

EE 508

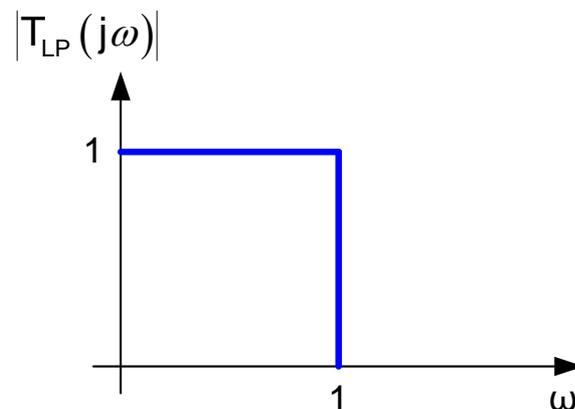
Lecture 8

The Approximation Problem

The Approximation Problem

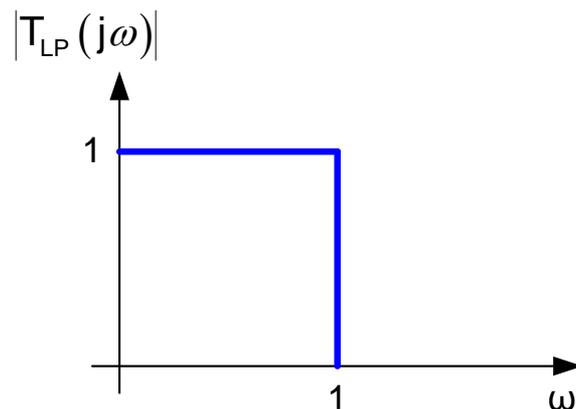
The goal in the approximation problem is simple, just want a function $T_A(s)$ or $H_A(z)$ that meets the filter requirements.

Will focus primarily on approximations of the standard normalized lowpass function



- Frequency scaling will be used to obtain other LP band edges
- Frequency transformations will be used to obtain HP, BP, and BR responses

The Approximation Problem



$$T_A(s) = ?$$

$T_A(s)$ is a rational fraction in s

$$T(s) = \frac{\sum_{i=0}^m a_i s^i}{\sum_{i=0}^n b_i s^i}$$

Rational fractions in s have no discontinuities in either magnitude or phase response

No natural metrics for $T_A(s)$ that relate to magnitude and phase characteristics (difficult to meaningfully compare $T_{A1}(s)$ and $T_{A2}(s)$)

Review from Last Time

Magnitude Squared Approximating Functions

$$T(s) = \frac{\sum_{i=0}^m a_i s^i}{\sum_{i=0}^n b_i s^i}$$

$$T(j\omega) = \frac{[F_1(\omega^2)] + j[\omega F_2(\omega^2)]}{[F_3(\omega^2)] + j[\omega F_4(\omega^2)]}$$

$$|T(j\omega)| = \sqrt{\frac{[F_1(\omega^2)]^2 + \omega^2 [F_2(\omega^2)]^2}{[F_3(\omega^2)]^2 + \omega^2 [F_4(\omega^2)]^2}}$$

Thus $|T(j\omega)|$ is an even function of ω

It follows that $|T(j\omega)|^2$ is a rational fraction in ω^2 with real coefficients

Since $|T(j\omega)|^2$ is a real variable, natural metrics exist for comparing approximating functions to $|T(j\omega)|^2$

Review from Last Time

Magnitude Squared Approximating Functions

$$T(s) = \frac{\sum_{i=0}^m a_i s^i}{\sum_{i=0}^n b_i s^i}$$

If a desired magnitude response is given, it is common to find a rational fraction in ω^2 with real coefficients, denoted as $H_A(\omega^2)$, that approximates the desired magnitude squared response and then obtain a function $T_A(s)$ that satisfies the relationship $|T_A(j\omega)|^2 = H_A(\omega^2)$

$H_A(\omega^2)$ is real so natural metrics exist for obtaining $H_A(\omega^2)$

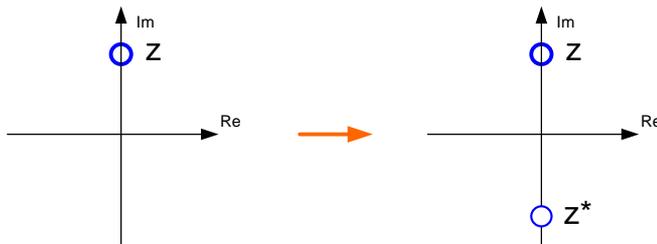
$$H_A(\omega^2) = \frac{\sum_{i=0}^{2l} c_i \omega^{2i}}{\sum_{i=0}^{2k} d_i \omega^{2i}}$$

Obtaining $T_A(s)$ from $H_A(\omega^2)$ is termed the inverse mapping problem

But how is $T_A(s)$ obtained from $H_A(\omega^2)$?

Review from Last Time

Observation: If z is a zero (pole) of $H_A(\omega^2)$, then $-z$, z^* , and $-z^*$ are also zeros (poles) of $H_A(\omega^2)$



Thus, roots come as quadruples if off of the axis and as pairs if they lay on the axis

Review from Last Time

Theorem: If $H_A(\omega^2)$ is a rational fraction with real coefficients with no poles or zeros of odd multiplicity on the real axis, then there exists a real number H_0 such that the function

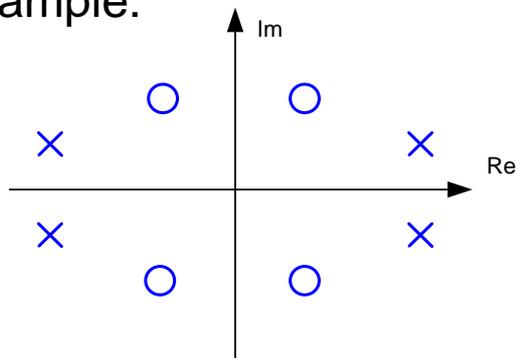
$$T_{AM}(s) = \frac{H_0 (s-jz_1)(s-jz_2) \cdots (s-jz_m)}{(s-jp_1)(s-jp_2) \cdots (s-jp_n)}$$

is a minimum phase rational fraction with real coefficients that satisfies the relationship

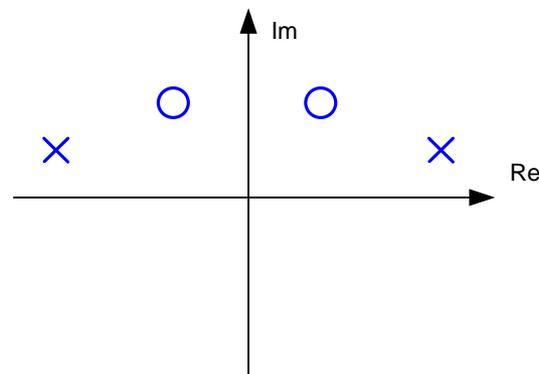
$$|T_{AM}(j\omega)| = \sqrt{H_A(\omega^2)}$$

where $\{z_1, z_2, \dots, z_m\}$ are the upper half-plane zeros of $H_A(\omega^2)$ and exactly half of the real axis zeros, and where $\{p_1, p_2, \dots, p_n\}$ are the upper half-plane poles of $H_A(\omega^2)$ and exactly half of the real axis poles.

