EE 508
Lecture 7

- Canonical Approximating Functions
- Functional Transformations
Approximations

- Magnitude Squared Approximating Functions – $H_A(\omega^2)$
- Inverse Transform - $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation
- Least Squares Approximations
- Pade Approximations
- Other Analytical Optimizations
- Numerical Optimization
- Canonical Approximations
  - Butterworth
  - Chebyschev
    - Elliptic
    - Bessel
    - Thompson
Review from last time

**Butterworth Approximation**

- All pole approximation
- Maximally flat magnitude response at $\omega=0$
  \[ |T(j\omega)|^2 = \frac{1}{1 + \varepsilon^2 \omega^{2n}} \]
- Poles Lie on Unit Circle
  Of radius $\frac{1}{\varepsilon^{1/n}}$
- Transition becomes steeper at band edge as $n$ increases
- Pole $Q$ is rather small
  \[ Q_{\text{max}} = \frac{1}{2 \sin \left( \frac{\pi}{2n} \right)} \]
Review from last time

Chebyschev Approximation

• All pole approximation
• Magnitude response equals 1 a maximum number of times in $[0,1]$

\[ |T(j\omega)|^2 = \frac{1}{1 + \varepsilon^2 (C(\omega))^2} \]

• Poles Lie on Ellipse
• Transition becomes steeper at band edge as $n$ increases
• Pole $Q$ is rather large

\[ Q_{\text{max}} = Q_{\text{imp max}} \cdot \sqrt{1 + \left( \frac{\cos\left(\frac{\pi}{2n}\right)}{\sinh\left(\frac{1}{n} \sin^{-1}\left(\frac{1}{\varepsilon}\right)\right)} \right)} \]
Fig. 17-3a  Magnitude of the maximally flat approximation ($\varepsilon = 1$)
Fig. 17-3b  Phase of the maximally flat approximation ($\epsilon = 1$)
Fig. 17-6a Fourth-order Chebyshev and Butterworth magnitude characteristics
Fig. 17-6b  Fourth-order Chebyshev and Butterworth phase characteristics
Comparison of BW and CC Responses

- CC slope at band edge much steeper than that of BW
  \[ \text{Slope}_{cc}(\omega = 1) = \left( \frac{-n}{2\sqrt{2}} \right) n = [\text{Slope}_{bw}(\omega = 1)] \cdot n \]

- Corresponding pole Q of CC much higher than that of BW

- Lower-order CC filter can often meet same band-edge transition as a given BW filter

- Both are widely used

- Cost of implementation of BW and CC for same order is about the same
Transitional BW-CC filters

\[ H_{ABW} (\omega^2) = \frac{1}{1 + \varepsilon^2 \omega^{2n}} \]
\[ H_{ACC} (\omega^2) = \frac{1}{1 + \varepsilon^2 \left( C_n (\omega) \right)^2} \]

\[ H_{ATRAN1} (\omega^2) = \frac{1}{1 + \varepsilon^2 \left( \omega^{2k} \right) \left( C_{n-k} (\omega) \right)} \]
\[ 0 \leq k \leq n \]

\[ H_{ATRAN2} (\omega^2) = \frac{1}{1 + \varepsilon^2 \left[ \theta \omega^{2n} + (1 - \theta) C_2^n (\omega) \right]} \]
\[ 0 \leq \theta \leq 1 \]

Other transitional BW-CC approximations exist as well.
Transitional BW-CC filters

\[ H_{ATRAN1} (\omega^2) = \frac{1}{1 + \varepsilon^2 (\omega^{2k}) C_{n-k}^2 (\omega)} \]

\[ H_{ATRAN2} (\omega^2) = \frac{1}{1 + \varepsilon^2 \left[ \theta \omega^{2n} + (1-\theta) C_n^2 (\omega) \right]} \]

Transitional filters will exhibit flatness at \( \omega=0 \), passband ripple, and intermediate slope characteristics at band-edge.
what about?

\[ H_n(\omega^2) = \frac{\theta}{1 + \epsilon^2 \omega^{2n}} + \frac{1 - \theta}{1 + \epsilon^2 [C_n(\omega)]^2} \]

- may not be an all-pole approximation
- does not preserve order
- order reduces to \(2n\) at \(\theta = 0, \theta = 1\) due to pole/zero cancellation
Transitional BW-CC Filters

\[ H_{BCn} = \frac{1}{1 + \varepsilon^2 \omega^{2k} \zeta_{n-k}^2(\omega)} \]

