s)=D_0(s)+\tau D_1(s)$ (from bilinear form of T(s))

Recall: $$\mathcal{S}_\tau^p = \frac{-D_1(p)}{\left.\frac{\partial D(s)}{\partial s}\right|_{s=p,\tau=0}}$$

Theorem: $$\Delta p \simeq \tau \mathcal{S}_\tau^p$$

Theorem: $$\Delta\alpha \simeq \tau \,\text{Re}\left(\mathcal{S}_\tau^p\right)$$
$$\Delta\beta \simeq \tau \,\text{Im}\left(\mathcal{S}_\tau^p\right)$$

Theorem:

$$\frac{\Delta\omega_0}{\omega_0} \simeq \frac{1}{2Q}\frac{\Delta\alpha}{\omega_0} + \sqrt{1-\frac{1}{4Q^2}}\frac{\Delta\beta}{\omega_0} \qquad \frac{\Delta Q}{Q} \simeq -2Q\left(1-\frac{1}{4Q^2}\right)\frac{\Delta\alpha}{\omega_0} + \sqrt{1-\frac{1}{4Q^2}}\frac{\Delta\beta}{\omega_0}$$

Claim: These theorems, with straightforward modification, also apply to other parameters (R, C, L, K, …) where, $D_0(s)$ and $D_1(s)$ will change since the parameter is different

Table 10-1 *KRC* Realization
(see Fig. 10-3b)

Equal-R, Equal-C

$$\omega_o = \frac{1}{RC}, \qquad Q = \frac{1}{3 - K_0}$$

$$\frac{V_o}{V_i} = \frac{\left(3 - \frac{1}{Q}\right)\omega_o^2}{s^2 + s\frac{\omega_o}{Q} + \omega_o^2 + \frac{\left(3 - \frac{1}{Q}\right)}{\text{GB}}s(s^2 + s3\omega_o + \omega_o^2)} \qquad \left(\omega_a \ll \frac{\omega_o}{2Q}\right)$$

$$-\frac{\Delta\alpha}{\omega_o} \cong \frac{1}{2Q}\left(3 - \frac{1}{Q}\right)^2 \frac{\omega_o}{\text{GB}}, \qquad \frac{\Delta\beta}{\omega_o} \cong -\frac{1}{2}\left(3 - \frac{1}{Q}\right)^2 \frac{\left(1 - \frac{1}{2Q^2}\right)}{\sqrt{1 - \frac{1}{4Q^2}}}\frac{\omega_o}{\text{GB}}$$

$$\frac{\Delta\omega_o}{\omega_o} \cong -\frac{1}{2}\left(3 - \frac{1}{Q}\right)^2 \frac{\omega_o}{\text{GB}}, \qquad \frac{\Delta Q}{Q} \cong \frac{1}{2}\left(3 - \frac{1}{Q}\right)^2 \frac{\omega_o}{\text{GB}}$$

Unity-gain, Equal-R

$$\omega_o = \frac{1}{R\sqrt{C_1C_2}}, \qquad Q = \frac{1}{2}\sqrt{\frac{C_1}{C_2}}$$

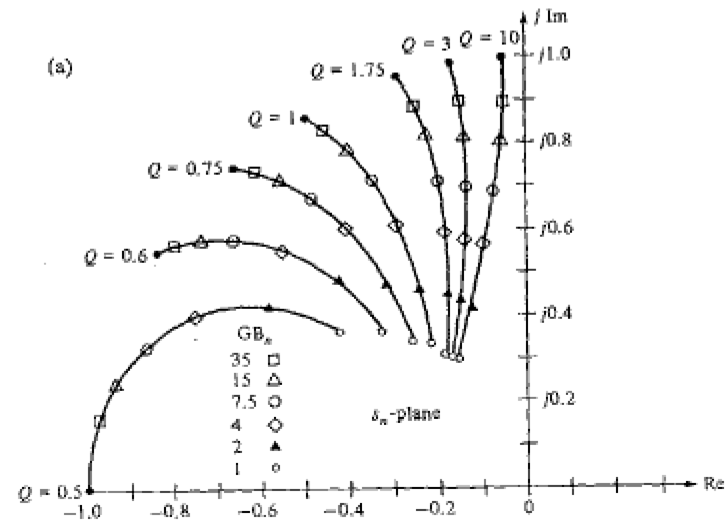
$$\frac{V_o}{V_i} = \frac{\omega_o^2}{s^2 + s\frac{\omega_o}{Q} + \omega_o^2 + \frac{s}{\text{GB}}\left[s^2 + s\omega_o\left(2Q + \frac{1}{Q}\right) + \omega_o^2\right]} \qquad \left(\omega_a \ll \frac{\omega_o}{2Q}\right)$$

