EE 508
Lecture 4
Filter Concepts/Terminology
Approximation Problem
Review from Last Time

Filter Design Process

Establish Specifications
- possibly $T_D(s)$ or $H_D(z)$
- magnitude and phase characteristics or restrictions
- time domain requirements

Approximation
- obtain acceptable transfer functions $T_a(s)$ or $H_a(z)$
- possibly acceptable realizable time-domain responses

Synthesis
- build circuit or implement algorithm that has response close to $T_a(s)$ or $H_a(z)$
- actually realize $T_a(s)$ or $H_a(z)$

Filter
Review from Last Time

Biquadratic Factorization

Pole and zero pairings can always be made so that all coefficients in the biquadratic factorizations are real.

In general, the biquadratic factorizations are not unique.

- If roots are real, multiple choices for first-order factor and remaining roots can be partitioned into groups of 2 in different ways.

- Complex conjugate root pairs are generally grouped together so that all coefficients are real.
Frequency Normalization / Denormalization

\[ T(s) \quad \text{Desired Transfer Function} \]

\[ T_n(s_n) \quad \text{Normalized Transfer Function} \]

\[ T_n(s_n) = T(s) \bigg|_{s=s_n} \]

where

\[ s_n = \frac{s}{\omega_0} \]

\( \omega_0 \) is termed the frequency normalization factor
Use of normalized functions in the design process

Theorem: A circuit with transfer function $T(s)$ can be obtained from a circuit with normalized transfer function $T_n(s_n)$ by denormalizing all frequency dependent components.

<table>
<thead>
<tr>
<th>Component</th>
<th>Denormalization</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>$R$</td>
</tr>
<tr>
<td>$C$</td>
<td>$C/\omega_0$</td>
</tr>
<tr>
<td>$L$</td>
<td>$L/\omega_0$</td>
</tr>
<tr>
<td>$A$</td>
<td>$A$</td>
</tr>
</tbody>
</table>
Impedance Scaling

Impedance scaling entails multiplying an impedance by a constant \( \Theta \).

\[
\begin{align*}
R & \rightarrow \Theta R \\
C & \rightarrow \Theta C \\
L & \rightarrow \Theta L \\
A & \rightarrow \begin{cases} 
\Theta A & \text{if } A \text{ is direct}
\Theta^{-1} A & \text{if } A \text{ is inverse}
\end{cases}
\end{align*}
\]

Claim: If all impedances in a circuit are scaled by the same constant \( \Theta \), then

\[
\begin{align*}
\frac{V_o(s)}{V_i(s)} & \rightarrow \frac{V_o(s)}{V_i(s)} \\
\frac{I_o(s)}{I_i(s)} & \rightarrow \frac{I_o(s)}{I_i(s)} \\
\frac{I_o(s)}{V_i(s)} & \rightarrow \frac{1}{\Theta} \frac{I_o(s)}{I_i(s)} \\
\frac{V_o(s)}{I_i(s)} & \rightarrow \Theta \frac{V_o(s)}{I_i(s)}
\end{align*}
\]
Recall

\[ T(s) = \frac{1}{1 + 2\pi f s} \]

\[ w_0 = (2\pi f) \times 10^3 \]

\[ f = 1000 \]

\[ C = \frac{1}{(2\pi f)^2} \times 10^{-6} = 0.159 \mu F \]

Practically likely scale to achieve a desired \( C \)

\[ \Theta = 10 \times 10^{-6} \times \frac{1}{\frac{1}{2\pi f} \times 10^3} \]
Summary

Typical Design Scenario.

1) Obtain Normalized Approximating Function

2) Synthesize the Normalized Approximating Function

3) Denormalize the circuit of Step 2

4) Impedance scale to acceptable component values.
Example: Design a 2nd-order lowpass Butterworth filter with a dc gain of 5 and a 3dB band edge of 4KHz.