Example:



Roots of $H_A(\omega^2)$



Roots that Appear in $T_{AM}(s)$

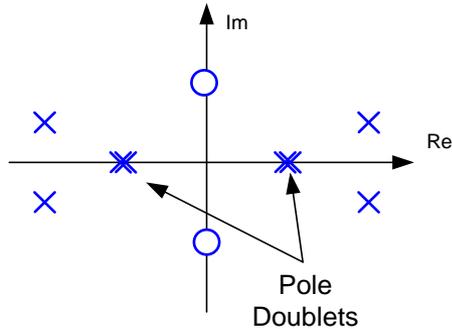
Review from Last Time

$$H_A(\omega^2) = \frac{H_0^2 \left[(\omega - z_1)(\omega - z_2) \cdots (\omega - z_m) \right] \cdot \left[(\omega + z_1)(\omega + z_2) \cdots (\omega + z_m) \right]}{\left[(\omega - p_1)(\omega - p_2) \cdots (\omega - p_n) \right] \cdot \left[(\omega + p_1)(\omega + p_2) \cdots (\omega + p_n) \right]}$$

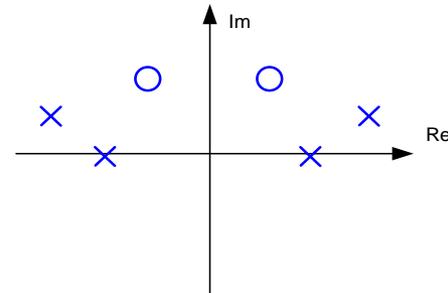
↓ If inverse exists

$$T_{AM}(s) = \frac{H_0 (s - jz_1)(s - jz_2) \cdots (s - jz_m)}{(s - jp_1)(s - jp_2) \cdots (s - jp_n)}$$

Example:

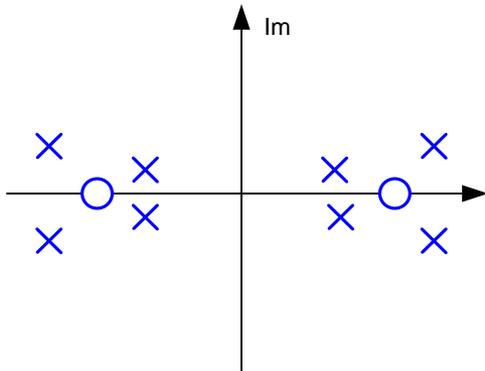


Roots of $H_A(\omega^2)$



Roots that appear in $T_{AM}(s)$

Example:



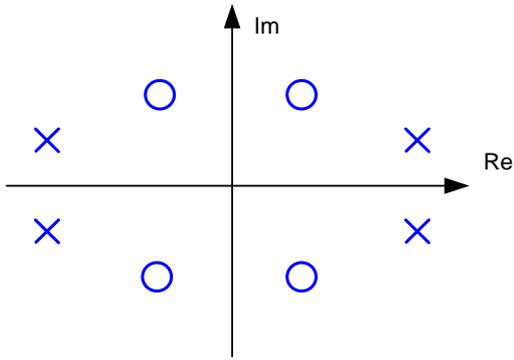
Inverse does not exist because zeros are of odd multiplicity on the real axis

Review from Last Time

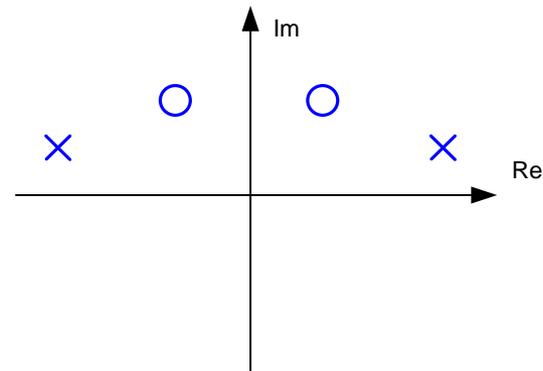
$$H_A(\omega^2) = \frac{H_0^2 \left[(\omega - z_1)(\omega - z_2) \cdots (\omega - z_m) \right] \cdot \left[(\omega + z_1)(\omega + z_2) \cdots (\omega + z_m) \right]}{\left[(\omega - p_1)(\omega - p_2) \cdots (\omega - p_n) \right] \cdot \left[(\omega + p_1)(\omega + p_2) \cdots (\omega + p_n) \right]}$$

↓ If inverse exists

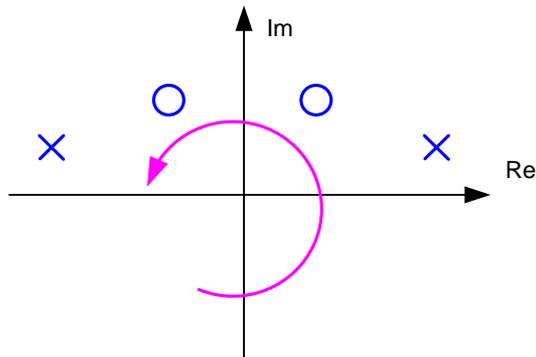
$$T_{AM}(s) = \frac{H_0 (s - jz_1)(s - jz_2) \cdots (s - jz_m)}{(s - jp_1)(s - jp_2) \cdots (s - jp_n)}$$



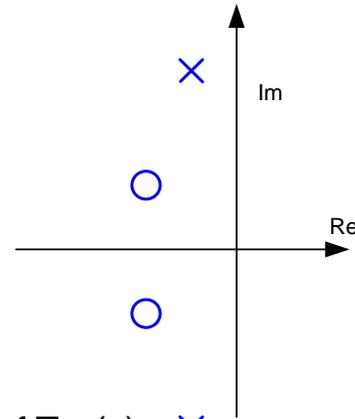
Roots of $H_A(\omega^2)$



Roots that appear in $T_{AM}(s)$



Rotate roots by 90°



Roots of $T_{AM}(s)$

Review from Last Time

$$H_A(\omega^2) = \frac{H_0^2 \left[(\omega - z_1)(\omega - z_2) \cdots (\omega - z_m) \right] \cdot \left[(\omega + z_1)(\omega + z_2) \cdots (\omega + z_m) \right]}{\left[(\omega - p_1)(\omega - p_2) \cdots (\omega - p_n) \right] \cdot \left[(\omega + p_1)(\omega + p_2) \cdots (\omega + p_n) \right]}$$



If inverse exists

$$T_{AM}(s) = \frac{H_0 (s - jz_1)(s - jz_2) \cdots (s - jz_m)}{(s - jp_1)(s - jp_2) \cdots (s - jp_n)}$$

Observations:

- Coefficients of $T_{AM}(s)$ are real
- If x is a root of $H_A(\omega^2)$, then jx is a root of $T_{AM}(s)$
- Multiplying a root by j is equivalent to rotating it by 90° cc in the complex plane
- Roots of $T_{AM}(s)$ are obtained from roots of $H_A(\omega^2)$ by multiplying by j
- Roots of $T_{AM}(s)$ are upper half-plane roots and exactly half of real axis roots all rotated cc by 90°
- If a root of $H_A(\omega^2)$ has odd multiplicity on the real axis, the inverse mapping does not exist
- Other (often many) inverse mappings exist but are not minimum phase
(These can be obtained by reflecting any subset of the zeros or poles around the imaginary axis into the RHP)

