Graduate Tutorial Notes 2004

Theory of Electromagnetic Nondestructive Evaluation

Chapter 1. Magnetic Fields due to Electric Current

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Chapter 1

Magnetic Fields and Electric Current

1.1 Magnetic Field Due to a Current

1.1.1 Vector Path Product

The magnetic field **H** is a vector field in space determine by the sources of the magnetic field namely electric current and magnetic materials. We shall examine the relationship between magnetic and electrical current but first, let us dispose of the issue of units. In SI units, electric charge is measured in Coulombs (C). An electric current I is the rate of flow of charge in Coulombs per second which are called Ampère's (abbreviated as Amp.= C. s^{-1}).

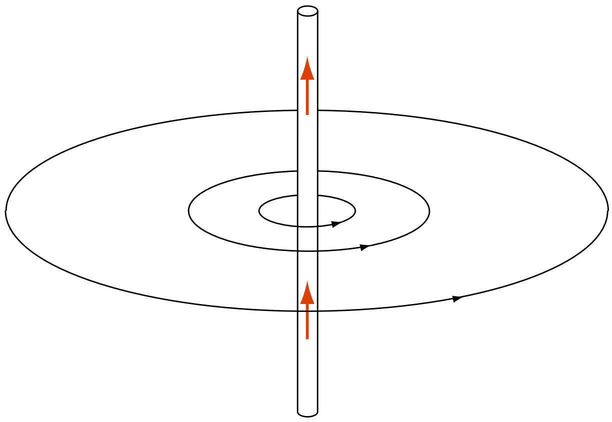

Figure 1.1: Magnetic field of a straight wire.

The magnetic field due to a straight current carrying wire is shown conventionally in Figure 1.1. The magnetic field here has only an azimuthal component H_ϕ whose strength is proportional to the current I and decreases with distance from the wire according to a relationship established by Ampère. Note that if a field line is followed round a circle of radius ρ, then the length of the closed path is $2\pi\rho$. Ampère established law which implies a simple rule:

> **The product of the path length round a circle coaxial with the wire and the magnetic field is a constant proportional to the current passing through the circle.**

Conveniently, in SI units the proportionality factor is one which means that $2\pi\rho H_\phi = I$. Hence the azimuthal field at a point a distance ρ from the wire is

$$H_\phi = \frac{I}{2\pi\rho}.$$

The current density $\mathbf{J}$, is a vector in the direction of the charge flow whose magnitude is the charge current per unit area (Amp m^{-2}). Thus with a wire of radius a aligned with the z-direction $\mathbf{J} = \hat{z}J_z = \hat{z}I/(\pi a^2)$.

Exercise 1: Given that Ampère's rule hold inside the wire what is the expression for the azimuthal magnetic field inside a wire of radius a?.

Magnetic field is defined in terms of charge current and not in terms of any other measurable physical properties of the field since magnetic observables (magnetic force, induced emf, the output of a magnetic field sensor etc.) all depend on the magnetic flux density vector $\mathbf{B}$ which is quite a different quantity. The magnetic field itself, if not directly observable. It is, in effect, a mathematical extension into the surrounding space of the electric current. Magnetic field does not reveal itself directly but is manifest through its relationship with magnetic flux.

1.1.2 Curl

Curl is a differential operator that is said to give the circulation of the field. The field in Figure 1.1 certainly seem to have some circulation. To quantifying it we use the expression for the curl operator in cylindrical polar co-ordinates.

$$\nabla \times \mathbf{V} = \left(\frac{1}{\rho}\frac{\partial V_z}{\partial \phi} - \frac{\partial V_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial \rho}\right)\hat{\phi} + \left(\frac{1}{\rho}\frac{\partial(\rho V_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial V_\rho}{\partial \phi}\right)\hat{z} \tag{1.1}$$

Exercise 2: Determine $\nabla \times \mathbf{H}$ for (a) the region outside the wire radius a in Figure 1.1 and (b) inside the wire expressing your answer in terms of the current density $\mathbf{J}$.

Exercise 3: A circulating fluid in a cylinder has a velocity $\mathbf{v} = \omega\rho$ where ω is a constant angular velocity. What is the curl of $\mathbf{v}$?

1.1.3 Ampère's Circuital Theorem

The magnetic field vector $\mathbf{H}$ in steady state conditions is determined by the charge current density $\mathbf{J}$. According to the differential form of Ampère's law,

$$\nabla \times \mathbf{H} = \mathbf{J}. \tag{1.2}$$

There are several ways of finding the magnetic field from the current. A basic starting point is to integrate the differential form of Ampère's law and use Stoke's theorem;

$$\int_{S_0} \nabla \times \mathbf{F}.\, d\mathbf{S} = \oint_{C_0} \mathbf{F}.\, d\mathbf{l} \tag{1.3}$$

which relates the integral of the curl of a vector over an open surface S_0 to the line integral around its closed boundary path C_0, $d\mathbf{l}$ being a line element of the path. Conventionally, the direction of the line integral is clockwise when viewed in the direction of the normal to the surface.

Integrating both sides of the quasi-static field equation, (1.2) over some chosen surface area S_0 gives

$$\int_{S_0} \nabla \times \mathbf{H}.\, d\mathbf{S} = \int_{S_0} \mathbf{J}.\, d\mathbf{S}.$$

Applying Stoke's theorem yields Ampère's circuital theorem:

$$\oint_{C_0} \mathbf{H}.\, d\mathbf{l} = I \tag{1.4}$$

where I, given by

$$I = \int_{S_0} \mathbf{J} \cdot d\mathbf{S}, \tag{1.5}$$

is the current through the surface S_0.