Solution: \[ T_n(s) = \left[ \frac{1}{S^2 + \sqrt{2}S + 1} \right] \]

Synthesis - a 2nd-order lowpass filter (2-integrator loop)

\[ T(s) = \frac{1}{S^2 + \frac{S}{Q} + 1} \]
\[ T_n(s) = \left( \frac{1}{s^2 + \sqrt{2}s + 1} \right) (5) \]

\[ Q = \sqrt{\frac{1}{\sqrt{2}}} = 0.707 \]

**Frequency denormalization**

\[ C_n = \frac{C_n}{(2\pi f)^4000} = 39.81 \mu F \]

**Impedance Scaling** (\( \Theta = \text{impedance scaling factor} \))

Let \( C_s = 0.01 \mu F \) \( \Rightarrow \) \( \Theta = \frac{C}{C_s} = 39.81 \)

\[ R_s = R_n \cdot \Theta = 3.981 \text{ k\Omega} \]
Gain of 5

Order if amplifiers/filters in a cascade is important as it can affect distortion, noise and offset.
INTRODUCTION TO THE
THEORY AND DESIGN OF
ACTIVE FILTERS

L. PHUELSMAN
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degenerate feedback path. The input signal $V_i$ is injected into the circuit through $R_4$. To analyze this filter we begin by finding $V_{bp}(s)/V_i(s)$. $V_{bp}(s)$ can be expressed as

$$V_{bp}(s) = \frac{1}{R_4 C_1} V_i(s) - \frac{1}{s + \frac{1}{R_1 C_1}} V_i(s) - \frac{1}{s + \frac{1}{R_1 C_1}} V_4(s)$$  \hspace{1cm} (17)

However, we see that

$$V_s(s) = -V_{lp}(s) = \frac{V_{bp}(s)}{sR_2C_2} \hspace{1cm} (18)$$

so that substituting (18) into (17) results in the bandpass transfer function of the resonator active filter.

$$\frac{V_{bp}(s)}{V_i(s)} = \frac{s}{s^2 + \frac{s}{R_1 C_1} + \frac{1}{R_2 R_3 C_1 C_2}} \hspace{1cm} (19)$$
This transfer function is attractive in its simplicity. The low-pass transfer function can now be readily derived by substituting (19) into (18) to obtain

\[
\frac{V_p(s)}{V_i(s)} = \frac{1}{s^2 + \frac{s}{R_1 C_1} + \frac{1}{R_3 R_5 C_1 C_2}}
\]

(20)

If the low-pass output is taken from the output of the inverter, then an inverting low-pass realization may also be obtained. The ideal integrator between \(x\) and \(y\) may be interchanged with the unity gain inverter between \(x\) and \(z\) to provide a noninverting bandpass transfer function at \(V_p\). The three-amplifier circuit of Fig. 52-4 cannot be used to provide a high-pass transfer function.

Equating the denominator of (19) to that of the standard second-order bandpass function of (1) of Sec. 42.2, we obtain

\[
\omega_n = \frac{1}{\sqrt{R_3 R_5 C_1 C_2}}
\]

(21a)

\[
Q = \frac{1}{R_1} \frac{R_3 R_5 C_2}{C_1}
\]

(21b)

From these relations we readily find that the \(Q\) and \(\omega_n\) sensitivities are very low, namely, they all have magnitudes of either one or \(\frac{1}{\sqrt{2}}\). The expressions for \(H_0\) are

**Low-pass:**

\[
H_0 = \frac{R_3}{R_4}
\]

(22a)

**Bandpass:**

\[
|H_{oc}| = \frac{R_4}{R_1}
\]

(22b)

A design procedure can now be developed by assuming that \(R_1 = R_3 = R\) and \(C_1 = C_2 = C\), and proceeding as follows:

1. Assume \(\omega_n\), \(Q\), and \(H_0\) are specified.
2. Let \(R_1 = R_3 = R\) and \(C_1 = C_2 = C\.
3. Select either \(R\) or \(C\) and solve for the other, using

\[
\omega_n = \frac{1}{RC}
\]

(23a)

4. Calculate:

**Low-pass:**

\[
R_4 = \frac{R}{H_0}
\]

(23b)

**Bandpass:**

\[
R_4 = \frac{R_1}{|H_{oc}|}
\]