$$-\frac{\Delta\alpha}{\omega_o} \cong \frac{\omega_o}{\text{GB}}, \qquad \frac{\Delta\beta}{\omega_o} \cong -Q\frac{\left(1 - \frac{1}{2Q^2}\right)}{\sqrt{1 - \frac{1}{4Q^2}}}\frac{\omega_o}{\text{GB}}$$

$$\frac{\Delta\omega_o}{\omega_o} \cong -Q\frac{\omega_o}{\text{GB}}, \qquad \frac{\Delta Q}{Q} \cong Q\frac{\omega_o}{\text{GB}}$$

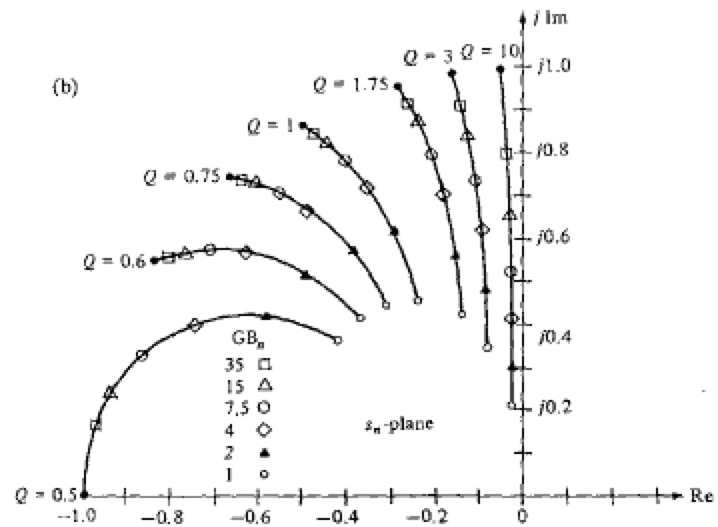
where

$$s_n = \frac{s}{\omega_o}, \qquad \text{GB}_n = \frac{\text{GB}}{\omega_o}.$$



◀Fig. 10-5a Plot of upper half-plane root of

$$s_n^3 + s_n^2\left(3+\frac{Q\,\mathrm{GB}_n}{3Q-1}\right) + s_n\left(1+\frac{\mathrm{GB}_n}{3Q-1}\right) + \frac{Q\,\mathrm{GB}_n}{3Q-1} = 0 \qquad \text{(Equal-}R\text{, equal-}C\text{)}$$



◀Fig. 10-5b Plot of upper half plane root of

$$s_n^3 + s_n^2\left(2Q+\frac{1}{Q}+\mathrm{GB}_n\right) + s_n\left(1+\frac{\mathrm{GB}_n}{Q}\right) + \mathrm{GB}_n = 0 \qquad \text{(Unity-gain, equal-}R\text{)}$$

c) Bridged-T structure

Table 10-3 Infinite-gain Realization
(see Fig. 10-10b)