The circuital theorem does not provide a general method for finding the magnetic field since one needs prior knowledge of the field in order to evaluate the line integral. However, in special cases one can use symmetry arguments.

Magnetic Field due to a Line Current

Consider a filamentary line current orientated in the z-direction and carrying a current I, as the sole source of the magnetic field. The axial symmetry of the problem means that the azimuthal magnetic field due to the current is constant on a circle of radius ρ centered at the wire. With the knowledge that H_ϕ is constant on a circular path, the line integral in (1.4) becomes the integral of a constant, $\mathbf{H} \cdot \hat{\phi}$ over a path of length $2\pi\rho$. Hence the line integral is $2\pi\rho H_\phi$ nd the surface integral is I. Hence the circuital theorem gives

$$\mathbf{H} = \frac{I}{2\pi\rho}\hat{\phi} \tag{1.6}$$

for the magnetic field.

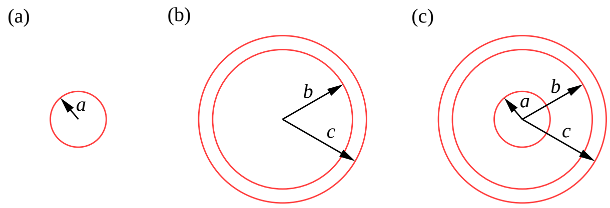

Figure 1.2: Cross-section of (a) a wire radius a (b) a cylindrical conducting screen (c) a coaxial cable consisting of wire core and screen.

Exercise 4: Coaxial cable (a) A wire radius a carries a current I in the positive z-direction sketch a graph showing the magnitude of the azimuthal magnetic field as a function of ρ showing the value of H_ϕ at the surface of the conductor. (b) Determine the magnetic field due to a screen carrying a current in the negative z-direction. (c) Combine the effects of the core and screen and sketch a graph of the field due to the current in the coaxial conductors Figure 1.2 as a function of ρ and give expressions for the magnetic field in each region.

Magnetic Field due to a Current Sheet

A current sheet of negligible thickness in the plane $y = 0$ carries current I per unit area in the z-direction. Applying the circuital theorem to a closed rectangular path bounding an area with dimension $a_x a_y$, in an arbitrary plane $y = y_0$, the line integral is $2a_x Hx$ and the surface integral $a_x I$. Hence the field is given by

$$H_x = \mp I/2 \tag{1.7}$$

where the plus sign applies below the current sheet and the minus sign above the sheet.

Magnetic Field due to a Long Uniformly Wound Solenoid

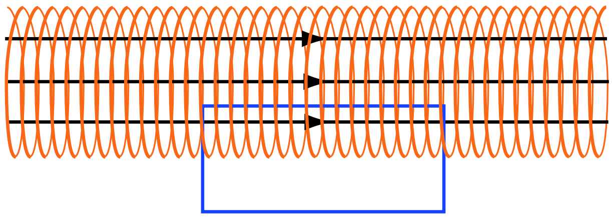

Figure 1.3: Magnetic field in a long solenoid with a closed rectangular path suitable for application of the circuital law.

A uniformly and tightly wound solenoid with n turns per unit length carrying a current I provides a uniform azimuthal current of current density nI per unit length. The longer it is, the weaker is the external field. In the limit of an infinitely long solenoid, the external field is zero and the internal field is parallel to the axis. Applying the circuital law to a rectangular surface $a_z \times a_\rho$ in an azimuthal plane shows that the internal field is uniform and given by

$$H_z = nI. \tag{1.8}$$

1.2 Magnetic Vector Potential

The general problem of finding the field **H** due to a current in free space by introducing a magnetic vector potential conventionally defined such that

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{1.9}$$

which means that for a nonmagnetic region ($\mathbf{B} = \mu_0 \mathbf{H}$),

$$\mathbf{H} = \frac{1}{\mu_0} \nabla \times \mathbf{A},$$

where μ_0 is the permeability of free space. Substituting this result back into (1.2) we find

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}$$

whence, upon use of the identity $\nabla \times \nabla \times = \nabla\nabla \cdot -\nabla^2$, and adopting the gauge condition $\nabla \cdot \mathbf{A} = 0$ (Coulomb gauge), we have

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{1.10}$$

which is a vector Poisson equation. Note that **A** and **J** are antiparallel.

1.2.1 Line Source

The magnetic vector potential due to a line current source can be found by applying the divergence theorem. This theorem can be stated as

$$\int_{\Omega_0} \nabla.\mathbf{F}(\mathbf{r})\, d\mathbf{r} = \oint_{S_0} \mathbf{F}(\mathbf{r})\, .\, d\mathbf{S} \tag{1.11}$$

where the arbitrary closed surface S_0 encloses the volume Ω_0. The theorem will be applied in two dimensions. First write

$$\nabla = \nabla_t + \frac{\partial}{\partial z} \tag{1.12}$$

where ∇_t is the tangential gradient. Then the 2D form of (1.11) is

$$\int_{S_0} \nabla_t.\mathbf{F}(\mathbf{r})\, dS = \oint_{C_0} \mathbf{F}(\mathbf{r})\, .\hat{n}\, dl \tag{1.13}$$

where C_0 is a closed path and $\hat{n}$ is unit outward vector normal to the path.