(23c)
in Fig. 11.24(b); for this summer we can write
\[ V_c = -\left( \frac{R_E V_{VP} + R_F V_{VP} + R_E V_P}{R_L} \right) \]
\[ = -V_c \left( \frac{R_E T_{VP} + R_F T_{VP} + R_E T_V}{R_L} \right) \] (11.65)
Substituting for \( T_{VP} \), \( T_{VP} \), and \( T_V \) from Eqs. (11.56), (11.59), and (11.60), respectively, gives the overall transfer function
\[ \frac{V_c}{V_c} = -K \left( \frac{(R_E/R_P)z^3}{z^3 + \text{Ma}(Q) + \text{Mab}} \right) \] (11.66)
from which we can see that different transmission zeros can be obtained by the appropriate selection of the values of the summing resistors. For instance, a notch is obtained by selecting \( R_P = \infty \) and
\[ \frac{R_E}{R_L} \approx \left( \frac{\omega_n}{\omega_0} \right)^2 \] (11.67)

An Alternative Two-Integrator-Loop Biquad Circuit

An alternative two-integrator-loop biquad circuit in which all three op amps are used in a single-ended mode can be developed as follows. Rather than using the inner summer to add signals with positive and negative coefficients, we can introduce an additional inverter, as shown in Fig. 11.25(a). Now all the coefficients of the summer have the same sign, and we can dispense with the summing amplifier altogether and perform the summation at the virtual ground input of the first integrator. The resulting circuit is shown in Fig. 11.25(b), from which we observe that the high-pass function is no longer available! This is the price paid for obtaining a circuit that utilizes all op amps in a single-ended mode. The circuit of Fig. 11.25(b) is known as the Tow–Thomas Biquad, after its originators.

Rather than using a fourth op amp to realize the finite transmission zeros required for the notch and all-pass functions, as was done with the KHN biquad, an economical feedforward scheme can be employed with the Tow–Thomas circuit. Specifically, the virtual ground available at the input of each of the three op amps in the Tow–Thomas circuit permits the input signal to be fed to all three op amps, as shown in Fig. 11.26. If \( V_c \) is taken at the output of the damped integrator, straightforward analysis yields the filter transfer function
\[ \frac{V_c}{V_c} = -\left( \frac{\omega_n^2}{\omega_0^2} \right) \frac{1}{s^2 + \frac{1}{QCR} + \frac{1}{C^2R^2}} \] (11.68)
which can be used to obtain the design data given in Table 11.2.
Fig. 11.25 Derivation of an alternative two-integrator-loop biquad in which all op amps are in a single-ended fashion. The resulting circuit in (b) is known as the Tow-Thomson biquad.

Fig. 11.26 The Tow-Thomson biquad with feedback. The transfer function of Eq. (11.68) is realized by feeding the input signal through appropriate components to the input of the three op amps. This circuit can realize all special second-order functions. The design equations are given in Table 11.2.
Approximations:
- Magnitude Squared Approximating Function
  \[ H_A(\omega^2) \]
- Inverse Transform
  \[ H_A(\omega^2) \rightarrow T_A(s) \]
- Collocation
- Least Squares
- Padé Approximation
- Other Analytical Optimization
- Numerical Optimization
- Canonical Approximations
  - BW
  - CC
  - Elliptic
- If there is a mapping to \( T(h) \), it is often not unique.
- The inverse mapping may not exist.
Observation:

\[ T(s) = \sum_{i=0}^{M^3} \frac{a_i s^i}{b_i s^i} \]

\[ T(s;w) = a_0 + a_1 w_1 - a_2 w^2 - a_3 w^3 + \ldots \]

\[ \frac{h_0 + h_1 w_1 - h_2 w^2 - h_3 w^3 + \ldots}{u_0 + u_1 w_1 - u_2 w^2 - u_3 w^3 + \ldots} \]

\[ T(s;\omega) = \left( \sum_{k=0}^{\infty} a_k \omega^k \right) + j \left( \sum_{k=0}^{\infty} a_k \omega^{k+1} \right) + \left( \sum_{k=0}^{\infty} b_k \omega^k \right) + j \left( \sum_{k=0}^{\infty} b_k \omega^{k+1} \right) \]

\[ T(s;\omega) = F_1(\omega^2) + j \omega F_2(\omega^2) \]

\[ \frac{F_3(\omega^2) + j \omega F_4(\omega^2)}{F_5(\omega^2) + j \omega F_6(\omega^2)} \]
\[ |T_c(i\omega)| = \sqrt{\frac{[F_1(i\omega)]^2 + \omega^2 [F_2(i\omega)]^2}{[F_3(i\omega)]^2 + \omega^2 [F_4(i\omega)]^2}} \]

\[ |T_c(i\omega)| \text{ is an even function of } \omega \]

\[ |T_c(i\omega)|^2 \text{ is a rational with real coefficients in } \omega^2 \]
If a desired magnitude response is given, it is most common to find a rational fraction in \( \omega^2 \) with real coefficients, \( H_A(\omega^2) \) that approximates the desired magnitude squared response and then obtain a function \( T_A(s) \) that satisfies the relationship

\[
|T_A(i\omega)|^2 = H_A(\omega^2)
\]

\[
H_A(\omega^2) = \sum_{i=0}^{2k} a_i \omega^{2i} - \sum_{i=0}^{2k} b_i \omega^{2i}
\]