\( 1 \leq k \leq n \)

\[ H_{BCn} = \frac{1}{1 + \varepsilon^2 \left[ \sigma \omega^{2n} + (1-\sigma)\xi_{n}^2(\omega) \right]} \]
Elliptic Filters

- Can be thought of as an extension of the CC approach by adding complex-conjugate zeros on the imaginary axis to increase the sharpness of the slope at the band edge.
Elliptic Filters

Conceptual:

[Graphs showing the behavior of elliptic filters, with one graph indicating a smooth roll-off and another showing a more oscillatory response.]
Elliptic Filters:

- $n$ even
  - add $n$ zeros on $j\omega$ axis

- $n$ odd
  - add $n-1$ zeros on $j\omega$ axis

$$H(\omega^2) = \frac{1}{1 + \epsilon^2 R^2(\omega^2)}$$

$R^2(\omega^2) \leq 1$ for $0 \leq \omega \leq 1$

$R^2(\omega^2) > H$ for $\omega > \Omega_s$

$R^2(\omega^2)$ is equal ripple in both passband and stopband.
As \leq 1

If elliptic approx. is of full order, the response is completely determined by

\[ \{ n, \varepsilon, \Omega_s, A_s \} \]

It can be shown that any 3 of these parameters are independent.

Typically \( \varepsilon, \Omega_s, \text{ and } A_s \) are fixed by specifications \( \therefore n \) can be determined
If $n$ odd.

\[ \begin{align*}
\text{deg. num} &= n-1 \\
\text{deg. den.} &= n \\
\frac{n-1}{2} &= \text{peaks in passband} \\
\frac{n-1}{2} &= \text{peaks in stop band} \\
\text{maximum occurs at } \omega = 0 \\
|T(j\omega)| &= 0
\end{align*} \]
If \( n \) even

1) deg. of num & den. equal to \( n 
2) \( n/2 \) peaks in passband & stopband
3) \( |H(\omega)| = \frac{1}{\sqrt{1+\varepsilon^2}} \neq 1
4) \( |H(\omega)| = A_s = \neq 0 \)
- Simple closed-form expressions for elliptic approx. do not exist

- Simple closed-form solution for obtaining $n$ does not exist (design tables for determining $n$)

- Approximations need not be of full order

- With 3 d.o.f., a simple table for elliptic filters does not exist
Comparison of Elliptic Filters with CC filters

- Elliptic Filters have steeper slope at band edge than CC
- Elliptic filters do not roll off as quickly in stop band
- For a given band edge slope requirement and a given stop band attenuation, elliptic filter will generally be of lower order
- Cost of implementation of elliptic filter comparable to that of CC filter if orders are the same
- Maximum pole Q of elliptic filter higher than that of CC filter if orders are the same
Canonical Approximating Functions

Butterworth
Chebyschev
Transitional BW-CC
Elliptic

Thompson
Bessel

Thompson and Bessel Approximating Functions are Two Different Names for the Same Approximation
Thompson and Bessel Approximations

- All-pole filters
- Maximally linear phase at $\omega=0$

Will first consider concepts of frequency distortion and group delay
Flat Passband / Stopband Filters

Non flat passband / stopband filter
Example: If \( \omega_1 \) and \( 3\omega_1 \) are in the passband of a filter and

\[
X_i = mx_0 \sin(\omega_1 t + \theta_1) + mx_2 \sin(3\omega_1 t + \theta_2)
\]

\( \theta_1 = \theta_2 = 0 \)

Note symmetry in \( X_i(t) \).
If passed through a filter with transfer function $T(s)$ then if no magnitude or phase distortion shifted in time and scaled in magnitude.

If magnitude distortion present but no phase distortion $\forall t_i \neq \forall t_j$
If phase distortion present but no magnitude distortion

- Note distortion in shape

- Phase distortion does not disturb the spectral components! and does not introduce new spectral components!

- Magnitude distortion disturbs the spectral components but does not introduce new spectral components!

- Nonlinear distortion introduces new spectral components!
Example: \[ X_i = X_m \sin(\omega_i t + \theta_i) + X_{m2} \sin(\omega_{2t} + \theta_2) \]

Spectral components of \( X_i \):

\[ X_{m1} \]
\[ X_{m2} \]

\[ \omega_i \quad \quad \omega_2 \]

\[ X_{m2} = \Delta X_{m1} \]

Note: Phase information completely lost in spectral component characterization.

