Degrees of Freedom
The Approximation Problem
Theorem: A circuit with transfer function $T(s)$ can be obtained from a circuit with normalized transfer function $T_n(s_n)$ by denormalizing all frequency dependent components.

\[ \begin{align*} 
C &\rightarrow \frac{C}{\omega_0} \\
L &\rightarrow \frac{L}{\omega_0} 
\end{align*} \]
Frequency normalization/scaling

The frequency scaled circuit can be obtained from the normalized circuit simply by scaling the frequency dependent impedances (up or down) by the scaling factor $\omega_0$.

Component denormalization by factor of $\omega_0$

<table>
<thead>
<tr>
<th>Normalized Component</th>
<th>Denormalized Component</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>C</td>
<td>$C/\omega_0$</td>
</tr>
<tr>
<td>L</td>
<td>$L/\omega_0$</td>
</tr>
<tr>
<td>Other Components</td>
<td>Unchanged</td>
</tr>
</tbody>
</table>

Component values of energy storage elements are scaled down by a factor of $\omega_0$. 

Review from Last Time
Example: Design a V-V passive 3\(^{rd}\)-order Lowpass Butterworth filter with a 3-db band-edge of 1K rad/sec and equal source and load terminations.

(from the BW approximation which will be discussed later:)

\[
T(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}
\]


Review from Last Time

Example: Design a V-V passive 3\textsuperscript{rd}-order Lowpass Butterworth filter with a band-edge of 1K Rad/Sec and equal source and load terminations.

\[
T(s) = K \frac{10^9}{s^3 + 2 \cdot 10^3 s^2 + 2 \cdot 10^8 s + 10^9}
\]

Is this solution practical?

Some component values are too big and some are too small!
Filter Concepts and Terminology

- Frequency scaling
- Frequency Normalization
- Impedance scaling
- Transformations
  - LP to BP
  - LP to HP
  - LP to BR
Impedance scaling of a circuit is achieved by multiplying ALL impedances in the circuit by a constant

\[ R \rightarrow \theta R \]
\[ C \rightarrow C/\theta \]
\[ L \rightarrow L\theta \]

\[ \theta A \] for transresistance gain
\[ A \rightarrow A \] for dimensionless gain
\[ A/\theta \] for transconductance gain
Impedance Scaling

Theorem: If all impedances in a circuit are scaled by a constant $\theta$, then

a) All dimensionless transfer functions are unchanged
b) All transresistance transfer functions are scaled by $\theta$
c) All transconductance transfer functions are scaled by $\theta^{-1}$
Review from Last Time

Example: Design a V-V passive 3\textsuperscript{rd}-order Lowpass Butterworth filter with a band-edge of 1K Rad/Sec and equal source and load terminations.

Is this solution practical?

Some component values are too big and some are too small!

Impedance scale by $\theta=1000$

\begin{align*}
R & \rightarrow \theta R \\ 
C & \rightarrow C/\theta \\ 
L & \rightarrow \theta L \\
\end{align*}

Component values more practical
Typical approach to lowpass filter design

1. Obtain normalized approximating function
2. Synthesize circuit to realize normalized approximating function
3. Denormalize circuit obtained in step 2
4. Impedance scale to obtain acceptable component values
Degrees of Freedom

Circuit has two design variables: \{R, C\}

Circuit has one key controllable performance characteristic: \( \omega_0 = \frac{1}{RC} \)

If \( \omega_0 \) is specified for a design, circuit has

- 2 design variables
- 1 constraint

1 Degree of Freedom

Performance/Cost strongly affected by how degrees of freedom in a design are used!
Example: Design a 2\textsuperscript{nd} order lowpass Butterworth filter with 3dB passband attenuation, a dc gain of 5, and a 3dB bandedge of 4KHz

Note: We have not discussed the Butterworth approximation yet so some details here will be based upon concepts that will be developed later

\[ T_{BWn} = \left( \frac{1}{s^2 + \sqrt{2}s + 1} \right) \cdot 5 \]

\[ \omega_0 = 1 \]
\[ Q = \frac{1}{\sqrt{2}} = 0.707 \]
Example: Design a 2\textsuperscript{nd} order lowpass Butterworth filter with 3dB passband attenuation, a dc gain of 5, and a 3dB band edge of 4KHz

\[ T(s) = \frac{1}{s^2 + s \left( \frac{1}{R_Q C_1} \right) + \frac{1}{R_2 R_1 C_1 C_2}} \]

\[ \omega_0 = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}} \]

\[ Q = \frac{R_Q}{\sqrt{R_1 R_2}} \sqrt{\frac{C_1}{C_2}} \]