But - how can \( T(s) \) be obtained from \( H_A(\omega^2) \)?
Observations:

If $Z$ is a zero (pole) of $H_A(w^2)$ then

a) $-Z$ is a zero (pole)
b) $Z^*$ is a zero (pole)
c) $-Z^*$ is a zero (pole)

Zeros are solutions of

\[ \sum_{i=0}^{2l} a_i w^{2i} = 0 \]
Obtaining $T_a(s)$ from $H_a(w)$

Theorem: If $H_a(w^2)$ is a rational fraction with real coefficients and no poles or zeros of $H_a(w^2)$ of odd multiplicity on the real axis, then the function

$$T_a(s) = \frac{H_0 (s-jz_1)(s-jz_2) \cdots (s-jz_k)}{(s-jp_1)(s-jp_2) \cdots (s-jp_k)}$$

is a minimum phase rational fraction with real coefficients that satisfies the relationship

$$|T_a(i\omega)|^2 = H_a(\omega^2)$$

where $z_1, \ldots, z_k$ are the upper half-plane zeros of $H_a(w^2)$ and exactly half of the real axis zeros and where $p_1, \ldots, p_k$ are the upper half-plane poles of $H_a(w^2)$ and exactly half of the real axis poles.
Minimum phase:

Observe: cc rotation of u.h.p. poles & zeros of \( H_a(w) \) by 90° assures all appear in u.h.p. (or on jw axis).

\( \cdot \) minimum phase criterion.
Real coefficients:

\[(S - i2i)(S - 1(-2i^*))\]

\[(S - i2i)(S + i2i^*)\]

\[S^2 + S \left( j \right)(2i^* - 2i) + \frac{2i2i^*}{\text{Real}}\]

This can be formalized to show that all coefficients are real.
\[ |T_{an}(i\omega)|^2 = H_{A}(\omega^2) \]

\[ |T_{an}(i\omega)|^2 = H_0 \frac{(\omega^2 - 2)(\omega^2 - 2^2)}{(\omega^2 - 2^2)(\omega^2 - z_1^2)} \cdots \]

\[ \frac{(\omega - p_1)(\omega - p_1^*)}{(\omega - p_1)(\omega - p_1^*)} \cdots \]
Claim: Generally there are many rational fractions $T_a(s)$ that satisfy

$$|T_a(iw)| = H_a(w^2)$$

if one rational fraction satisfies this relationship.

These are all obtainable from $T_{Am}(s)$ by replacing any number of complex conjugate pole and/or zero pairs by the negatives of the corresponding poles or zeros.

Example: 

$$|T_{An}(iw)|^2 = H_a(w^2) = \frac{1+w^2}{4+w^4}$$

$T_{Am}(s)$ = ?

zeros of $H_a(w^2)$: $w = \pm j$

poles of $H_a(w^2)$: $w = \pm i \pm j$
\[ T_{AA}(s) : \text{poles at } -1+i, -1-i \]
\[ -2\text{poles at } -1 \]

\[ T_{AA}(s) = \frac{s+1}{(s+1-i)(s+1+i)} = \frac{s+1}{s^2+2s+1} \]

\[ T_{A1}(s) = \frac{s-1}{s^2+2s+1} \]

\[ T_{A2}(s) = \frac{s+1}{s^2-2s+1} \]

\[ T_{A3}(s) = \frac{s-1}{s^2-2s+1} \]
Observation:

It may be the case that $T_{AM}(s)$ does not exist.
\[ H_\lambda(\omega^2) = \frac{20\omega^2 - 8\omega}{1 - 16\omega^2} \]

Zeros: \( \omega = \pm 2 \)

Poles: \( \omega = \pm \frac{1}{4} \)