Equal-R

$$\omega_o = \frac{1}{R\sqrt{C_1 C_2}}; \qquad Q = \frac{1}{3}\sqrt{\frac{C_1}{C_2}}$$

$$\frac{V_o}{V_i} \simeq -\frac{\omega_o^2}{s^2 + s\frac{\omega_o}{Q} + \omega_o^2 + \frac{s}{\text{GB}}\left[s^2 + s\omega_o\left(3Q + \frac{1}{Q}\right) + 2\omega_o^2\right]} \qquad \left(\omega_o \ll \frac{\omega_p}{2Q}\right)$$

$$-\frac{\Delta\alpha}{\omega_o} \simeq \frac{\omega_o}{\text{GB}}, \qquad \frac{\Delta\beta}{\omega_o} \simeq -\frac{1}{2}\frac{3Q - \frac{1}{Q}}{\sqrt{1 - \frac{1}{4Q^2}}}\frac{\omega_o}{\text{GB}}$$

$$\frac{\Delta\omega_o}{\omega_o} \simeq -\frac{3Q}{2}\frac{\omega_o}{\text{GB}}, \qquad \frac{\Delta Q}{Q} \simeq \frac{Q}{2}\frac{\omega_o}{\text{GB}}$$

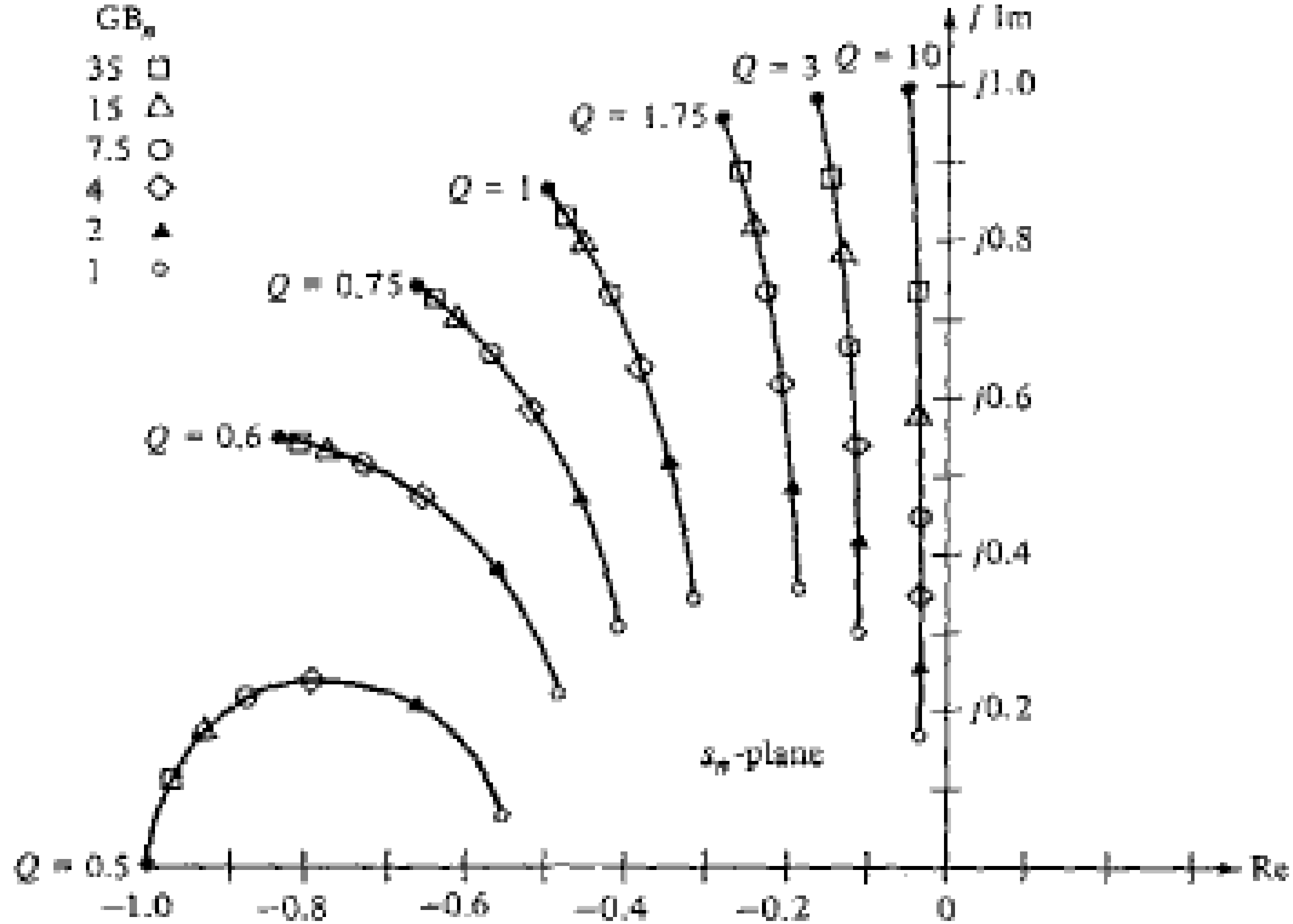


Fig. 10-12 Plot of upper half-plane root of

$$s_n^3 + s_n^2\left(3Q + \frac{1}{Q} + \mathrm{GB}_n\right) + s_n\left(2 + \frac{\mathrm{GB}_n}{Q}\right) + \mathrm{GB}_n = 0$$