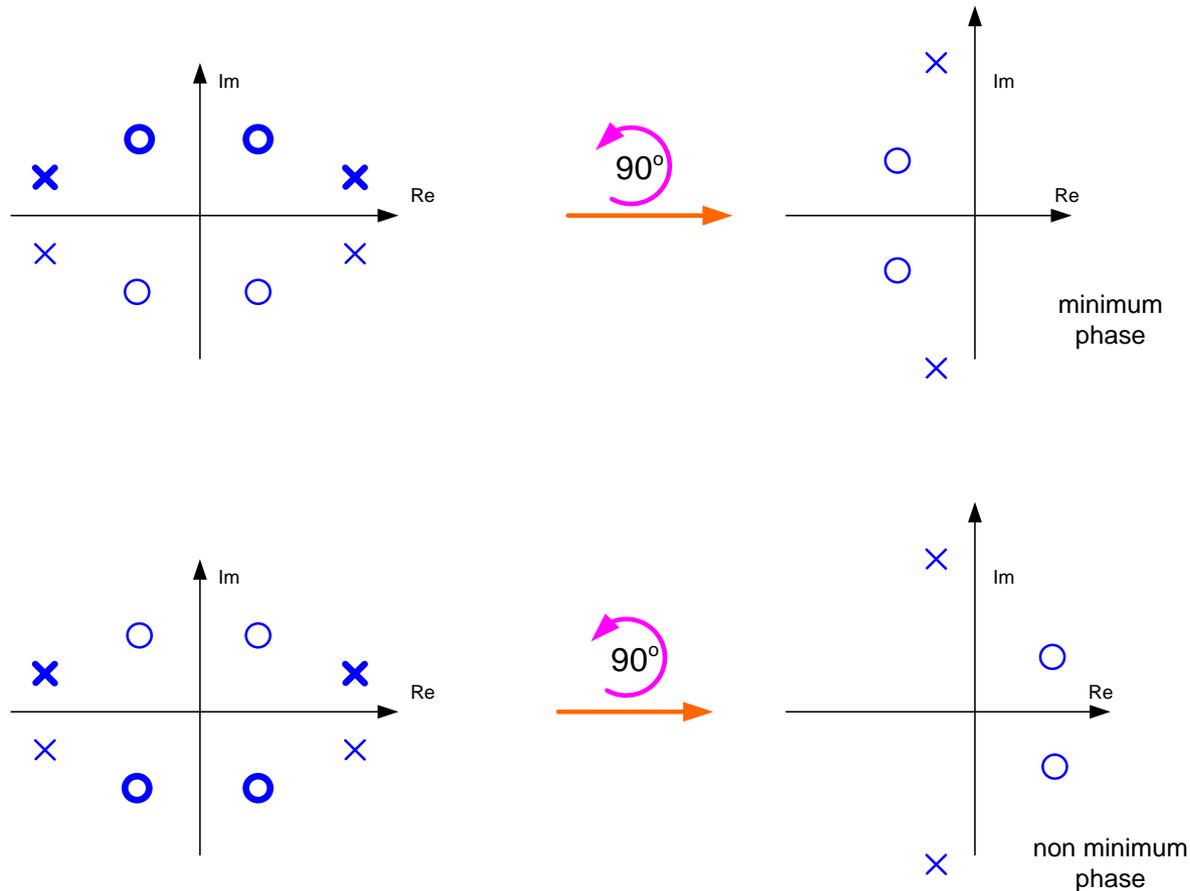
Review from Last Time

$$H_A(\omega^2) = \frac{H_0^2 \left[(\omega - z_1)(\omega - z_2) \cdots (\omega - z_m) \right] \cdot \left[(\omega + z_1)(\omega + z_2) \cdots (\omega + z_m) \right]}{\left[(\omega - p_1)(\omega - p_2) \cdots (\omega - p_n) \right] \cdot \left[(\omega + p_1)(\omega + p_2) \cdots (\omega + p_n) \right]}$$



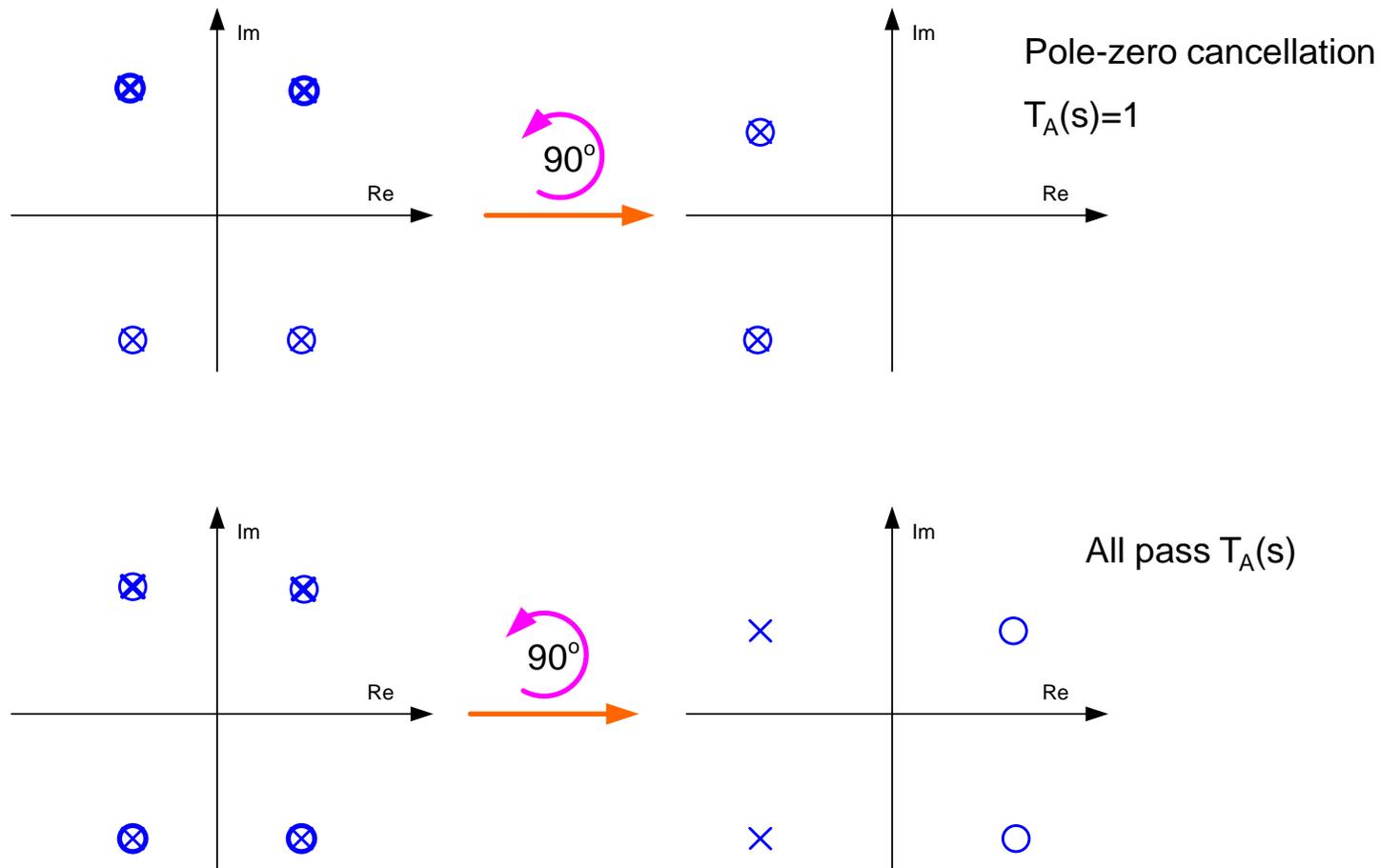
If inverse exists

$$T_{AM}(s) = \frac{H_0 (s - jz_1)(s - jz_2) \cdots (s - jz_m)}{(s - jp_1)(s - jp_2) \cdots (s - jp_n)}$$



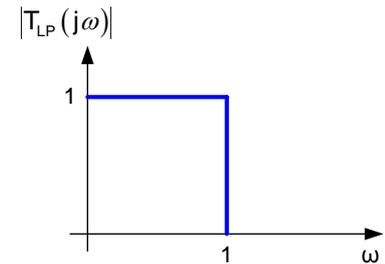
Review from Last Time

All pass functions (and factors)



- Must not allow cancellations to take place in $H_A(\omega^2)$ to obtain all-pass $T_A(s)$
- Must keep upper HP poles and lower HP zeros in $H_A(\omega^2)$ to obtain all-pass $T_A(s)$
- All-pass $T_A(s)$ is not minimum phase

The Approximation Problem



Approach we will follow:

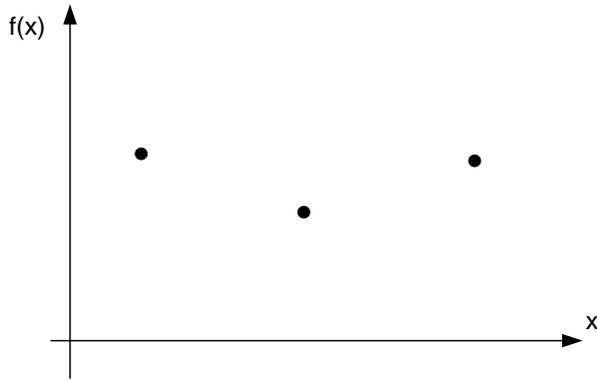
- Magnitude Squared Approximating Functions $H_A(\omega^2)$
- Inverse Transform $H_A(\omega^2) \rightarrow T_A(s)$



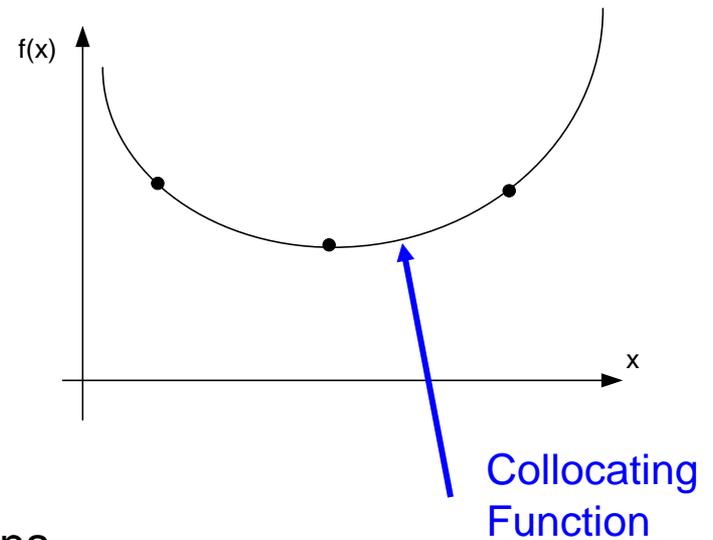
- Collocation
- Least Squares
- Pade Approximations
- Other Analytical Optimization
- Numerical Optimization
- Canonical Approximations
 - Butterworth (BW)
 - Chebyshev (CC)
 - Elliptic
 - Thompson

Collocation

Collocation is the fitting of a function to a set of points (or measurements) so that the function agrees with the sample at each point in the set.