7 design variables and only two constraints (ignoring the gain right now)

Circuit has 5 Degrees of Freedom!
Example: Design a $2^{nd}$ order lowpass Butterworth filter with 3dB passband attenuation, a dc gain of 5, and a 3dB band edge of 4KHz

If $C_1=C_2=C$ and $R_1=R_2=R_0=R$, this reduces to

$$T(s) = \frac{1}{(RC)^2} \frac{1}{s^2 + s \left( \frac{R}{R_Q \cdot R} \right) + \frac{1}{(RC)^2}}$$

How many degrees of freedom remain? 2
Example: Design a 2\textsuperscript{nd} order lowpass Butterworth filter with 3dB passband attenuation, a dc gain of 5, and a 3dB bandedge of 4KHz

\[
T(s) = \frac{1}{s^2 + s \left( \frac{R}{R_Q \cdot RC} \right) + \left( \frac{1}{(RC)^2} \right)} \quad \omega_0 = \frac{1}{RC} \quad Q=\frac{R_Q}{R}
\]

Normalizing by the factor $\omega_0$, we obtain

\[
T(s_n) = \frac{1}{s^2 + s \left( \frac{1}{Q} \right) + 1}
\]

Lets now use up the two degrees of freedom in the circuit:

Setting $R=R_3=1$ obtain the following normalized circuit
Example: Design a 2\textsuperscript{nd} order lowpass Butterworth filter with 3dB passband attenuation, a dc gain of 5, and a 3dB bandedge of 4KHz

Setting $R=R_3=1$ obtain the following circuit

![A Popular Second-Order Lowpass Filter](image)

The two constraints become

$$\omega_0 = \frac{1}{RC} = \frac{1}{C}$$

$$Q = \frac{R_Q}{R} = R_Q$$

This leaves 2 unknowns, $R_Q$ and $C$ and two constraints (i.e. no remaining degrees of freedom)
Example: Design a 2nd order lowpass Butterworth filter with 3dB passband attenuation, a dc gain of 5, and a 3dB bandedge of 4KHz

\[ T(s_n) = \frac{1}{s^2 + s \left( \frac{1}{Q} \right) + 1} \]

\[ \omega_0 = 1 \quad Q_N = \frac{1}{\sqrt{2}} \]

To satisfy the 2 constraints, must now set \( R_Q = Q \quad C = 1 \)

Now we can do frequency scaling

\[ C \rightarrow C/\omega_0 \]

\[ L \rightarrow L/\omega_0 \]

\[ C=1 \rightarrow \frac{1}{(2\pi \cdot 4K)} = 39.8\text{uF} \]
Example: Design a 2\textsuperscript{nd} order lowpass Butterworth filter with 3dB passband attenuation, a dc gain of 5, and a 3dB bandedge of 4KHz

Denormalized circuit with bandedge of 4 KHz

This has the right transfer function (but unity gain)

Can now do impedance scaling to get more practical component values

\[
\begin{align*}
R &\rightarrow 0R \\
C &\rightarrow C/\theta \\
L &\rightarrow 0L
\end{align*}
\]

A good impedance scaling factor may be $\theta=1000$

\[
\begin{align*}
R &\rightarrow 1K \\
C &\rightarrow 39.8\text{nF}
\end{align*}
\]
Example: Design a 2\textsuperscript{nd} order lowpass Butterworth filter with 3dB passband attenuation, a dc gain of 5, and a 3dB bandedge of 4KHz

Denormalized circuit with bandedge of 4 KHz

This has the right transfer function (but unity gain)

To finish the design, proceed or follow this circuit with an amplifier with a gain of 5 to meet the dc gain requirements
Filter Concepts and Terminology

- Frequency scaling
- Frequency Normalization
- Impedance scaling

Transformations
- LP to BP
- LP to HP
- LP to BR

It can be shown the standard HP, BP, and BR approximations can be obtained by a frequency transformation of a standard LP approximating function.