Filtered output:
If \( \hat{X}_{m2} = \Delta \hat{X}_{m1} \), then no amplitude distortion.
Example: Nonlinear Distortion

\[ y_i = x_{m1} \sin (\omega_1 t + \theta_1) \]

\[ y_0 = \hat{x}_{m1} \sin (\omega_1 t + \theta_1 + \Delta \theta_1) \]

\[ + \sum_{k=2}^{\infty} b_k \sin (k\omega_1 t + \theta_1 + \Theta_k) \]

Nonlinear Distortion Component

Input Spectrum

Output Spectrum

Nonlinear Distortion Components
Amplitude and Phase Distortion

(Not to be confused with nonlinear distortion)

\[ X_i \rightarrow T(s) \rightarrow X_0 \]

If \( X_0(t) = K(X_i(t-t_0)) \), then we say there is no amplitude or phase distortion of \( X_i(t) \).

Consider \( X_i = A_1 \sin(w_1 t) + A_2 \sin(w_2 t) \)

In steady state:

\[ X_0(t) = A_1 |T(|i,w_1)| \sin(w_1 t + \angle T(|i,w_1|)) \]

\[ + A_2 |T(|i,w_2)| \sin(w_2 t + \angle T(|i,w_2|)) \]
\[ x_0(t) = A_1 |T(i\omega_1)| \sin \left( \omega_1 \left[ t - \frac{\triangle T(i\omega_1)}{\omega_1} \right] \right) + \]
\[ A_2 |T(i\omega_2)| \sin \left( \omega_2 \left[ t - \frac{\triangle T(i\omega_2)}{\omega_2} \right] \right) \]

If \( |T(i\omega_1)| = |T(i\omega_2)| \)

\[ x_0(t) = |T(i\omega)| \left( [A_1 \sin (\omega_1 \left[ t - \frac{\triangle T(i\omega)}{\omega_1} \right] ) + \right. \]
\[ \left. [A_2 \sin (\omega_2 \left[ t - \frac{\triangle T(i\omega_2)}{\omega_2} \right] )] \right) \]

If \( \triangle T(i\omega) = k \omega \)

\[ x_0(t) = |T(i\omega)| \left( A_1 \sin (\omega_1 \left[ t - k \right] ) + A_2 \sin (\omega_2 \left[ t - k \right] ) \right) \]

\[ y_0(t) = |T(i\omega)| \times (t - k) \]
If \( \omega_1 \) and \( \omega_2 \) are any two spectral components of an input signal in which \( |T(i\omega_1)| \neq |T(i\omega_2)| \), then the filter exhibits amplitude distortion.

If \( \omega_1 \) and \( \omega_2 \) are any two spectral components of an input signal, then the input signal exhibits phase distortion if \( \angle T(i\omega_2) \neq \frac{\omega_2}{\omega_1} \angle T(i\omega_1) \).

Magnitude and phase distortion are often of concern in filter applications requiring a flat passband and a flat zero-magnitude stop band.

Phase distortion is usually of little concern in the stop band.

Magnitude distortion is usually of little concern in the stop band.
It can be shown that the only way to avoid magnitude and phase distortion respectively for signals that have energy components in the interval \(0 \leq \omega \leq \omega_y\) is to have constants, \(k_1\) and \(k_2\) such that

a) \( |T(i\omega)| = k_1 \quad 0 \leq \omega \leq \omega_x \)

b) \( \angle T(i\omega) = k_2\omega \quad 0 \leq \omega \leq \omega_x \)

If \( \angle T(i\omega) = k_2\omega \), we say \( T(s) \) is a linear phase function.
Defn: Group Delay is the negative of the phase derivative.

\[ \tau_G = -\frac{\partial (\angle T(i\omega))}{\partial \omega} \]

Observation: Phase is linear iff \( \tau_G \) is constant.