d) Two integrator loop architecture

Table 10-4 Three-Amplifier Realization
(see Fig. 10-16)

Equal-R (except R_Q) and Equal-C

$$\omega_o = \frac{1}{RC}, \qquad Q = \frac{R_Q}{R}$$

$$\frac{V_B}{V_1} \cong \frac{\omega_o^2\left(\frac{2}{GB}s + 1\right)}{s^2 + s\frac{\omega_o}{Q} + \omega_o^2 + \frac{1}{GB}\left\{4s\left[s^2 + s\omega_o\left(\frac{1}{2} + \frac{1}{Q}\right) + \frac{\omega_o^2}{4Q}\right]\right\}} \qquad \left(\omega_o \ll \frac{\omega_a}{2Q}\right)$$

$$-\frac{\Delta\alpha}{\omega_o} \simeq 2\left(1 + \frac{1}{4Q}\right)\frac{\omega_o}{GB}, \qquad \frac{\Delta\beta}{\omega_o} \cong -\frac{\left(1 - \frac{1}{Q} - \frac{1}{4Q^2}\right)}{\sqrt{1 - \frac{1}{4Q^2}}}\frac{\omega_o}{GB}$$

$$\frac{\Delta\omega_o}{\omega_o} \simeq -\frac{\omega_o}{GB}, \qquad \frac{\Delta Q}{Q} \simeq 4Q\frac{\omega_o}{GB}$$