Often consider critically constrained functions



The function that is of interest for using collocation when addressing the approximation problem is $H_A(\omega^2)$

Collocation

Example: Collocation points $\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$

Polynomial collocating function (critically constrained)

$$f(x) = a_0 + a_1x + a_2x^2$$

Unknowns: $\{a_1, a_2, a_3\}$

Set of equations:

$$y_1 = a_0 + a_1x_1 + a_2x_1^2$$
$$y_2 = a_0 + a_1x_2 + a_2x_2^2$$
$$y_3 = a_0 + a_1x_3 + a_2x_3^2$$

These equations are linear in the unknowns $\{a_1, a_2, a_3\}$

Can be expressed in matrix form

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$\mathbf{Y} = \mathbf{X} \cdot \mathbf{A}$$

Solution:

$$\mathbf{A} = \mathbf{X}^{-1} \cdot \mathbf{Y}$$

Closed form solution exists when collocating to a polynomial

Collocation

Is it possible to get a closed-form solution when collocating to a rational fraction?

$$\{(x_1, y_1), (x_2, y_2) \dots (x_k, y_k)\} \quad f(x) = \frac{a_0 + a_1x + a_2x^2 + \dots + a_mx^m}{1 + b_1x + b_2x^2 + \dots + b_nx^n}$$

where $k=m+n+1$

The rational fraction is nonlinear in x !

$$y_1(1 + b_1x_1 + b_2x_1^2 + \dots + b_nx_1^n) = a_0 + a_1x_1 + a_2x_1^2 + \dots + a_mx_1^n$$

This can be expressed as

$$y_1 = a_0 + a_1x_1 + a_2x_1^2 + \dots + a_mx_1^n - b_1x_1y_1 - b_2x_1^2y_1 - \dots - b_nx_1^ny_1$$

Note this equation is linear in the unknowns $\{a_0, a_1, \dots, a_m, b_1, b_2, \dots, b_n\}$

Collocation

Is it possible to get a closed-form solution when collocating to a rational fraction?

$$\{(x_1, y_1), (x_2, y_2) \dots (x_k, y_k)\} \quad f(x) = \frac{a_0 + a_1x + a_2x^2 + \dots + a_mx^m}{1 + b_1x + b_2x^2 + \dots + b_nx^n}$$

where $k=m+n+1$

$$y_1 = a_0 + a_1x_1 + a_2x_1^2 + \dots + a_mx_1^m - b_1x_1y_1 - b_2x_1^2y_1 - \dots - b_nx_1^ny_1$$

$$y_2 = a_0 + a_1x_2 + a_2x_2^2 + \dots + a_mx_2^m - b_1x_2y_2 - b_2x_2^2y_2 - \dots - b_nx_2^ny_2$$

•

•

•

$$y_k = a_0 + a_1x_k + a_2x_k^2 + \dots + a_mx_k^m - b_1x_ky_k - b_2x_k^2y_k - \dots - b_nx_k^ny_k$$

Collocation

Is it possible to get a closed-form solution when collocating to a rational fraction?

$$\{(x_1, y_1), (x_2, y_2) \dots (x_k, y_k)\} \quad f(x) = \frac{a_0 + a_1x + a_2x^2 + \dots + a_mx^m}{1 + b_1x + b_2x^2 + \dots + b_nx^n}$$

$$\begin{aligned} y_1 &= a_0 + a_1x_1 + a_2x_1^2 + \dots + a_mx_1^m - b_1x_1y_1 - b_2x_1^2y_1 - \dots - b_nx_1^ny_1 \\ y_2 &= a_0 + a_1x_2 + a_2x_2^2 + \dots + a_mx_2^m - b_1x_2y_2 - b_2x_2^2y_2 - \dots - b_nx_2^ny_2 \\ &\vdots \\ &\vdots \\ &\vdots \\ y_k &= a_0 + a_1x_k + a_2x_k^2 + \dots + a_mx_k^m - b_1x_ky_k - b_2x_k^2y_k - \dots - b_nx_k^ny_k \end{aligned}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \bullet \\ \bullet \\ y_k \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^m & -x_1y_1 & -x_1^2y_1 & \dots & -x_1^ny_1 \\ 1 & x_2 & x_2^2 & \dots & x_2^m & -x_2y_2 & -x_2^2y_2 & \dots & -x_2^ny_2 \\ \bullet & & & & & \bullet & & & \\ \bullet & & & & & \bullet & & & \\ 1 & x_k & x_k^2 & \dots & x_k^m & -x_ky_k & -x_k^2y_k & \dots & -x_k^ny_k \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ \dots \\ a_m \\ b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$$

$$Y = Z \cdot C$$

$$C = Z^{-1} \cdot Y$$

Closed form solution when collocating to a rational fraction !



Collocation

Applying to $H_A(\omega^2)$

$$\{(\omega_1, y_1), (\omega_2, y_2) \dots (\omega_k, y_k)\} \quad H_A(\omega^2) = \frac{a_0 + a_1\omega^2 + a_2\omega^4 + \dots + a_m\omega^{2m}}{1 + b_1\omega^2 + b_2\omega^4 + \dots + b_n\omega^{2n}}$$

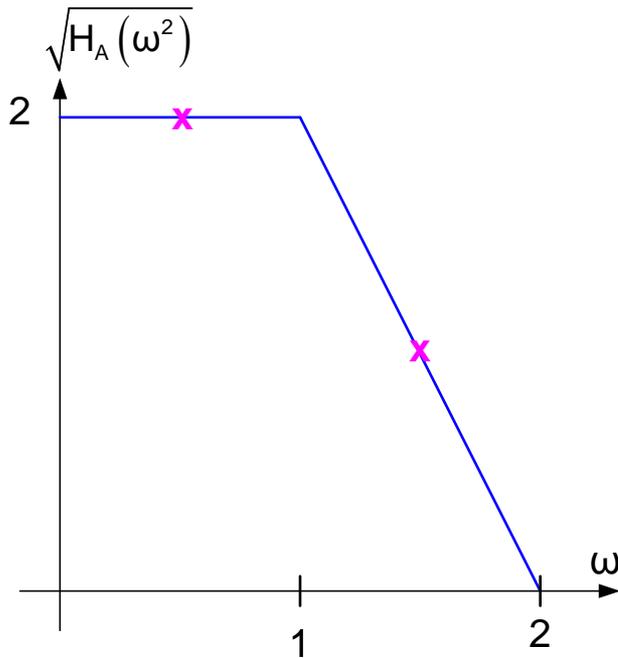
$$\begin{bmatrix} y_1 \\ y_2 \\ \bullet \\ \bullet \\ y_k \end{bmatrix} = \begin{bmatrix} 1 & \omega_1^2 & \omega_1^4 & \dots & \omega_1^{2m} & -\omega_1^2 y_1 & -\omega_1^4 y_1 & -\dots & -\omega_1^{2n} y_1 \\ 1 & \omega_2^2 & \omega_2^4 & \dots & \omega_2^{2m} & -\omega_2^2 y_1 & -\omega_2^4 y_1 & -\dots & -\omega_2^{2n} y_1 \\ \bullet & & & & & & & & \\ \bullet & & & & & & & & \\ 1 & \omega_k^2 & \omega_k^4 & \dots & \omega_k^{2m} & -\omega_k^2 y_1 & -\omega_k^4 y_1 & -\dots & -\omega_k^{2n} y_1 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ \dots \\ a_m \\ b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$$

$$\mathbf{Y} = \mathbf{Z} \cdot \mathbf{C}$$

$$\mathbf{C} = \mathbf{Z}^{-1} \cdot \mathbf{Y}$$

Collocation

Example:



x denotes collocation points

$$H_A(\omega^2) = \frac{a_0}{1+b_1\omega^2}$$

$$4 = \frac{a_0}{1+b_1\left(\frac{1}{2}\right)^2}$$

$$1 = \frac{a_0}{1+b_1\left(\frac{3}{2}\right)^2}$$

\Rightarrow

$$H_A(\omega^2) = \frac{32/5}{1+(12/5)\omega^2}$$

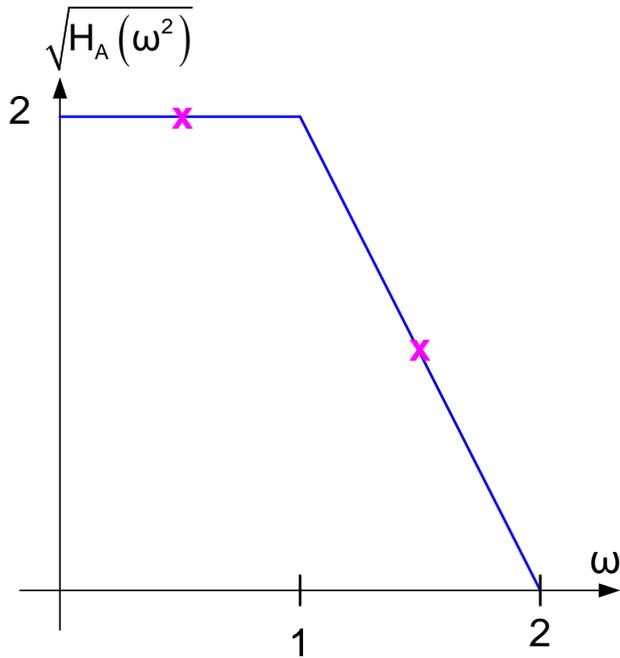
poles at

$$s = \pm j\sqrt{5/12}$$

Collocation

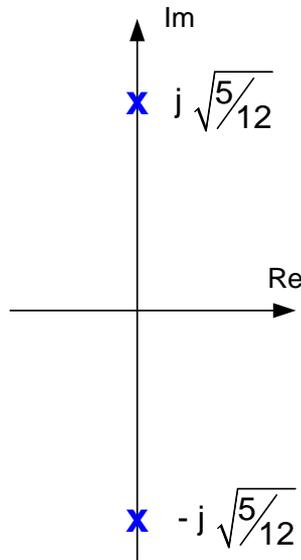
Example:

poles at
 $s = \pm j \sqrt{5/12}$

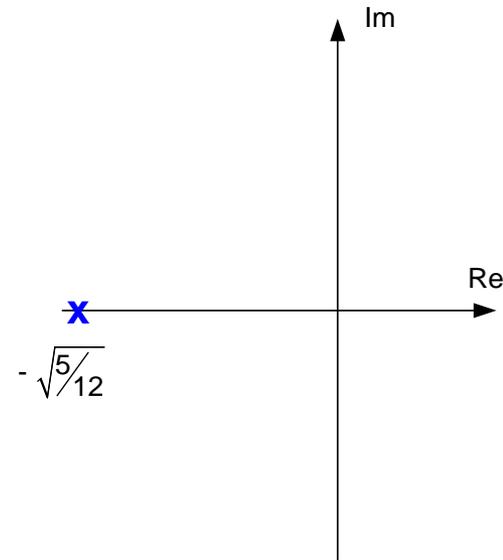


x denotes collocation points

Roots of $H_A(\omega^2)$



Roots of $T_{AM}(s)$



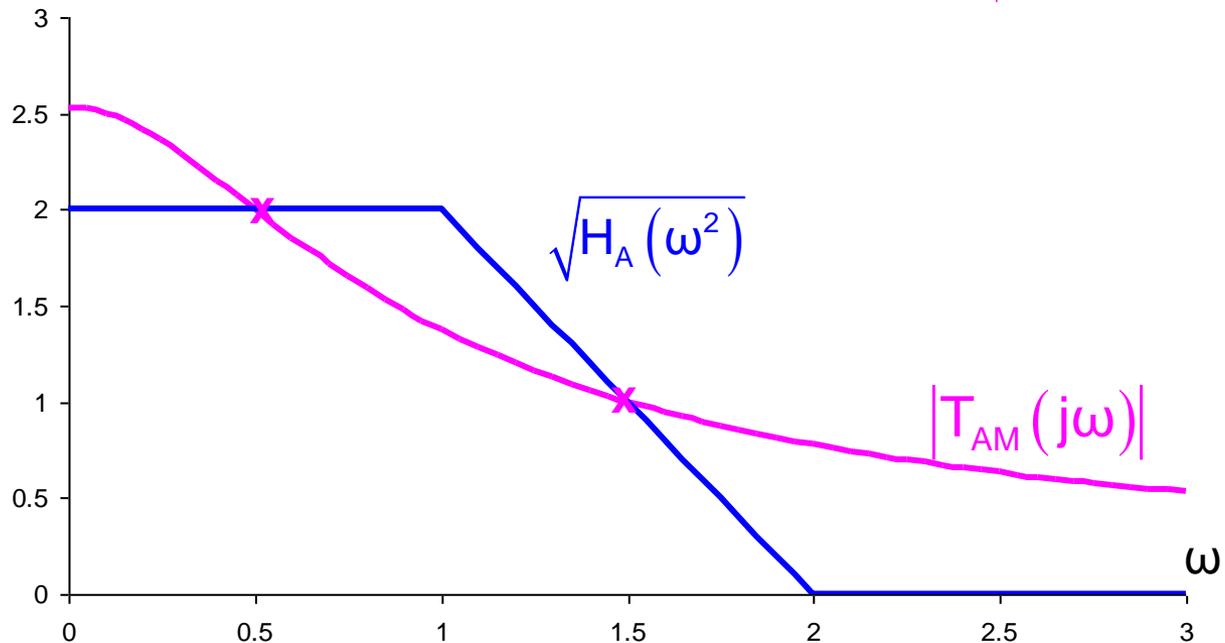
$$T_{AM}(s) = \frac{\sqrt{8/3}}{s + \sqrt{5/12}}$$

Collocation

Example:

$$H_A(\omega^2) = \frac{32/5}{1 + (12/5)\omega^2}$$

$$T_{AM}(s) = \frac{\sqrt{8/3}}{s + \sqrt{5/12}}$$



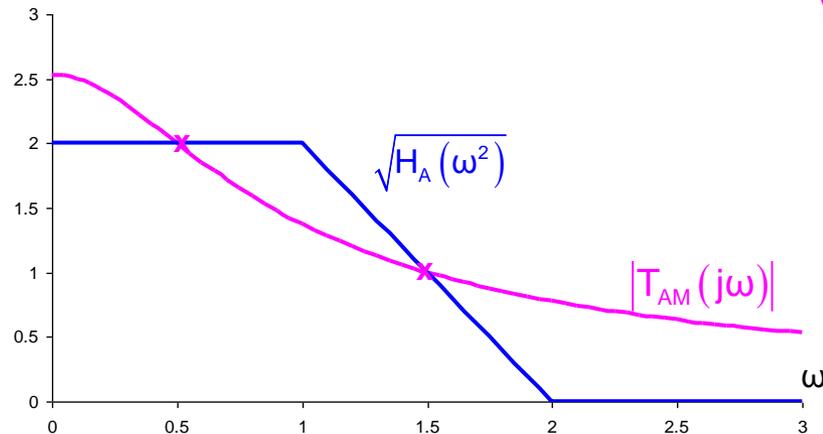
The approximation is reasonable but not too good