To apply (1.13), note first that (1.10) can be written as

$$\nabla.\nabla\mathbf{A} = -\mu_0\mathbf{J}. \tag{1.14}$$

but since there is unformity in the z-direction,

$$\nabla_t.\nabla_t\mathbf{A} = -\mu_0\mathbf{J}. \tag{1.15}$$

Integrating this equation over a circle radius ρ, in a $z = z_0$ plane, where z_0 is a constant, with the source passing through its center and applying the 2D divergence theorem gives

$$2\pi\rho\frac{\partial A}{\partial \rho} = -\mu_0 I \tag{1.16}$$

where $A = \hat{z} \cdot \mathbf{A}$. Hence

$$A = -\frac{\mu_0 I}{2\pi} \ln \rho. \tag{1.17}$$

Taking the curl of $\mathbf{A} = \hat{z}A$ in cylindrical coordinates gives (1.6).

1.2.2 Green's Function for Two-Dimensional Poisson Problems

Note that the argument used above can be adapted to find the Green's function for a two-dimensional Poisson problem. Thus we see that the unbounded domain solution of

$$\nabla^2 g = -\delta(\boldsymbol{\rho} - \boldsymbol{\rho}') \tag{1.18}$$

is

$$g = -\frac{1}{2\pi} \ln \rho \tag{1.19}$$

where $\rho = |\boldsymbol{\rho} - \boldsymbol{\rho}'| = \sqrt{(x - x')^2 + (y - y')^2}$.

1.2.3 Unidirectional Current

A current flowing in the z-direction and therefore of the form $\mathbf{J}(x, y) = \hat{z}J(x, y)$ gives rise to a vector potential $\mathbf{A} = \hat{z}A$ which is a solution of

$$\nabla^2 A = -\mu_0 J \tag{1.20}$$

This is a scalar partial differential equation, a Poisson equation, in two variables. The common methods for solving this and other partial differential equations are to use the Green's function method or transformation techniques.

Green's function method

In Green's function method, the solution is expressed in the form

$$A(\boldsymbol{\rho}) = \mu_0 \int_S g(\boldsymbol{\rho} - \boldsymbol{\rho}')J(\boldsymbol{\rho}')\, d\boldsymbol{\rho}' \tag{1.21}$$

where the Greens function is given by (1.19) and the current density $J(\boldsymbol{\rho}')$ varies in an arbitrary manner over the area S. Is is assumed that there are no nearby boundaries and that the field vanishes as $|\boldsymbol{\rho}| \to \infty$.

Fourier Transform Method

The partial differential equation can often be reduced to a differential equation in only one variable by suitable transformation. The Fourier transform is defined, together with its inverse, by

$$\tilde{f}(u) = \int_{-\infty}^{\infty} f(x)e^{jux}\, dx \tag{1.22}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(u)e^{-jux}\, du \tag{1.23}$$

Transforming (1.20) in this way gives

$$\left(\frac{\partial^2}{\partial y^2} - u^2\right)\tilde{A} = -\mu_0 \tilde{J} \tag{1.24}$$

Thus reducing the derivative with respect to x to an algebraic form.

1.2.4 Current Strip

For the case of a uniform current in a strip, width $2a$, carrying current in the z-direction, the current density is expressed as

$$J = \begin{cases} \delta(y)I & \text{if } -a \le x \le a \\ 0 & \text{otherwise} \end{cases} \tag{1.25}$$

where I is the current per unit width of the strip.

Solution Using the Green's function method

Finding the solution, using (1.21), is made easier by differentiating A with respect to y to get H_x and x to get $-H_y$ prior to integration. Thus following the trivial y' integration

$$H_x = \frac{1}{2\pi}\int_{-a}^{a} \frac{y}{(x-x')^2+y^2}\,dx' \qquad \text{and} \qquad H_y = -\frac{1}{2\pi}\int_{-a}^{a} \frac{x-x'}{(x-x')^2+y^2}\,dx' \tag{1.26}$$

Let $(x'-x)/y = w$ to give

$$H_x = \frac{1}{2\pi}\int_{-(a+x)/y}^{(a-x)/y} \frac{1}{w^2+1}\,dw \qquad \text{and} \qquad H_y = -\frac{1}{2\pi}\int_{-(a+x)/y}^{(a-x)/y} \frac{w}{w^2+1}\,dw \tag{1.27}$$

From which we find [1]

$$H_x = \mp\frac{I_0}{2\pi}\left[\arctan\frac{2ay}{x^2+y^2-a^2}+s\pi\right], \qquad \text{and} \qquad H_y = \frac{I_0}{4\pi}\ln\left[\frac{(a-x)^2+y^2}{(a+x)^2+y^2}\right]. \tag{1.28}$$

where $s=1$ if $x^2+y^2-a^2<0$ and zero otherwise. The upper sign in the expression for H_x refers to $y>0$ and the lower sign to $y<0$

Solution by Fourier Transform Method

A solution of (1.20) is found using the Fourier transform. For a current strip defined with a current density $\delta(y)I(x)$, the Fourier transform of the current density is

$$\tilde{J}(u) = \delta(y)\int_{-a}^{a} I(x)e^{jux}\,dx. \tag{1.29}$$

For a conducting strip carrying a uniform current, $I(x)$ is constant for $-a \le x < a$ but, in this case, the Fourier integral form for $\tilde{A}$ does not converge. Therefore we seek instead the derivative of $\tilde{A}$ with respect to y which is in fact the x-component of the magnetic field. At points not at the current source,

$$\left(\frac{\partial^2}{\partial y^2}-u^2\right)\frac{\partial\tilde{A}}{\partial y} = 0. \tag{1.30}$$