Will address the LP approximation first, and then provide details about the frequency transformations.
Filter Design
Process

Establish Specifications
- possibly $T_D(s)$ or $H_D(z)$
- magnitude and phase characteristics or restrictions
- time domain requirements

Approximation
- obtain acceptable transfer functions $T_A(s)$ or $H_A(z)$
- possibly acceptable realizable time-domain responses

Synthesis
- build circuit or implement algorithm that has response close to $T_A(s)$ or $H_A(z)$
- actually realize $T_R(s)$ or $H_R(z)$

Filter
The Approximation Problem

The goal in the approximation problem is simple, just want a function $T_A(s)$ or $H_A(z)$ that meets the filter requirements.

Will focus primarily on approximations of the standard normalized lowpass function

\[ |T_{LP}(j\omega)| \]

- Frequency scaling will be used to obtain other LP band edges
- Frequency transformations will be used to obtain HP, BP, and BR responses
The Approximation Problem

\[ |T_{LP}(j\omega)| \]

\[ T_A(s) = \text{?} \]

\( T_A(s) \) is a rational fraction in \( s \)

\[ T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i} \]

Rational fractions in \( s \) have no discontinuities in either magnitude or phase response.

No natural metrics for \( T_A(s) \) that relate to magnitude and phase characteristics (difficult to meaningfully compare \( T_{A1}(s) \) and \( T_{A2}(s) \))
The Approximation Problem

Approach we will follow:

- Magnitude Squared Approximating Functions $H_A(\omega^2)$
  - Inverse Transform $H_A(\omega^2) \rightarrow T_A(s)$
  - Collocation
  - Least Squares
  - Pade Approximations
  - Other Analytical Optimization
  - Numerical Optimization
  - Canonical Approximations
    → Butterworth (BW)
    → Chebyshev (CC)
    → Elliptic
    → Thompson
Magnitude Squared Approximating Functions

\[ T(s) = \sum_{i=0}^{m} a_i s^i \]
\[ T(s) = \sum_{i=0}^{n} b_i s^i \]

\[ T(j\omega) = \frac{\sum_{i=0}^{m} a_i (j\omega)^i}{\sum_{i=0}^{n} b_i (j\omega)^i} \]

\[ T(j\omega) = \frac{a_0 + a_1 (j\omega) + a_2 (j\omega)^2 + \ldots + a_m (j\omega)^m}{b_0 + b_1 (j\omega) + b_2 (j\omega)^2 + \ldots + b_n (j\omega)^n} \]

\[ T(j\omega) = \left[ \sum_{0 \leq k \leq m \text{ keven}} a_k \omega^k \right] + j \left[ \omega \sum_{0 \leq k \leq m \text{ kodd}} a_k \omega^{k-1} \right] \]

\[ T(j\omega) = \left[ \sum_{0 \leq k \leq n \text{ keven}} b_k \omega^k \right] + j \left[ \omega \sum_{0 \leq k \leq n \text{ kodd}} b_k \omega^{k-1} \right] \]

\[ T(j\omega) = \left[ \frac{F_1(\omega^2)}{F_3(\omega^2)} \right] + j \left[ \frac{\omega F_2(\omega^2)}{\omega F_4(\omega^2)} \right] \]

where \( F_1, F_2, F_3 \) and \( F_4 \) are even functions of \( \omega \)
Magnitude Squared Approximating Functions

\[ T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i} \]

\[ T(j\omega) = \frac{\left[ F_1(\omega^2) \right] + j \left[ \omega F_2(\omega^2) \right]}{\left[ F_3(\omega^2) \right] + j \left[ \omega F_4(\omega^2) \right]} \]

\[ |T(j\omega)| = \sqrt{\left[ F_1(\omega^2) \right]^2 + \omega^2 \left[ F_2(\omega^2) \right]^2} \]

\[ \sqrt{\left[ F_3(\omega^2) \right]^2 + \omega^2 \left[ F_4(\omega^2) \right]^2} \]

Thus \(|T(j\omega)|\) is an even function of \(\omega\)

It follows that \(|T(j\omega)|^2\) is a rational fraction in \(\omega^2\) with real coefficients

Since \(|T(j\omega)|^2\) is a real variable, natural metrics exist for comparing approximating functions to \(|T(j\omega)|^2\)
Magnitude Squared Approximating Functions

\[
T(s) = \sum_{i=0}^{m} a_i s^i \div \sum_{i=0}^{n} b_i s^i
\]

If a desired magnitude response is given, it is common to find a rational fraction in \(\omega^2\) with real coefficients, denoted as \(H_A(\omega^2)\), that approximates the desired magnitude squared response and then obtain a function \(T_A(s)\) that satisfies the relationship 

\[
|T_A(j\omega)|^2 = H_A(\omega^2)
\]