d) Two integrator loop architecture

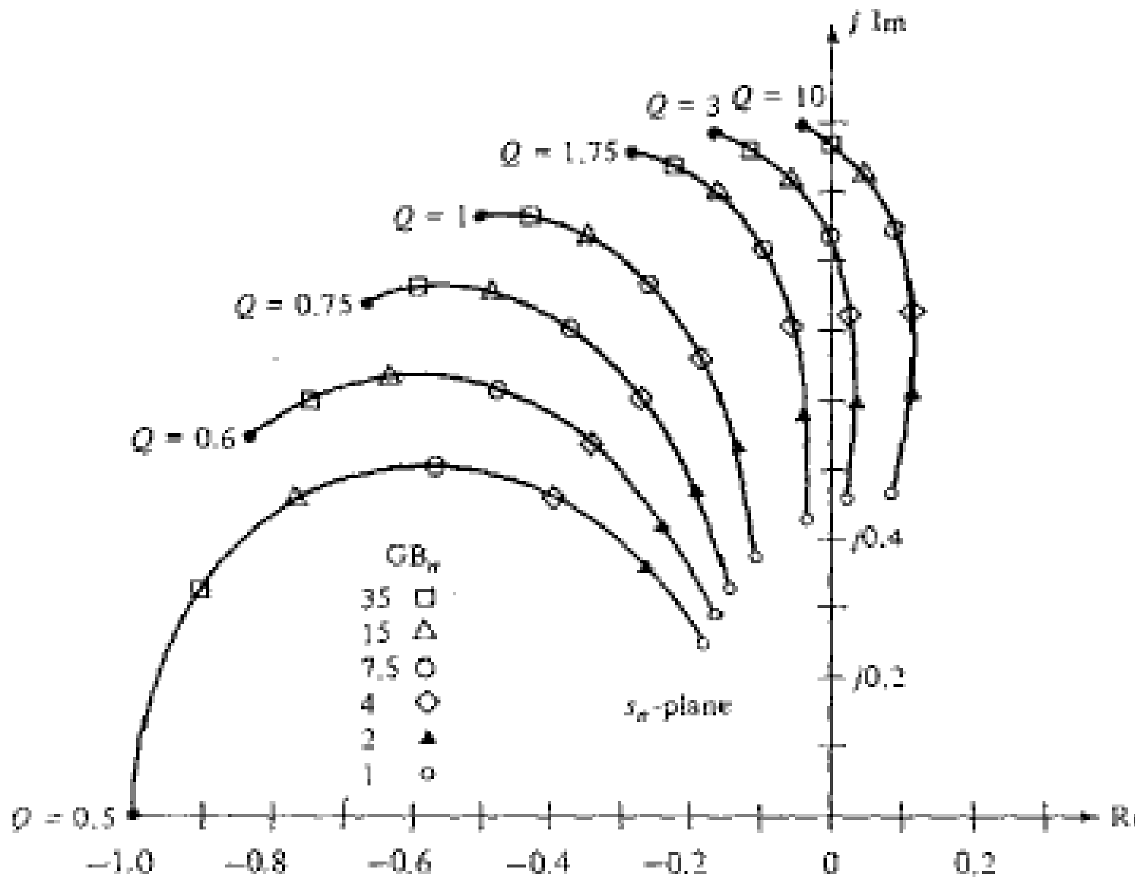


Fig. 10-17 Plot of upper half-plane root of

$$s_n^3 + s_n^2\left(\frac{1}{2} + \frac{1}{Q} + \frac{GB_n}{4}\right) + s_n\frac{1}{4Q}\left(1 + GB_n\right) + \frac{GB_n}{4} = 0$$