The required solution must vanish as $|y|\to\infty$. Therefore a solution that varies as e^{-uy} will behave correctly if u and y are either both positive or both negative. However, it is necessary to consider also what happens when u and y are of opposite sign and in addition ensure that the solution has odd parity with respect to y. The solution with the correct far-field behavior and parity has the form

$$\frac{\partial\tilde{A}}{\partial y} = \begin{cases} F(u)e^{-|u|y} & \text{if } y>0 \\ -F(u)e^{|u|y} & \text{if } y<0. \end{cases} \tag{1.31}$$

It is known from the circuital theorem that there is a discontinuity in the magnetic field at a current sheet or strip that is equal to the current per unit length. Denote the field just above the strip as H_x^+ and that below as H_x^-. Taking the negative z-direction as the direction of the surface normal in Stoke's theorem, the path of integral following a clockwise path round the strip is in the positive x-direction above the strip and the negative x-direction below it. Hence

$$H_x^+ - H_x^- = \begin{cases} -I_0 & \text{if } -a \le x \le a \\ 0 & \text{otherwise} \end{cases} \tag{1.32}$$

To satisfy this condition in the Fourier domain,

$$2F(u) = -I_0 \int_{-a}^{a} e^{jux}\, dx = -\frac{2I_0}{u} \sin(au) \tag{1.33}$$

Relation (1.32) determines $F(u)$, and, therefore

$$\frac{\partial A}{\partial y} = \mp \frac{I_0}{2\pi} \int_{-\infty}^{\infty} \frac{1}{u} \sin(au) e^{-jux-|u||y|}\, du, \tag{1.34}$$

where the upper sign refers to $y > 0$ and the lower sign to $y < 0$. Hence

$$H_x = \frac{\partial A}{\partial y} = \mp \frac{I_0}{\pi} \int_{0}^{\infty} \frac{1}{u} \sin(au) \cos(xu) e^{-u|y|}\, du. \tag{1.35}$$

In addition, it is found that

$$H_y = -\frac{\partial A}{\partial x} = \frac{I_0}{\pi} \int_{0}^{\infty} \frac{1}{u} \sin(au) \sin(xu) e^{-u|y|}\, du. \tag{1.36}$$

From Gradshteyn and Ryzhik [1], integrals 3.947 and 3.948, it is found that

$$H_x = \mp \frac{I_0}{2\pi} \left[\arctan \frac{2ay}{x^2 + y^2 - a^2} + s\pi \right], \tag{1.37}$$

where $s = 1$ if $x^2 + y^2 - a^2 < 0$ and zero otherwise. Also

$$H_y = \frac{I_0}{4\pi} \ln \left[\frac{(a-x)^2 + y^2}{(a+x)^2 + y^2} \right]. \tag{1.38}$$

1.3 Field due to a Circular Current Filament

The magnetic vector potential satisfies

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{1.39}$$

For a circular filament radius ρ_0 in the plane $z = z_0$

$$\mathbf{J} = I\delta(\rho - \rho_0)\delta(z - z_0)\hat{\phi}. \tag{1.40}$$

The solution will be found using a combination of Fourier transform and Green's function methods. Define the the 2D Fourier transform of $\tilde{A}$ as

$$\tilde{A}(u, v, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(\mathbf{r}) e^{-iux-ivy}\, dx\, dy, \tag{1.41}$$

its inverse as

$$A(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(u, v, z) e^{iux+ivy}\, du\, dv \tag{1.42}$$

For a circular filament the magnetic vector potential can be written $\mathbf{A} = A\hat{\phi}$ and for a current loop in the plane $z = z_0$, we can write $\mathbf{J} = J\hat{\phi}\delta(z - z_0)$ where $J = I\delta(\rho - \rho_0)$. Taking the 2D Fourier transform of (1.20) gives

$$\left(\frac{\partial^2}{\partial z^2} - \kappa^2 \right) \tilde{A}(u, v, z) = -\mu_0 \tilde{J}(u, v)\delta(z - z_0). \tag{1.43}$$

where $\kappa^2 = u^2 + v^2$. The solution will be written as

$$\tilde{A} = \mu_0 \tilde{J} g(z - z_0) \tag{1.44}$$

where $g(z - z')$ is a one dimensional Green's function satisfying

$$\left(\frac{\partial^2}{\partial z^2} - \kappa^2 \right) g(z - z') = -\delta(z - z'). \tag{1.45}$$

The solution is thus found through the following steps:

- determine the 1D Greens function $g(z - z')$
- find the 2D Fourier transform of the current $\tilde{J}$
- evaluate the inverse Fourier transform of $\tilde{A} = \mu_0 \tilde{J} g(z - z_0)$.

Although the result could be computed numerically using a fast Fourier transform algorithm, the inverse transform, in this case, can be performed analytically.