Collocation

Example:

$$H_A(\omega^2) = \frac{32/5}{1 + (12/5)\omega^2}$$

$$T_{AM}(s) = \frac{\sqrt{8/3}}{s + \sqrt{5/12}}$$

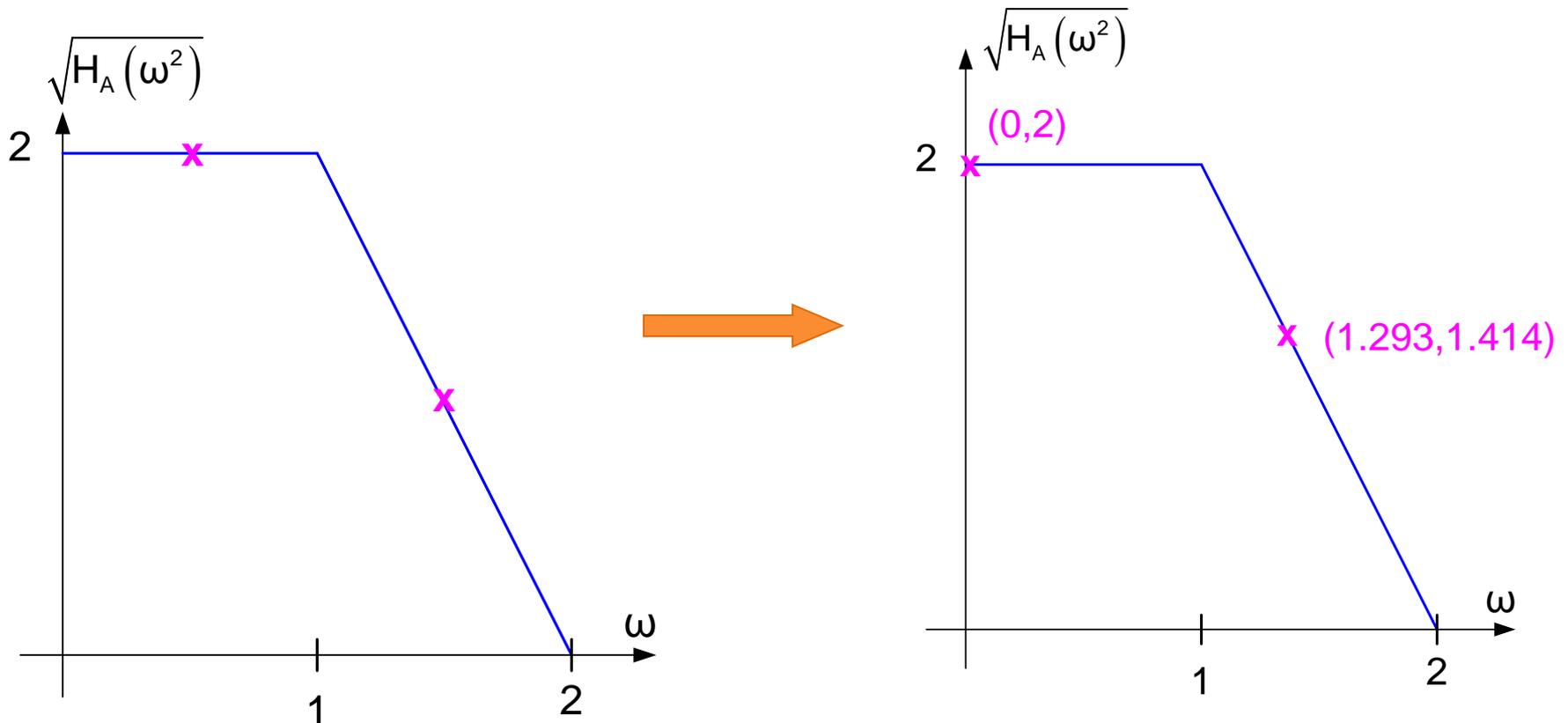


- The problem was critically constrained from a function viewpoint (two variables and two equations)
- Highly under-constrained as an approximation technique since the collocation points are also variables

Collocation

Example: same $H_A(\omega^2)$ but with different collocation points

$$H_A(\omega^2) = \frac{a_0}{1+b_1\omega^2}$$



Collocation

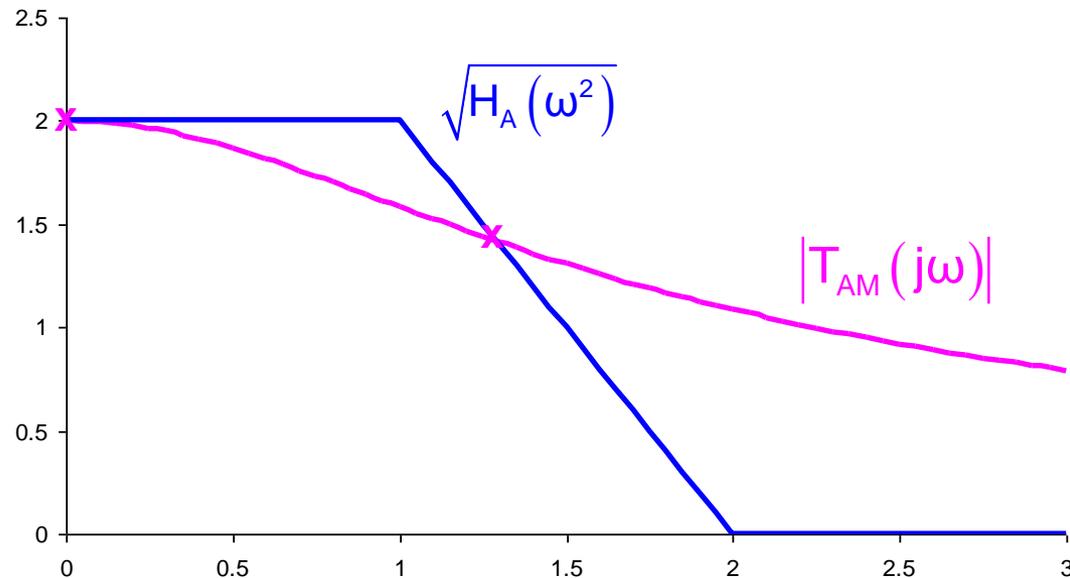
Example: same $H_A(\omega^2)$ but with different collocation points

$$H_A(\omega^2) = \frac{a_0}{1+b_1\omega^2}$$

$$\left. \begin{aligned} 4 &= \frac{a_0}{1+b_1(0)^2} \\ 2 &= \frac{a_0}{1+b_1(1.293)^2} \end{aligned} \right\}$$

$$\Rightarrow H_A(\omega^2) = \frac{4}{1+(.598)\omega^2}$$

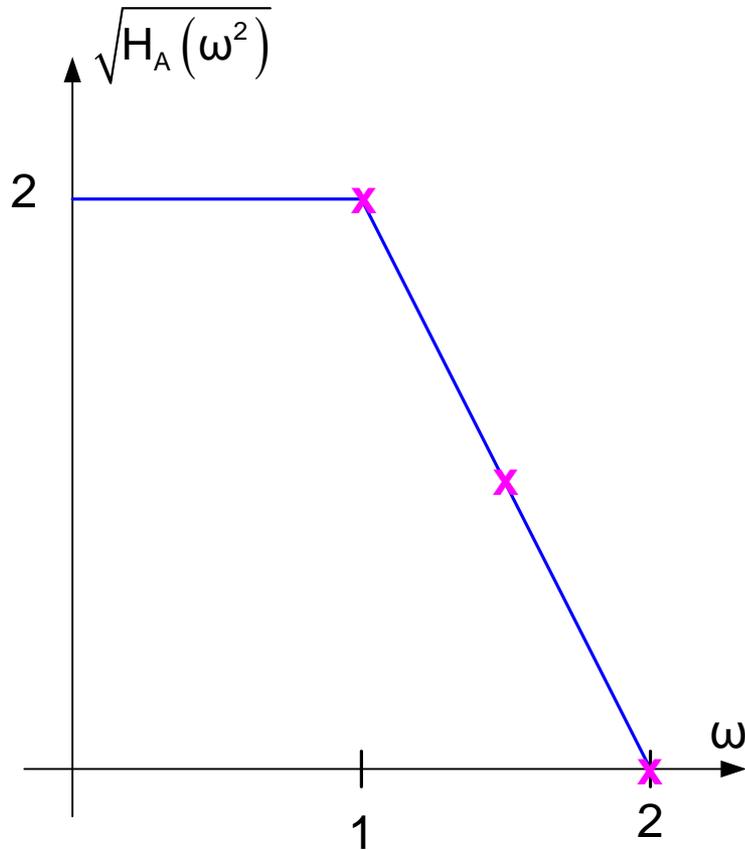
$$T_{AM}(s) = \frac{1.293}{s + 1.293}$$



Choice of collocation points plays a big role on the approximation

Collocation

Example: same $H_A(\omega^2)$ but with different collocation points and different approximating function



$$H_A(\omega^2) = \frac{a_0 + a_1\omega^2}{1 + b_1\omega^2}$$

$$\left. \begin{aligned} 4 &= \frac{a_0 + a_1}{1 + b_1} \\ 1 &= \frac{a_0 + a_1(3/2)^2}{1 + b_1(3/2)^2} \\ 0 &= \frac{a_0 + a_1(4)}{1 + b_1(4)} \end{aligned} \right\} \Rightarrow H_A(\omega^2) = \frac{-80 + 20\omega^2}{1 + -16\omega^2}$$