\(H_A(\omega^2)\) is real so natural metrics exist for obtaining \(H_A(\omega^2)\)

\[
H_A(\omega^2) = \sum_{i=0}^{2l} c_i \omega^{2i} \div \sum_{i=0}^{2k} d_i \omega^{2i}
\]

Obtaining \(T_A(s)\) from \(H_A(\omega^2)\) is termed the inverse mapping problem

But how is \(T_A(s)\) obtained from \(H_A(\omega^2)\) ?
Inverse mapping problem:

\[ T_A(s) \quad \rightarrow \quad H_A(\omega^2) \quad \quad H_A(\omega^2) = |T_A(j\omega)|^2 \]

Consider an example:

\[ T_1(s) = s + 1 \quad \rightarrow \quad H_A(\omega^2) = 1 + \omega^2 \]

\[ T_1(s) = s - 1 \]

Thus, the inverse mapping in this example is not unique!
Inverse mapping problem:

\[ T_A(s) \quad \rightarrow \quad H_A(\omega^2) \quad \quad H_A(\omega^2) = |T_A(j\omega)|^2 \]

\[ T_A(s) \quad \quad ? \quad \quad H_A(\omega^2) \]

Some observations:

- If an inverse mapping exists, it is not necessarily unique

- If an inverse mapping exists, then a minimum phase inverse mapping exists and it is unique (within all-pass factors)

- The mapping from \( T_A(s) \) to \( H_A(\omega^2) \) increases order by a factor of 2

- Any inverse mapping from \( H_A(\omega^2) \) to \( T_A(s) \) will reduce order by a factor of 2 (within all-pass factors)
Example:

\[ H_A(\omega^2) = \frac{2\omega^2 + 1}{\omega^4 + 2\omega^2 + 1} \rightarrow T_A(s) = \frac{\sqrt{2}s + 1}{(s + 1)(s + 1)} \]

Example:

\[ H_A(\omega^2) = \frac{\omega^2 - 1}{\omega^4 + 2\omega^2 + 1} \rightarrow ? \]

Inverse mapping does not exist!

It can be shown that many even rational fractions in \( \omega^2 \) do not have an inverse mapping back to the s-domain!

Often these functions have a magnitude squared response that does a good job of approximating the desired filter magnitude response.

If an inverse mapping exists, there are often several inverse mappings that exist.
Observation: If $z$ is a zero (pole) of $H_A(\omega^2)$, then $-z$, $z^*$, and $-z^*$ are also zeros (poles) of $H_A(\omega^2)$.

Thus, roots come as quadruples if off of the axis and as pairs if they lay on the axis.
Observation: If \( z \) is a zero (pole) of \( H_A(\omega^2) \), then \(-z, z^*, \) and \(-z^*\) are also zeros (poles) of \( H_A(\omega^2) \)

Proof:
Consider an even polynomial in \( \omega^2 \) with real coefficients

\[
P(\omega^2) = \sum_{i=0}^{m} a_i \omega^{2i}
\]

At a root, this polynomial satisfies the expression

\[
P(\omega^2) = \sum_{i=0}^{m} a_i \omega^{2i} = 0
\]

Replacing \( \omega \) with \(-\omega\), we obtain

\[
P(-\omega^2) = \sum_{i=0}^{m} a_i \omega^{2i} = \sum_{i=0}^{m} a_i (-1)^i \omega^{2i} = \sum_{i=0}^{m} a_i \omega^{2i} = 0 \quad \rightarrow \quad -\omega \text{ is a root of } P(\omega^2)
\]

Recall \((xy)^* = x^*y^*\) and \((x^n)^* = (x^*)^n\)

Taking the complex conjugate of \( P(\omega^2) = 0 \) we obtain

\[
P(\omega^2)^* = \sum_{i=0}^{m} (a_i \omega^{2i})^* = \sum_{i=0}^{m} (a_i^*) \omega^{2i} = \sum_{i=0}^{m} (a_i^*) (\omega^*)^{2i} = 0
\]

Since \( a_i \) is real for all \( i \), it thus follows that

\[
\sum_{i=0}^{m} (a_i^*) (\omega^*)^{2i} = 0 \quad \rightarrow \quad \omega^* \text{ is a root of } P(\omega^2)
\]
End of Lecture 6