1.3.1 Evaluation of the Green's Function

The solution of (1.45) can be written in terms of exponentials $e^{\pm\kappa(z-z')}$. One can ensure the solution vanishes as $|z - z'| \to \infty$ by writing

$$g(z - z') = F(\kappa) e^{-\kappa|z-z'|} \tag{1.46}$$

then the unknown function $F(\kappa)$ is found from the jump in the derivative of g as given by integrating (1.45) between $z' - \epsilon$ and $z' + \epsilon$ and letting ϵ tend to zero to give

$$\left[\frac{\partial g}{\partial z} \right] = \left. \frac{\partial g}{\partial z} \right|_+ - \left. \frac{\partial g}{\partial z} \right|_- = -1 \tag{1.47}$$

where the $\pm$ sign refer to the approach to z' from the positive and negative side. From this condition we find that $F(\kappa) = 1/2\kappa$ and hence

$$g(z - z') = \frac{1}{2\kappa} e^{-\kappa|z-z'|} \tag{1.48}$$

1.3.2 Evaluation of the Fourier Transform of the Current

The x- and $y-$components of $J\hat{\phi}$ are

$$J_x = -I\delta(\rho - \rho_0)\sin\phi \qquad \text{and} \qquad J_y = I\delta(\rho - \rho_0)\cos\phi \tag{1.49}$$

Take the 2D Fourier transform

$$\begin{bmatrix} \tilde{J}_x \\ \tilde{J}_y \end{bmatrix} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \begin{bmatrix} J_x(\mathbf{r}) \\ J_y(\mathbf{r}) \end{bmatrix} e^{-iux-ivy}\, dx\, dy \tag{1.50}$$

Change to cylindrical co-ordinates by using the substitutions

$$\begin{array}{ll} x = \rho\cos\phi & y = \rho\sin\phi \\ u = \kappa\cos\beta & v = \kappa\sin\beta \end{array}$$

then

$$ux + vy = \kappa\rho\cos(\phi - \beta) \qquad \text{and} \qquad dx\ dy = \rho\ d\rho\ d\phi \tag{1.51}$$

hence

$$\begin{bmatrix} \tilde{J}_x \\ \tilde{J}_y \end{bmatrix} = \int_0^\infty \int_0^{2\pi} \begin{bmatrix} J_x(\mathbf{r}) \\ J_y(\mathbf{r}) \end{bmatrix} e^{-i\kappa\rho\cos(\phi-\beta)}\, \rho\; d\rho\; d\phi \tag{1.52}$$

By making the substitution $\phi = \theta + \beta$, and using

$$J_1(z) = \frac{1}{i\pi} \int_0^\pi e^{-iz\cos\theta} \cos\theta\, d\theta \tag{1.53}$$

we find that

$$\begin{bmatrix} \tilde{J}_x \\ \tilde{J}_y \end{bmatrix} = 2\pi i \rho_0 I \begin{bmatrix} -\frac{v}{\kappa} \\ \frac{u}{\kappa} \end{bmatrix} J_1(\kappa\rho_0) \tag{1.54}$$

Hence

$$\tilde{J} = 2\pi i \rho_0 I J_1(\kappa\rho_0) \tag{1.55}$$

1.3.3 Solution by Inverse Fourier Transform

Consider the inverse Fourier transform giving the x-component of A :

$$\begin{bmatrix} A_x \\ A_y \end{bmatrix} = \mu_0 I \rho_0 \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{2\kappa} e^{-\kappa|z|} \begin{bmatrix} -\frac{iv}{\kappa} \\ \frac{iu}{\kappa} \end{bmatrix} J_1(\kappa\rho_0) e^{iux+ivy}\; du\; dv \tag{1.56}$$

Transforming to cylindrical co-ordinates gives

$$\begin{aligned} \begin{bmatrix} A_x \\ A_y \end{bmatrix} &= \mu_0 I \rho_0 \int_0^\infty \int_0^\pi J_1(\kappa\rho_0) e^{-\kappa|z-z_0|} e^{i\kappa\rho\cos(\phi-\beta)} \begin{bmatrix} -i\sin\beta \\ i\cos\beta \end{bmatrix} d\kappa\; d\beta \\ &= \pi\mu_0 I \begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix} \rho_0 \int_0^\infty J_1(\kappa\rho_0) J_1(\kappa\rho) e^{-\kappa|z-z_0|}\; d\kappa \end{aligned} \tag{1.57}$$

Hence

$$A = \pi\mu_0 I \rho_0 \int_0^\infty J_1(\kappa\rho_0) J_1(\kappa\rho) e^{-\kappa|z-z_0|}\; d\kappa \tag{1.58}$$

Exercise 5: Evaluate (1.58) using 6.162 of Gradshteyn and Ryzhik [1]. Write the solution in terms of elliptical functions using 8.3.27 of Abramowitz and Stegun [2] and check the results with the text by Van Bladel [27].

Exercise 6: Write the filament solution (1.58) as $\mu_0 I G(\rho, z|\rho_0, z_0)$ and the vector potential of a coil as

$$A(\rho, z) = \mu_0 \int_{S_c} G(\rho, z|\rho_0, z_0) J(\rho_0, z_0)\, dS_0 \tag{1.59}$$

where S_c is the cross-sectional area of the coil and

$$G(\rho, z|\rho_0, z_0) = \pi\rho_0 \int_0^\infty J_1(\kappa\rho_0) J_1(\kappa\rho) e^{-\kappa|z-z_0|}\; d\kappa. \tag{1.60}$$

Determine the vector potential for a tightly wound coil carry a current I whose turns density n is unform over a rectangular cross section such that $z_2 < z_0 < z_1$ and $a_2 < \rho_0 < a_1$.