e) -KRC

Equal-R, Equal-C

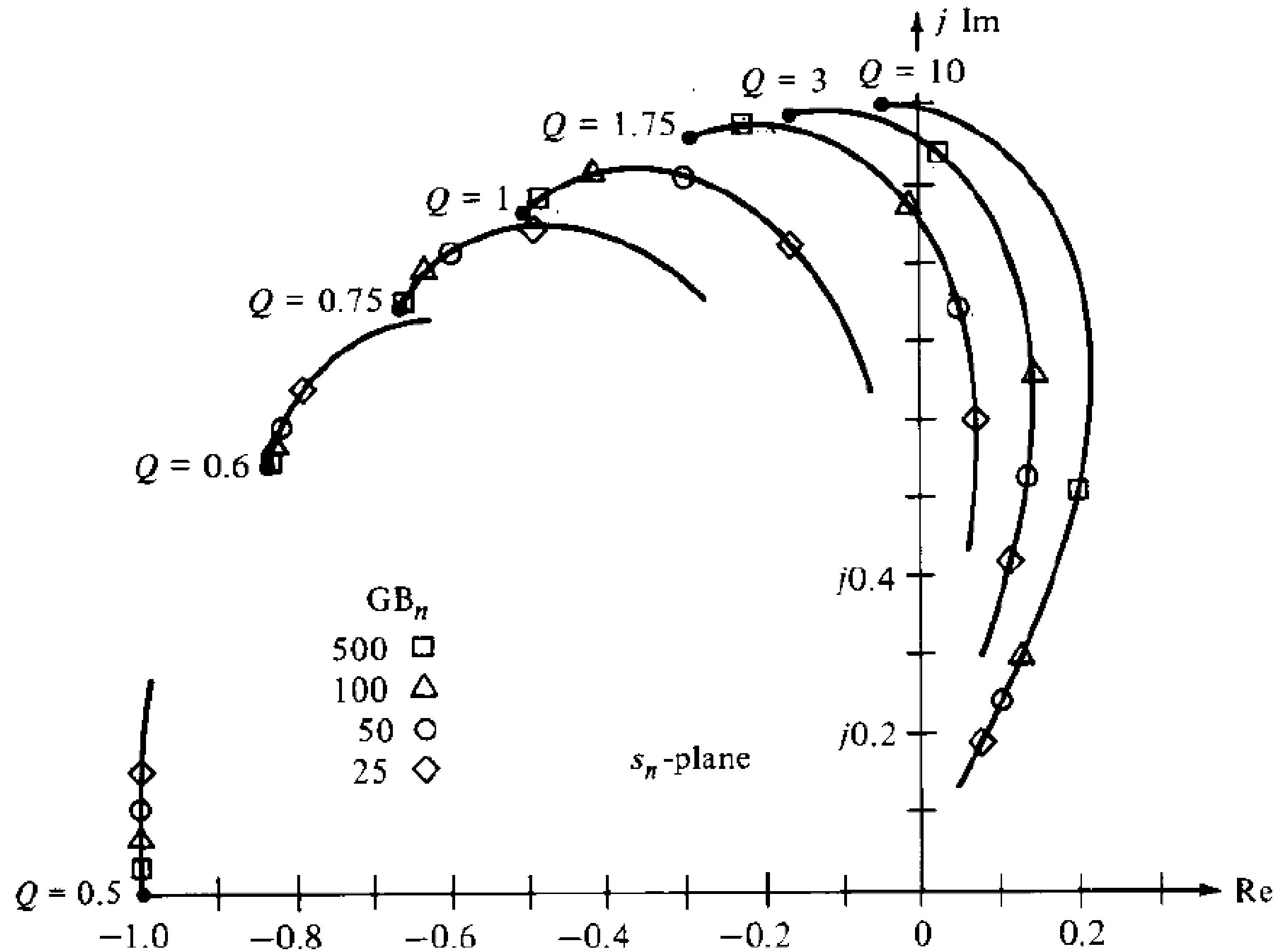
$$\omega_o = \frac{\sqrt{5 + K_0}}{RC}, \qquad Q = \frac{\sqrt{5 + K_0}}{5}$$

$$\frac{V_o}{V_i} = -\frac{\omega_o^2\left(1 - \frac{1}{5Q^2}\right)}{s^2 + s\frac{\omega_o}{Q} + \omega_o^2 + \frac{s}{GB}\left[s^2(25Q^2 - 4) + s\omega_o\left(20Q - \frac{3}{Q}\right) + \left(2 - \frac{1}{5Q^2}\right)\omega_o^2\right]}$$

$$\left(\omega_a \ll \frac{\omega_o}{2Q}\right)$$

$$-\frac{\Delta\alpha}{\omega_o} \cong \frac{25Q^2}{2}\left(1 - \frac{1}{5Q^2}\right)\left(1 - \frac{6}{25Q^2}\right)\frac{\omega_o}{GB}, \qquad \frac{\Delta\beta}{\omega_o} \cong \frac{35Q}{4}\frac{\left(1 - \frac{1}{5Q^2}\right)\left(1 - \frac{6}{35Q^2}\right)}{\sqrt{1 - \frac{1}{4Q^2}}}\frac{\omega_o}{GB}$$

$$\frac{\Delta\omega_o}{\omega_o} \cong \frac{5Q}{2}\left(1 - \frac{1}{5Q^2}\right)\frac{\omega_o}{GB}, \qquad \frac{\Delta Q}{Q} \cong 25Q^3\left(1 - \frac{1}{5Q^2}\right)\left(1 - \frac{7}{5Q^2}\right)\frac{\omega_o}{GB}$$



j Im
Q = 10
Q = 3
Q = 1.75
Q = 1
Q = 0.75
Q = 0.6
Q = 0.5
GB_n
500
100
50
25
s_n-plane
j0.4
j0.2
Re
−1.0
−0.8
−0.6
−0.4
−0.2
0
0.2

