VCO-Derived Filters

Some additional observations
Poles of \[ D(s) = s^n + I_0^n \]
Some useful theorems

Theorem: A rational fraction \( T(s) = \frac{N(s)}{\prod_{i=1}^{n}(s-p_i)} \) with simple poles can be expressed in partial fraction form as

\[
T(s) = \sum_{i=1}^{n} \frac{A_i}{s-p_i}
\]

where \( A_i = (s-p_i)T(s)|_{s=p_i} \) for \( 1 \leq j \leq n \)

Theorem: The impulse response of a rational fraction \( T(s) \) with simple poles can be expressed as

\[
T(s) = \sum_{i=1}^{n} A_i e^{p_i t}
\]

where the numbers \( A_i \) are the coefficients in the partial fraction expansion of \( T(s) \)
Theorem: If \( p_i \) is a simple complex pole of the rational fraction \( T(s) \), then the partial fraction expansion terms in the impulse response corresponding to \( p_i \) and \( p_i^* \) can be expressed as:

\[
\frac{A_i}{s-p_i} + \frac{A_i^*}{s-p_i^*}
\]

Theorem: If \( p_i = \alpha_i + j\beta_i \) is a simple pole with non-zero imaginary part of the rational fraction \( T(s) \), then the impulse response terms corresponding to the poles \( p_i \) and \( p_i^* \) in the partial fraction expansion can be expressed as:

\[
|A_i| e^{\alpha_i t} \cos(\beta_i t + \theta_i)
\]

where \( \theta_i \) is the angle of the complex quantity \( A_i \).
Theorem: If all poles of an n-th order rational fraction $T(s)$ are simple and have a non-zero Imaginary part, then the impulse response can be expressed as

$$\sum_{i=1}^{n/2} A_i |e^{\alpha_t^i}\cos(\beta_t^i+\theta_i)$$

where $\theta_i, A_i, \alpha_i, \text{ and } \beta_i$ are as defined before.

Theorem: If an odd-order rational fraction has one pole on the negative real axis at $\alpha_0$ and n simple poles that have a non-zero Imaginary part, then the impulse response can be expressed as

$$A_0 e^{\alpha_0 t} + \sum_{i=1}^{n/2} A_i |e^{\alpha_t^i}\cos(\beta_t^i+\theta_i)$$

where $\theta_i, A_i, \alpha_i, \text{ and } \beta_i$ are as defined before.
Consider the following

\[
\text{Poles of } D(s) = s^n + I_0^n
\]

Consider the following

\[
\begin{align*}
0.5 & \quad -0.866025404 \\
0.5 & \quad 0.866025404 \\
-1 & \quad 3.67545E-16
\end{align*}
\]

\[\alpha = 0.5 \ I_0\]

\[\beta = 0.866 \ I_0\]

frequency of oscillation:

\[
|A_i| e^{\alpha t} \cos(\beta t + \theta_i)
\]

Starts at \(\omega = 0.866 I_0\) and will slow down as nonlinearities limit amplitude
Consider the following
\[ \beta = 0.866 \, I_0 \]
\[ \alpha = 0.5 \, I_0 - \Delta \alpha \]
\[ \omega_0 = \sqrt{(\alpha - \Delta \alpha)^2 + \beta^2} \]

So, to get a high \( \omega_0 \), want \( \beta \) as large as possible
Consider now the filter by adding a loss of $\alpha_L$ to the integrator

Will now determine $\alpha_L$ and $I_0$ needed to get a desired pole $Q$ and $\omega_0$

The values of $\alpha$ and $\beta$ are dependent upon $I_0$ but the angle $\theta$ is only dependent upon the number of integrators in the VCO

$$\alpha + j\beta = I_0 (\cos \theta + j \sin \theta)$$

Define the location of the filter pole to be

$$\alpha_F + j\beta_F$$

It follows that

$$\beta_F = \beta \quad \alpha_F = \alpha - \alpha_L$$

The relationship between the filter parameters is well known

$$\beta_F = \frac{\omega_0}{2Q} \sqrt{4Q^2 - 1} \quad \alpha_F = - \frac{\omega_0}{2Q}$$

Thus

$$I_0 = \frac{\omega_0}{(\sin \theta)2Q} \sqrt{4Q^2 - 1} \quad \alpha_L = \frac{\omega_0}{2Q} + I_0 \cos \theta = \frac{\omega_0}{2Q} + \frac{\omega_0}{2Q(\tan \theta)} \sqrt{4Q^2 - 1}$$
Will a two-stage structure give the highest frequency of operation?

\[ \omega_0 = \sqrt{(\alpha - \Delta \alpha)^2 + \beta^2} \]

Even though the two-stage structure may not oscillate, can work as a filter!

Can add phase lead if necessary
What will happen with a circuit that has two pole-pairs in the RHP?

The impulse response will have three decaying exponential terms and two growing exponential terms

\[ A_0 e^{\alpha_0 t} + \sum_{i=1}^{n/2} |A_i| e^{\alpha_i t} \cos(\beta_i t + \theta_i) \]
What will happen with a circuit that has two pole-pairs in the RHP?