The integral for the coil magnetic vector potential is

$$A(\rho, z) = \mu_0 n I \int_{z_2}^{z_1} \int_{a_2}^{a_1} \pi\rho_0 \int_0^\infty J_1(\kappa\rho_0) J_1(\kappa\rho) e^{-\kappa|z-z_0|} \; d\kappa\, d\rho_0\, dz_0 \tag{1.61}$$

The radial integral will be expressed in terms of a function

$$\mathcal{J}_1(r_1, r_2) = \int_{r_2}^{r_1} r J_1(r)\, dr \tag{1.62}$$

and the $z-$integral gives

$$f(\kappa, z) = \int_{z_2}^{z_1} e^{-\kappa|z-z_0|}\, dz = \begin{cases} \frac{1}{\kappa}[e^{\kappa z_1} - e^{\kappa z_2}]e^{-\kappa z} & z > z_1 \\ \frac{1}{\kappa}[2 - e^{-\kappa(z-z_2)} - e^{-\kappa(z_1-z)}] & z_1 > z > z_2 \\ \frac{1}{\kappa}[e^{-\kappa z_1} - e^{-\kappa z_2}]e^{\kappa z} & z_2 > z \end{cases} \tag{1.63}$$

Thus

$$A(\rho, z) = \mu_0 \pi n I \int_0^\infty \frac{1}{\kappa^2} f(\kappa, z) \mathcal{J}_1(\kappa a_1, \kappa a_2) J_1(\kappa\rho)\, d\kappa \tag{1.64}$$

Exercise 7: Coil Impedance
The impedance of a coil is given by

$$I^2 Z = -\int_\Omega \mathbf{E}(\mathbf{r}) \cdot \mathbf{J}(\mathbf{r})\, d\mathbf{r}. \tag{1.65}$$

Determine the free space impedance of a coil of rectangular cross-section.

Exercise 8: Magnetic Shell Representation
(a) A coil current source $\mathbf{J}$ can be replaced by a magnetic dipole distribution $\mathbf{M}$ which gives the same field. Determine $\mathbf{M}$ in the form of a magnetic shell in the plane of a circular current filament radius ρ_0 carrying a current I such that the current density is expressed as

$$\mathbf{J} = \nabla \times \mathbf{M} \tag{1.66}$$

(b) Write the vector potential for the problems as $\mu_0 \nabla \times (\hat{z}\psi)$ and determine the transverse electric scalar potential ψ.

1.3.4 Integral Solution

Assuming $\mathbf{A}(\mathbf{r})$ vanishes as $|\mathbf{r}| \to \infty$, the free-space (unbounded domain) solution of equation (1.10) may be written as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\Omega_J} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \tag{1.67}$$

where the integration is carried out over the electric current source region. Here we have simply stated the solution in integral form with a Green's function kernel, $\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$. Next we shall elaborate in some detail on how this solution is derived.

1.3.5 Static Green's function

To find a solution of (1.10) in free space which vanishes as $|\mathbf{r}| \to \infty$, note that there is no coupling between components. Hence each independent component of the equation can be treated separately as the solution of a scalar problem. The scalar problem can be solved using the solution for a point source in three-dimensional space and representing the solution for a distributed source as a superposition of point sources.

A point source with a coordinate $\mathbf{r}'$ is represented by a delta distribution; $\delta(\mathbf{r} - \mathbf{r}')$. The position of the field point is given by the vector $\mathbf{r}$. The delta distribution is zero for all field points (for all $\mathbf{r}$) except where $\mathbf{r} = \mathbf{r}'$. When the source and field points coincide, the delta distribution does not have a finite numerical value (which is why it is called a distribution and not a function), but it is given the following defining property

$$U(\mathbf{r}) = \int_{\Omega} U(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')\, d\mathbf{r}', \tag{1.68}$$

where Ω is a region containing the point $\mathbf{r}'$ and $U(\mathbf{r})$ is a function of position in space. Note that if we choose for example $U(\mathbf{r}) = 1$ everywhere, it can be seen that

$$\int_{\Omega} \delta(\mathbf{r} - \mathbf{r}')\, d\mathbf{r}' = 1 \tag{1.69}$$

Thus the delta distribution in three-dimensional space is zero almost everywhere but integration over it gives the value 1.

Now seek a function $G(\mathbf{r}, \mathbf{r}')$ which vanishes as $|\mathbf{r}| \to \infty$ and satisfies

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'). \tag{1.70}$$

A solution is deduced using the divergence theorem (1.11). First write $\nabla^2 G(\mathbf{r}, \mathbf{r}') = \nabla.\nabla G(\mathbf{r}, \mathbf{r}')$ and then apply the divergence theorem to equation (1.70) for a spherical region centered at $\mathbf{r}'$. This gives

$$\int_{S_0} \nabla G(\mathbf{r}, \mathbf{r}') \cdot d\mathbf{S} = -1 \tag{1.71}$$

Spherical symmetry means that $\nabla G(\mathbf{r}, \mathbf{r}')$ is radially directed, a function of $R = |\mathbf{r} - \mathbf{r}'|$ and therefore constant over a surface of fixed radius. Hence

$$\nabla G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi R^2}\hat{R} \tag{1.72}$$

where $\hat{R}$ is a unit vector in the radial direction. The result

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \tag{1.73}$$

can be confirmed by differentiation to get (1.72). Having established that (1.73) is the required solution of (1.70), note the similarity of form between (1.70) and (1.10). Because of this similarity, any vector component of the magnetic vector potential can be written as a superposition of scalar point source solutions. The full vector potential is a superposition of these vector component solutions as expressed in equation (1.67).