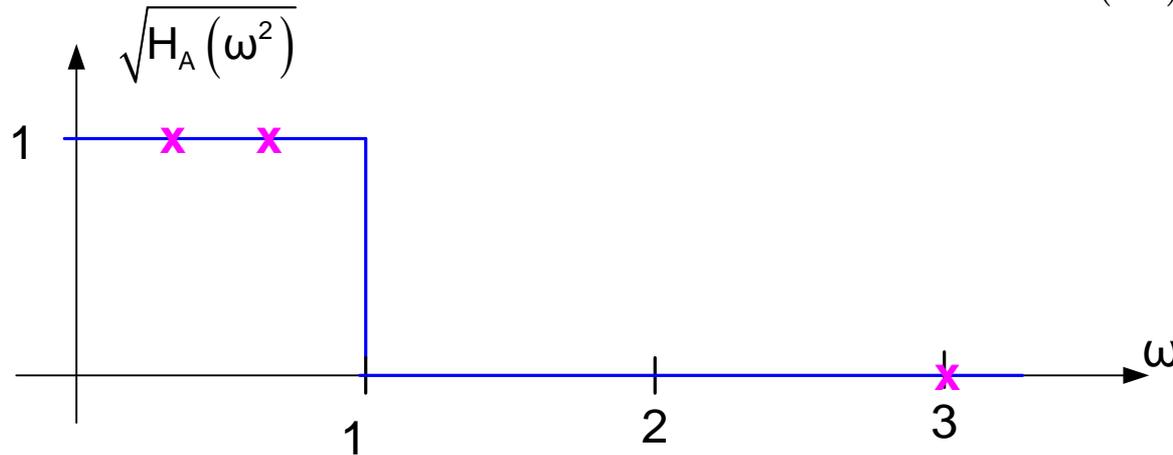
$$a_0 = -80, \quad a_1 = 20, \quad b_1 = -16$$

Inverse mapping does not exist because roots of odd multiplicity on real axis

Collocation

Example:

$$H_A(\omega^2) = \frac{a_0 + a_1\omega^2}{1 + b_1\omega^2}$$



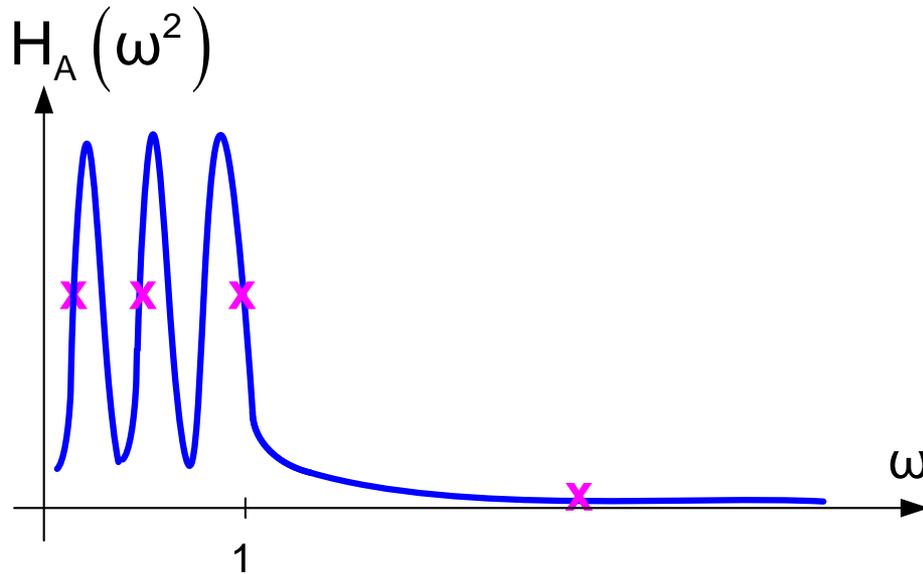
$$\left. \begin{aligned} 1 &= \frac{a_0 + a_1(1/9)}{1 + b_1(1/9)} \\ 1 &= \frac{a_0 + a_1(4/9)}{1 + b_1(4/9)} \\ 0 &= \frac{a_0 + a_1(9)}{1 + b_1(9)} \end{aligned} \right\} \Rightarrow H_A(\omega^2) = \frac{1 + (-27/243)\omega^2}{1 + (-27/243)\omega^2}$$

$a_0=1, a_1=-27/243, b_1=-27/243$

- This solution is equal to 1 at all frequencies except $\omega=3$ where it is undefined
- Thus there is no solution with these collocation points

Collocation

Example:



In some situations, collocation causes a lot of ripple between the collocation points

Collocation Observations

Fitting an approximating function to a set of data or points (collocation points)

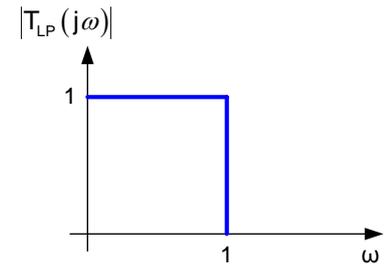
- Closed-form matrix solution for fitting to a rational fraction in ω^2
- Can be useful when somewhat nonstandard approximations are required
- Quite sensitive to collocation points
- Although function is critically constrained, since collocation points are variables, highly under constrained as an optimization approach
- Although fit will be perfect at collocation points, significant deviation can occur close to collocation points
- Inverse mapping to $T_A(s)$ may not exist
- Solution may not exist at specified collocation points

Collocation

What is the major contributor to the limitations observed with the collocation approach?

- Totally dependent upon the value of the desired response at a small but finite set of points (no consideration for anything else)
- Highly dependent upon value of approximating function at a single point or at a small number of points
- Highly dependent upon the collocation points

The Approximation Problem



Approach we will follow:

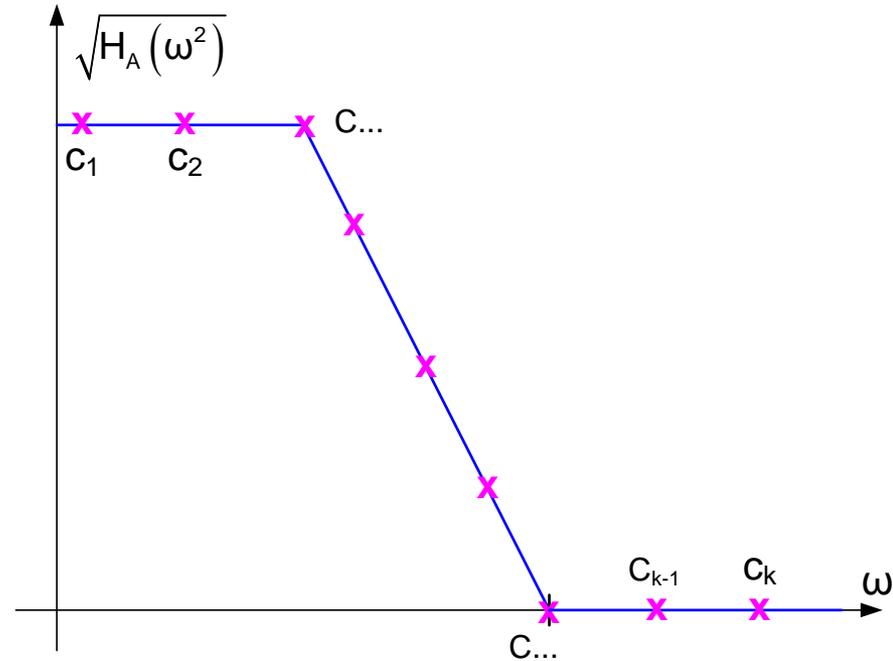
- Magnitude Squared Approximating Functions $H_A(\omega^2)$
- Inverse Transform $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation
-  Least Squares
- Pade Approximations
- Other Analytical Optimization
- Numerical Optimization
- Canonical Approximations
 - Butterworth (BW)
 - Chebyshev (CC)
 - Elliptic
 - Thompson

Least Squares Approximation

To minimize the heavy dependence on a small number of points, will consider many points thus creating an over-constrained system

$$H_A(\omega^2) = \frac{\sum_{i=0}^m a_i \omega^{2i}}{1 + \sum_{i=1}^n b_i \omega^{2i}}$$