Example: \( T(s) = \frac{1}{s+1} \Rightarrow T(i\omega) = \frac{1}{i\omega+1} \)

\[ \angle T(i\omega) = -\tan^{-1}\omega \]

Observation: Phase is analytically very complicated.

\[ \tau_g = -\frac{\partial (\angle T(i\omega))}{\partial \omega} \]

Recall \[ \frac{\partial (\tan^{-1}u)}{\partial x} = \frac{1}{1+u^2} \frac{\partial u}{\partial x} \]

\[ \therefore \tau_g = (-)(-) \frac{1}{1+w^2} = \frac{1}{1+w^2} \]
Group Delay is a rational function in $\omega^2$. 
In this example, the group delay is a rational fraction in $\omega^2$.

Claim: The group delay for any filter is a rational fraction in $\omega^2$.

Proof: (will consider all pole case for convenience)

\[
T(s) = \frac{1}{\sum_{k=0}^{n} a_k s^k}
\]

\[
T(j\omega) = \frac{1}{(1 - a_2 \omega^2 + a_4 \omega^4 + \ldots) + j\omega (a_1 - a_3 \omega^2 + a_5 \omega^4 + \ldots)}
\]

\[
T(j\omega) = \frac{1}{F_{p1}(\omega^2) + j\omega F_{p2}(\omega^2)}
\]

\[
\angle T(j\omega) = -\tan^{-1} \left( \frac{\omega F_{p2}(\omega^2)}{F_{p1}(\omega^2)} \right)
\]
\[ t_g = \frac{1}{1 + \left[ \frac{\omega F_{e_2}(\omega^2)}{F_{e_1}(\omega^2)} \right]^2} \]

Consider the derivative term:

\[ \frac{\partial}{\partial \omega} \left[ \frac{\omega F_{e_2}(\omega^2)}{F_{e_1}(\omega^2)} \right] = F_{e_1}(\omega^2) \frac{\partial}{\partial \omega} \left[ \frac{\omega F_{e_1}(\omega^2)}{F_{e_1}(\omega^2)} \right] - \frac{\omega F_{e_2}(\omega^2)}{F_{e_1}(\omega^2)} \frac{\partial}{\partial \omega} \left[ F_{e_1}(\omega^2) \right]^2 \]

Note the RHS of this function is even so \( t_g \) is an even rational function in \( \omega^2 \).

Observation: group delay much more convenient to work with than the phase!
The phase is linear iff the group delay is constant (i.e. independent of \( \omega \))
Example:

\[ T(s) = \frac{1}{s+1} \]

\[ T(\omega) = \frac{1}{i\omega+1} \]

\[ \angle T(i\omega) = -\tan^{-1} \omega \]

Phase is analytically quite complicated.
Thompson Filters

1) All pole filters

2) Maximaly linear phase at $w = 0$

3) Maximaly constant group delay at $w = 0$

$t_g = 1$ at $w = 0$

\[ T_A(s) = \frac{1}{1 + a_1 s + a_2 s^2 + \ldots + a_n s^n} \]

\[ T_A(iw) = \frac{1}{(1 - q_2 w^2 + \ldots) + i\omega (a_1 - q_1 w^2 + \ldots)} \]

\[ t_g = \frac{a_1 + \omega^2 (a_2 - 3a_3) + \omega^4 (5a_5 - 3a_4 + a_2 a_3) + \ldots}{1 + \omega^2 (a_2 - 2a_3) + \omega^4 (a_2^2 - 2a_4 + a_3^2) + \ldots} \]

Assume $t_g = 1$ at $w = 0 \implies a_1 = 1$

To be maximaly constant, want

\[ a_2 - 3a_3 = 1 - 2q_2 \]
\[ 5a_5 - 3a_4 + a_2 a_3 = a_2^2 - 2a_3 + 2a_4 \]
\[ \vdots \]

\[ \vdots \]
It can be shown that

\[ a_k = \frac{(2n-k)!}{H \, 2^{n-k} \, k! \, (n-k)!} \quad k = 1, \ldots, n-1 \]

\[ a_n = \frac{1}{H} \]

\[ H = \frac{(2n)!}{2^n \, n!} \]

Alternatively

\[ B_0 = s+1 \]
\[ B_2 = s^2 + 3s + 3 \]
\[ B_n = (2n-1)B_{n-1}(s) + s^2B_{n-2}(s) \]

\[ T_n(s) = \frac{1}{B_n(s)} \cdot B_n(s) \]

\(B_n(s)\) are Bessel Functions so sometimes the filters are termed Bessel Filters.
Claim: Thompson approximation exhibits relatively poor magnitude characteristic.