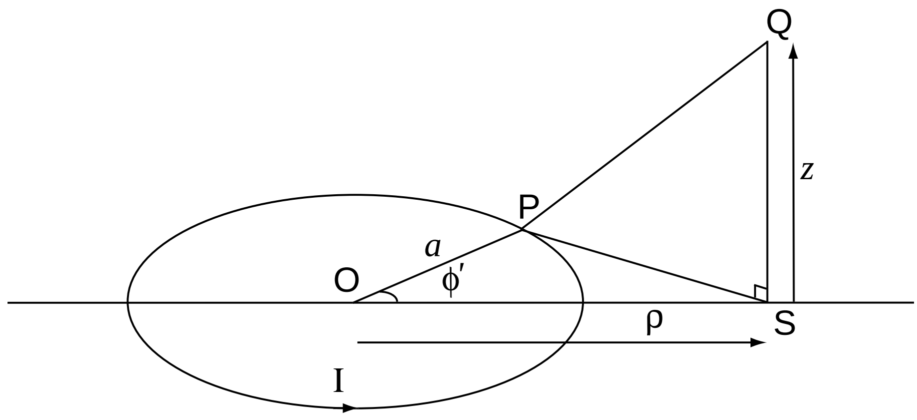

Figure 1.4: Circular current filament

1.4 Magnetic Field Due to a Circular Current Filament

Figure 1.4 shows a circular filament carrying a current I. This is a circular coil with one turn. The magnetic field due to this loop can be deduced from (1.67) and a knowledge of elliptical integrals. In order to evaluate (1.67) four things are needed: (i) an expression for the elemental volume $d\mathbf{r}'$ in cylindrical coordinates, (ii) an expression for the current density, (iii) a convenient expression for the distance $R = |\mathbf{r} - \mathbf{r}'|$ and (iv) a relationship between the direction of the source and the direction of the field.

1.4.1 Volume Element

In cylindrical coordinates,

$$d\mathbf{r}' = \rho' d\phi' \, d\rho' \, dz'. \tag{1.74}$$

1.4.2 Current Density

The current density for the filament will be represented using the delta distribution for a single scalar variable z, say. Thus $\delta(z - z')$ is defined such that

$$f(z) = \int_{z_a}^{z_b} f(z')\delta(z - z') \, dz'$$

where z' is between z_a and z_b. Using this delta distribution the current density for the loop with radius a in the plane $z' = 0$ is written

$$\mathbf{J}(\mathbf{r}') = I\delta(\rho' - a)\delta(z')\hat{\phi}' \tag{1.75}$$

where $\hat{\phi}'$ is the azimuthal unit vector.

1.4.3 Distance from source point to field point

Note from Figure 1.4 that OP has length a and that from the cosine rule, the square of the length os PS is $a^2 + \rho^2 - 2a\rho\cos\phi'$. Consequently, if R is the length of PQ,

$$R^2 = a^2 + \rho^2 + z^2 - 2a\rho\cos\phi'.$$

Hence

$$R = \sqrt{a^2 + \rho^2 + z^2 - 2a\rho\cos\phi'}. \tag{1.76}$$

1.4.4 Field direction

Without loss of generality, the vector potential at a point Q is considered for which the azimuthal coordinate $\phi = 0$. Note that with $\phi = 0$,

$$\hat{\phi}' = \hat{\phi}\cos\phi' - \hat{\rho}\sin\phi' \tag{1.77}$$

1.4.5 Vector potential

Combining the expressions (1.74) to (1.77) into the integral (1.67) gives

$$\begin{aligned}\mathbf{A}(\rho, z) &= \frac{\mu_0 I}{4\pi}\int_{-\pi}^{\pi}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{\hat{\phi}\cos\phi' - \hat{\rho}\sin\phi'}{\sqrt{a^2+\rho^2+z^2-2a\rho\cos\phi'}}\delta(\rho'-a)\delta(z')\rho' d\phi'\, d\rho'\, dz' \\ &= \hat{\phi}\frac{\mu_0 a I}{2\pi}\int_0^{\pi}\frac{\cos\phi'}{\sqrt{a^2+\rho^2+z^2-2a\rho\cos\phi'}}\, d\phi'. \end{aligned} \tag{1.78}$$

The radial term vanishes because $\sin\phi'$ gives rise to a part of the integrand that is an odd function about the mid-point $\phi' = 0$. Being unable to decide between two methods we shall give them both.

Method 1: Seek to form the standard integral (reference [4] equation 3.674.3)

$$\int_0^{\pi}\frac{\cos\phi}{\sqrt{1+p^2-2p\cos\phi}}\, d\phi = \frac{2}{p}[K(p) - E(p)], \qquad p^2 < 1, \tag{1.79}$$

where $K(p)$ and $E(p)$ are complete elliptical integrals of the first and second kinds respectively. Write

$$\mathbf{A}(\rho, z) = \hat{\phi}\frac{\mu_0 I}{2\pi}\sqrt{\frac{ap}{\rho}}\int_0^{\pi}\frac{\cos\phi'}{\sqrt{p(a^2+\rho^2+z^2)/a\rho - 2p\cos\phi'}}\, d\phi,'$$

where p is a factor determined from

$$p(a^2+\rho^2+z^2)/a\rho = 1 + p^2$$

as

$$p = \frac{1}{2a\rho}\left[(a^2+\rho^2+z^2) - \sqrt{(a+\rho)^2+z^2}\sqrt{(a-\rho)^2+z^2}\right].$$

Thus

$$\mathbf{A}(\rho, z) = \hat{\phi}\frac{\mu_0 I}{\pi}\sqrt{\frac{a}{p\,\rho}}\,[K(p) - E(p)]. \tag{1.80}$$