Consider the growing exponential terms and normalize to $I_0 = 1$

$$A_1 e^{\alpha_1 t} \cos(\beta_1 t + \theta_1) + A_2 e^{\alpha_2 t} \cos(\beta_2 t + \theta_2)$$

At $t=145$ (after only 10 periods of the lower frequency signal)

$$r = \left. \frac{e^{\alpha_2 t}}{e^{\alpha_1 t}} \right|_{t=145} = \frac{e^{0.9009 \cdot 145}}{e^{0.2225 \cdot 145}} = 5.2 \times 10^{42}$$

The lower frequency oscillation will completely dominate!
What will happen with a circuit that has two pole-pairs in the RHP?

Can only see the lower frequency component!

Thanks to Chen for these plots

Can only see the lower frequency component!
What will happen with a circuit that has two pole-pairs in the RHP?

After even only two periods of the lower frequency waveform, it completely dominates!
How do we guarantee that we have a net coefficient of +1 in \( D(s) \)?

\[
D(s) = s^n + I_0^n
\]

\[
\begin{array}{c}
\frac{I_0}{s} \\
\frac{I_0}{s} \\
\vdots \\
\frac{I_0}{s}
\end{array}
\xrightarrow{a_1} \xrightarrow{a_2} \ldots \xrightarrow{a_n}
\]

\[
\chi_{out} = \left( \prod_{i=1}^{n} a_i \left( \frac{I_0}{s} \right) \right) \chi_{out} \quad a_i \in \{-1, 1\}
\]

\[
D(s) = s^n - \left( \prod_{i=1}^{n} a_i \right) I_0^n \quad \prod_{i=1}^{n} a_i = -1
\]

Must have an odd number of inversions in the loop!

If \( n \) is odd, all stages can be inverting and identical!
How do we guarantee that we have a net coefficient of +1 in $D(s)$?

$$D(s) = s^n + I_0^n$$

If fully differential or fully balanced, must have an odd number of crossings of outputs

Applicable for both even and odd order loops
A lossy integrator stage

\[ I(s) = \frac{-g_{m1}/C_X}{s + g_{m2}/C_X} \]

\[ I_0 = g_{m1}/C_X \]

\[ \alpha_L = g_{m2}/C_X \]
A fully-differential voltage-controlled integrator stage

\[ I_{0d} = \frac{g_{m1}}{C_X} \]

Will need CMFB circuit
A fully-differential voltage-controlled integrator stage with loss

\[ I_0(s) = \frac{g_{m1}}{sC_X + g_{m3}} \]

Will need CMFB circuit
**Example:**

Using the single-stage lossy integrator, design the integrator to meet a given $\omega_0$ and Q requirement.

Recall:

\[
I_0 = \frac{\omega_0}{(\sin \theta) 2Q} \sqrt{4Q^2 - 1} \quad (1)
\]

\[
\alpha_L = \frac{\omega_0}{2Q} + \frac{\omega_0}{2Q(\tan \theta)} \sqrt{4Q^2 - 1} \quad (2)
\]

Substituting for $I_0$ and $\alpha_L$ we obtain:

\[
g_{m1} = \frac{\omega_0}{C_X} \frac{\omega_0}{(\sin \theta) 2Q} \sqrt{4Q^2 - 1} \quad (3)
\]

\[
g_{m2} = \frac{\omega_0}{2Q} + \frac{\omega_0}{2Q(\tan \theta)} \sqrt{4Q^2 - 1} \quad (4)
\]
Example:

Using the single-stage lossy integrator, design the integrator to meet a given \( \omega_0 \) and Q requirement.

Expressing \( g_{m1} \) and \( g_{m2} \) in terms of design parameters:

\[
\frac{\mu C_{OX} W_1 V_{EB1}}{L_1 C_X} = \frac{\omega_0}{(\sin \theta) 2Q} \sqrt{4Q^2 - 1} \quad (5)
\]

\[
\frac{\mu C_{OX} W_2 V_{EB2}}{L_2 C_X} = \frac{\omega_0}{2Q} + \frac{\omega_0}{2Q (\tan \theta)} \sqrt{4Q^2 - 1} \quad (6)
\]

If we assume \( I_B = 0 \), equating drain currents obtain:

\[
V_{EB2} = V_{EB1} \sqrt{\frac{W_1 L_2}{W_2 L_1}} \quad (7)
\]

Thus the previous two expressions can be rewritten as:

\[
\frac{\mu C_{OX} V_{EB1}}{C_X} \left[ \frac{W_1}{L_1} \right] = \frac{\omega_0}{(\sin \theta) 2Q} \sqrt{4Q^2 - 1} \quad (8)
\]

\[
\frac{\mu C_{OX} V_{EB1}}{C_X} \left[ \frac{W_1 W_2}{L_1 L_2} \right] = \frac{\omega_0}{2Q} + \frac{\omega_0}{2Q (\tan \theta)} \sqrt{4Q^2 - 1} \quad (9)
\]
Example:

Using the single-stage lossy integrator, design the integrator to meet a given $\omega_0$ and $Q$ requirement

$$\frac{\mu C_{OX} V_{EB1}}{C_X} \left[ \frac{W_1}{L_1} \right] = \frac{\omega_0}{(\sin \theta)2Q} \sqrt{4Q^2 - 1}$$  \hspace{1cm} (8)

$$\frac{\mu C_{OX} V_{EB1}}{C_X} \left[ \frac{W_1 W_2}{L_1 L_2} \right] = \frac{\omega_0}{2Q} + \frac{\omega_0}{2Q(\tan \theta)} \sqrt{4Q^2 - 1}$$  \hspace{1cm} (9)

Taking the ratio of these two equations we obtain:

$$\frac{W_2}{L_2} = \frac{\sin \theta + \cos \theta \sqrt{4Q^2 - 1}}{\sqrt{4Q^2 - 1}}$$  \hspace{1cm} (10)

Observe that the pole $Q$ is determined by the dimensions of the lossy device!
Example:

Using the single-stage lossy integrator, design the integrator to meet a given $\omega_0$ and Q requirement

$$\frac{\mu C_{OX} V_{EB1}}{C_X} \left[ \frac{W_1}{L_1} \right] = \frac{\omega_0}{(\sin \theta)2Q} \sqrt{4Q^2-1}$$  \hspace{1cm} (8)

$$\frac{W_2}{L_2} = \frac{\sin \theta + \cos \theta \sqrt{4Q^2-1}}{\sqrt{4Q^2-1}}$$  \hspace{1cm} (10)

Still must obtain $W_1/L_1$, $V_{EB1}$, and $C_X$ from either of these equations

Although it appears that there might be 3 degrees of freedom left and only one constraint (one of these equations), if these integrators are connected in a loop, the operating point (Q-point) will be the same for all stages and will be that value where $V_{out} = V_{in}$. So, this adds a second constraint.

Setting $V_{out} = V_{in}$, and assuming $V_{T1} = V_{T2}$, we obtain from KVL

$$V_{DD} = V_{EB1} + V_{EB2} + 2V_T$$  \hspace{1cm} (11)

But $V_{EB1}$ and $V_{EB2}$ are also related in (7)
Example:

Using the single-stage lossy integrator, design the integrator to meet a given \( \omega_0 \) and Q requirement

\[
\frac{\mu C_{OX} V_{EB1}}{C_X} \begin{bmatrix} \frac{W_1}{L_1} \end{bmatrix} = \frac{\omega_0}{(\sin \theta) 2Q} \sqrt{4Q^2 - 1} \tag{8}
\]

\[
\frac{W_2}{L_2} = \frac{\sin \theta + \cos \theta \sqrt{4Q^2 - 1}}{\sqrt{4Q^2 - 1}} \tag{10}
\]

Still must obtain \( W_1 / L_1, V_{EB1}, \) and \( C_X \) from either of these equations

\[
V_{DD} = V_{EB1} + V_{EB2} + 2V_T \tag{11}
\]

\[
V_{EB2} = V_{EB1} \sqrt{\frac{W_1 L_2}{W_2 L_1}} \tag{7}
\]

Substituting (10) into (12) and then into (8) we obtain

\[
\frac{\mu C_{OX}}{C_X} \begin{bmatrix} \frac{W_1}{L_1} \end{bmatrix} \left( \frac{V_{DD} - 2V_T}{1 + \sqrt{\left( \frac{W_1}{L_1} \right)^{-1} \left( \frac{\sin \theta + \cos \theta \sqrt{4Q^2 - 1}}{\sqrt{4Q^2 - 1}} \right)}} \right) = \frac{\omega_0}{(\sin \theta) 2Q} \sqrt{4Q^2 - 1} \tag{13}
\]
Example:

Using the single-stage lossy integrator, design the integrator to meet a given $\omega_0$ and Q requirement

$$\frac{W_2}{L_2} = \frac{\sin\theta + \cos\theta \sqrt{4Q^2 - 1}}{\sqrt{4Q^2 - 1}}$$ \tag{10}

$$\frac{\mu C_{OX}}{C_X} \left[ \frac{W_1}{L_1} \right] \left( \frac{V_{DD} - 2V_T}{1 + \left( \frac{W_1}{L_1} \right)^{-1} \left( \frac{\sin\theta + \cos\theta \sqrt{4Q^2 - 1}}{\sqrt{4Q^2 - 1}} \right)} \right) = \frac{\omega_0}{(\sin\theta)2Q} \sqrt{4Q^2 - 1}$$ \tag{13}

There is still one degree of freedom remaining. Can either pick $W_1/L_1$ and solve for $C_X$ or pick $C_X$ and solve for $W_1/L_1$.

Explicit expression for $W_1/L_1$ not available

Tradeoffs between $C_X$ and $W_1/L_1$ will often be made

Since $V_{OUTQ} = V_T + V_{EB1}$, it may be preferred to pick $V_{EB1}$, then solve (12) for $W_1/L_1$ and then solve (13) for $C_X$

Adding $I_B$ will provide one additional degree of freedom and will relax the relationship between $V_{OUTQ}$ and $W_1/L_1$ since (7) will be modified