Active Sensitivity Comparisons

	$\frac{\Delta\omega_0}{\omega_0}$	$\frac{\Delta Q}{Q}$
Passive RLC		
+KRC		
Equal R, Equal C (K=3-1/Q)	$-\frac{1}{2}\left(3-\frac{1}{Q}\right)^2 \tau\omega_0$	$-\frac{1}{2}\left(3-\frac{1}{Q}\right)^2 \tau\omega_0$
Equal R, K=1 ($C_1=4Q^2C_2$)	$-Q\tau\omega_0$	$Q\tau\omega_0$
Bridged-T Feedback	$-\frac{3}{2}Q\tau\omega_0$	$\frac{1}{2}Q\tau\omega_0$
Two-Integrator Loop	$-\tau\omega_0$	$4Q\tau\omega_0$
-KRC	$\frac{5}{2}Q\tau\omega_0$	$25Q^3\tau\omega_0$

Substantial Differences Between Architectures

Are these passive sensitivities acceptable?

	$\left\| S_x^{\omega_0} \right\|$	$\left\| S_x^{Q} \right\|$
Passive RLC	$\leq \frac{1}{2}$	**1,1/2**
+KRC		
Equal R, Equal C (K=3-1/Q)	**0,1/2**	**Q, 2Q, 3Q**
Equal R, K=1 ($C_1=4Q^2C_2$)	**0,1/2**	**0,1/2, $2Q^2$**
Bridged-T Feedback	**0,1/2**	**1/3,1/2, 1/6**
Two-Integrator Loop	**0,1/2**	**1,1/2, 0**
-KRC	**less than or equal to 1/2**	**less than or equal to 1/2**

Are these active sensitivities acceptable?

Active Sensitivity Comparisons

	$\frac{\Delta\omega_0}{\omega_0}$	$\frac{\Delta Q}{Q}$
Passive RLC		
+KRC		
Equal R, Equal C (K=3-1/Q)	$-\frac{1}{2}\left(3-\frac{1}{Q}\right)^2\tau\omega_0$	$-\frac{1}{2}\left(3-\frac{1}{Q}\right)^2\tau\omega_0$
Equal R, K=1 ($C_1=4Q^2C_2$)	$-Q\tau\omega_0$	$Q\tau\omega_0$
Bridged-T Feedback	$-\frac{3}{2}Q\tau\omega_0$	$\frac{1}{2}Q\tau\omega_0$
Two-Integrator Loop	$-\tau\omega_0$	$4Q\tau\omega_0$
-KRC	$\frac{5}{2}Q\tau\omega_0$	$25Q^3\tau\omega_0$

Are these sensitivities acceptable?

Passive Sensitivities:

$$\frac{\Delta\omega_0}{\omega_0} \simeq S_x^{\omega_0}\frac{\Delta x}{x}$$

In integrated circuits, Δ R/R and Δ C/C due to process variations can be K 30% or larger due to process variations

Many applications require $\Delta\omega_0/\omega_0 < .001$ or smaller and similar requirements on $\Delta Q/Q$

Even if sensitivity is around ½ or 1, variability is often orders of magnitude too large

Active Sensitivities:

All are proportional to $\tau\omega_0$

Some architectures much more sensitive than others

Can reduce $\tau\omega_0$ by making GB large but this is at the expense of increased power and even if power is not of concern, process presents fundamental limits on how large GB can be made

End of Lecture 23