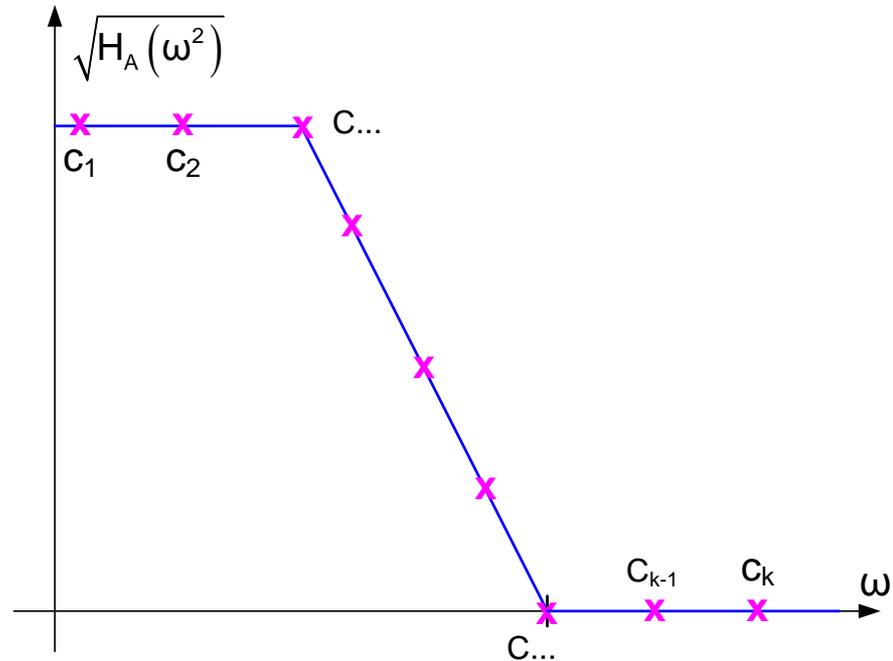
$$k > m+n+1$$



Approximating function can not be forced to go through all points
But, it can be “close” to all points in some sense

Least Squares Approximation

$$H_A(\omega^2) = \frac{\sum_{i=0}^m a_i \omega^{2i}}{1 + \sum_{i=1}^n b_i \omega^{2i}}$$



Define the error at point i by

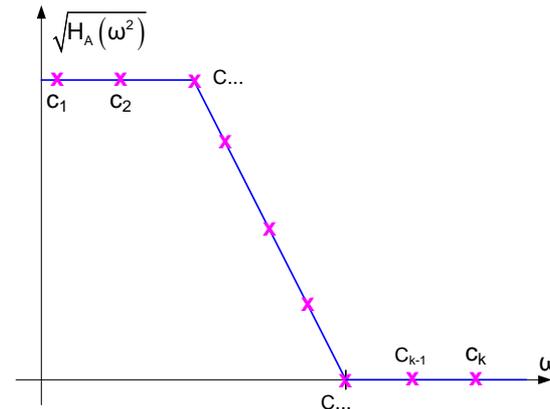
$$\varepsilon_i = H_D(\omega_i) - H_A(\omega_i)$$

where $H_D(\omega_i)$ is the desired magnitude squared response at ω_i and where $H_A(\omega_i)$ is the magnitude squared response of the approximating function

Least Squares Approximation

$$H_A(\omega^2) = \frac{\sum_{i=0}^m a_i \omega^{2i}}{1 + \sum_{i=1}^n b_i \omega^{2i}}$$

$$\varepsilon_i = H_D(\omega_i) - H_A(\omega_i)$$



Goal is to minimize some metrics associated with ε_i at a large number of points

Some possible cost functions

$$C_1 = \sum_{i=1}^N |\varepsilon_i|$$

$$C_2 = \sum_{i=1}^N \varepsilon_i^2$$

$$C_3 = \sum_{i=1}^N w_i \varepsilon_i^2$$

w_i a weighting function

- Reduces emphasis on individual points
- Some much better than others from performance viewpoint
- Some much better than others from computation viewpoint

Least Squares Approximation

$$H_A(\omega^2) = \frac{\sum_{i=0}^m a_i \omega^{2i}}{1 + \sum_{i=1}^n b_i \omega^{2i}}$$

$$\varepsilon_i = H_D(\omega_i) - H_A(\omega_i)$$

$$C_3 = \sum_{i=1}^N w_i \varepsilon_i^2$$

w_i a weighting function

Least Mean Square (LMS) based cost functions have minimums that can be analytically determined for some useful classes of approximating functions $H_A(\omega^2)$

Regression Analysis Review

Consider an n th order polynomial in x

$$F(x) = \sum_{k=0}^n a_k x^k$$

Consider N samples of a function $\tilde{F}(x)$

$$\hat{F}(x) = \left\langle \tilde{F}(x_i) \right\rangle_{i=1}^N$$

where the sampling coordinate variables are

$$X = \left\langle x_i \right\rangle_{i=1}^N$$

Define the summed square difference cost function as

$$C = \sum_{i=0}^N \left(F(x_i) - \tilde{F}(x_i) \right)^2$$

A standard regression analysis can be used to minimize C with respect to $\{a_0, a_1, \dots, a_n\}$

To do this, take the $n+1$ partials of C wrt the a_i variables

Regression Analysis Review

$$C = \sum_{i=0}^N \left(F(x_i) - \tilde{F}(x_i) \right)^2 \quad F(x) = \sum_{k=0}^n a_k x^k$$

$$C = \sum_{i=0}^N \left(\sum_{k=0}^n a_k x_i^k - \tilde{F}(x_i) \right)^2$$

Taking the partial of C wrt each coefficient and setting to 0, we obtain the set of equations

$$\left. \begin{aligned} \frac{\partial C}{\partial a_0} &= 2 \sum_{i=0}^N \left(\sum_{k=0}^n a_k x_i^k - \tilde{F}(x_i) \right) = 0 \\ \frac{\partial C}{\partial a_1} &= 2 \sum_{i=0}^N x_i^1 \left(\sum_{k=0}^n a_k x_i^k - \tilde{F}(x_i) \right) = 0 \\ \frac{\partial C}{\partial a_2} &= 2 \sum_{i=0}^N x_i^2 \left(\sum_{k=0}^n a_k x_i^k - \tilde{F}(x_i) \right) = 0 \\ &\dots \\ \frac{\partial C}{\partial a_n} &= 2 \sum_{i=0}^N x_i^n \left(\sum_{k=0}^n a_k x_i^k - \tilde{F}(x_i) \right) = 0 \end{aligned} \right\}$$

This is linear in the a_k s.

$$\mathbf{X} \bullet \mathbf{A} = \mathbf{F}$$

$$\mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ \dots \\ a_n \end{bmatrix}$$

Solution is

$$\mathbf{A} = \mathbf{X}^{-1} \bullet \mathbf{F}$$

Regression Analysis Review

A few details about regression analysis:

$$\mathbf{X} \bullet \mathbf{A} = \mathbf{F}$$

$$\mathbf{A} = \mathbf{X}^{-1} \bullet \mathbf{F}$$

$$\mathbf{X} = \begin{bmatrix} N+1 & \sum_{i=0}^N X_i & \sum_{i=0}^N X_i^2 & \dots & \sum_{i=0}^N X_i^n \\ \sum_{i=0}^N X_i & \sum_{i=0}^N X_i^2 & \dots & \dots & \sum_{i=0}^N X_i^{n+1} \\ \sum_{i=0}^N X_i^2 & \dots & \dots & \dots & \sum_{i=0}^N X_i^{n+2} \\ \dots & \dots & \dots & \dots & \dots \\ \sum_{i=0}^N X_i^n & \sum_{i=0}^N X_i^n & \dots & \dots & \sum_{i=0}^N X_i^{2n} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ \dots \\ a_n \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} \sum_{i=0}^N \tilde{F}(x_i) \\ \sum_{i=0}^N x_i \tilde{F}(x_i) \\ \sum_{i=0}^N x_i^2 \tilde{F}(x_i) \\ \dots \\ \sum_{i=0}^N x_i^n \tilde{F}(x_i) \end{bmatrix}$$

Regression Analysis Review

$$C = \sum_{i=0}^N \left(F(x_i) - \tilde{F}(x_i) \right)^2 \quad F(x) = \sum_{k=0}^n a_k x^k$$

$$C = \sum_{i=0}^N \left(\sum_{k=0}^n a_k x_i^k - \tilde{F}(x_i) \right)^2$$

$$\mathbf{A} = \mathbf{X}^{-1} \bullet \mathbf{F}$$

Observations about Regression Analysis:

- Closed form solution
- Requires inversion of a (n+1) dimensional square matrix
- Not highly sensitive to any single measurement
- Widely used for fitting a set of data to a polynomial model
- Points need not be uniformly distributed
- Adding weights does not complicate solution

End of Lecture 8