Method 2: Following Van Bladel [3] let $k^2 = 4a\rho/[(a+\rho)^2 + z^2]$. Then

$$\begin{aligned}\mathbf{A}(\rho, z) &= \hat{\phi}\frac{\mu_0 kI}{4\pi}\left(\frac{a}{\rho}\right)^{1/2}\int_0^{\pi}\frac{\cos\phi'}{\sqrt{1-\frac{1}{2}k^2(1+\cos\phi')}}\,d\phi' \\ &= \hat{\phi}\frac{\mu_0 kI}{4\pi}\left(\frac{a}{\rho}\right)^{1/2}\int_0^{\pi}\frac{2\cos^2(\phi'/2)-1}{\sqrt{1-k^2\cos^2(\phi'/2)}}\,d\phi'.\end{aligned}$$

If $\theta = (\pi - \phi')/2$,

$$\mathbf{A}(\rho, z) = \hat{\phi}\frac{\mu_0 kI}{2\pi}\left(\frac{a}{\rho}\right)^{1/2}\int_0^{\pi/2}\frac{2\sin^2\theta - 1}{\sqrt{1-k^2\sin^2\theta}}\,d\theta.$$

Expressing the integral using standard forms

$$K(k) = \int_0^{\pi/2}(1-k^2\sin^2\theta)^{-1/2}d\theta \tag{1.81}$$

and

$$E(k) = \int_0^{\pi/2}(1-k^2\sin^2\theta)^{1/2}d\theta \tag{1.82}$$

gives

$$\mathbf{A}(\rho, z) = \hat{\phi}\frac{\mu_0 I}{k\pi}\left(\frac{a}{\rho}\right)^{1/2}\left[\left(1-\frac{k^2}{2}\right)K(k) - E(k)\right]. \tag{1.83}$$

1.5 Magnetic Shell Formulation

1.5.1 Scalar Magnetic Potential

In a non-conducting region the magnetic field is related to a current source by Ampère's law:

$$\nabla \times \mathbf{H} = \mathbf{J}. \tag{1.84}$$

In regions where the current is zero $\nabla \times \mathbf{H} = 0$ in which case one can write the magnetic field as a gradient of a scalar field called the magnetic scalar potential. Thus

$$\mathbf{H} = -\nabla\Phi \tag{1.85}$$

A vector field that is written in this way is called conservative. The scalar gradient has the property that its line integral round a closed path is zero:

$$\oint \nabla\Phi . d\mathbf{s} = 0 \tag{1.86}$$

Therefore the same property applies to the magnetic field.

Of course, the scalar potential does not provide a complete description of the magnetic field because it does not apply in the current region and if a line integral of the magnetic field encloses a current it will not be zero as is known from the Circuital Theorem. The forgoing discussion raises a question. How can the scalar representation be modified to account for the current?

1.5.2 Equivalent Magnetic Source

As a first step to finding a complete scalar formulation for the magnetic field, replace the current source by a magnetic dipole density $\mathbf{M}$ giving the same field. Thus let

$$\mathbf{J} = \frac{1}{\mu_0}\nabla \times \mathbf{M}. \tag{1.87}$$

Note that $\mathbf{M}$ is not uniquely defined by this relationship and in fact there is considerable flexibility in its choice as we shall see. Next substitute into Ampère's law to give

$$\nabla \times \mathbf{H} = \frac{1}{\mu_0}\nabla \times \mathbf{M}. \tag{1.88}$$

From which it is clear that

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{M} - \nabla\Phi, \tag{1.89}$$

where Φ is a magnetic scalar potential. Take the divergence of (1.89) to give

$$\nabla^2\Phi = \frac{1}{\mu_0}\nabla.\mathbf{M}. \tag{1.90}$$

Note that $\mathbf{H}$ is uniquely defined by Φ alone in regions where $\mathbf{M}$ is zero. Inside the magnetic source region the magnetic field is made up of contributions from both $\mathbf{M}$ and $\nabla\Phi$. Hence the gradient of a scalar Laplacian can be added to $\mathbf{M}$ and subtracted from $\nabla\Phi$ without changing $\mathbf{H}$ or the solution of (1.90). A choice must be made of the most suitable form for the magnetic source. For a current filament this choice is usually taken to be an infinitesimally thin magnetic shell.

1.5.3 Example: Circular Current Filament

For a circular current filament, radius a carrying a current I in the plane $z = 0$, the current density is represented as

$$\mathbf{J} = \delta(\rho - a)\delta(z)I\hat{\phi}. \tag{1.91}$$

Consider an equivalent magnetic shell enclosed and bounded by the filament and having the form

$$\mathbf{M} = \mu_0 f(\rho)\delta(z)\hat{z} \qquad \text{then} \qquad \frac{1}{\mu_0}\nabla \times \mathbf{M} = -\frac{\partial f}{\partial \rho}\delta(z)\hat{\phi} \tag{1.92}$$

and, in view of the relationship (1.87), $\frac{\partial f}{\partial \rho} = -\delta(\rho - a)I$. Adopting the condition that $f(\rho) \to 0$ as $\rho \to \infty$ gives

$$f(\rho) = \begin{cases} I & \text{if } \rho < a \\ 0 & \text{otherwise} \end{cases} \qquad \mathbf{M} = \begin{cases} \mu_0 I\delta(z)\hat{z} & \text{if } \rho < a \\ 0 & \text{otherwise} \end{cases} \tag{1.93}$$

Note that the shell need not be planar. For example a shell on a hemispherical surface can serve as the equivalent source or indeed a magnetic shell on any other open surface bounded by the filament could be used. To keep the calculations simple it makes sense to use a flat shell.

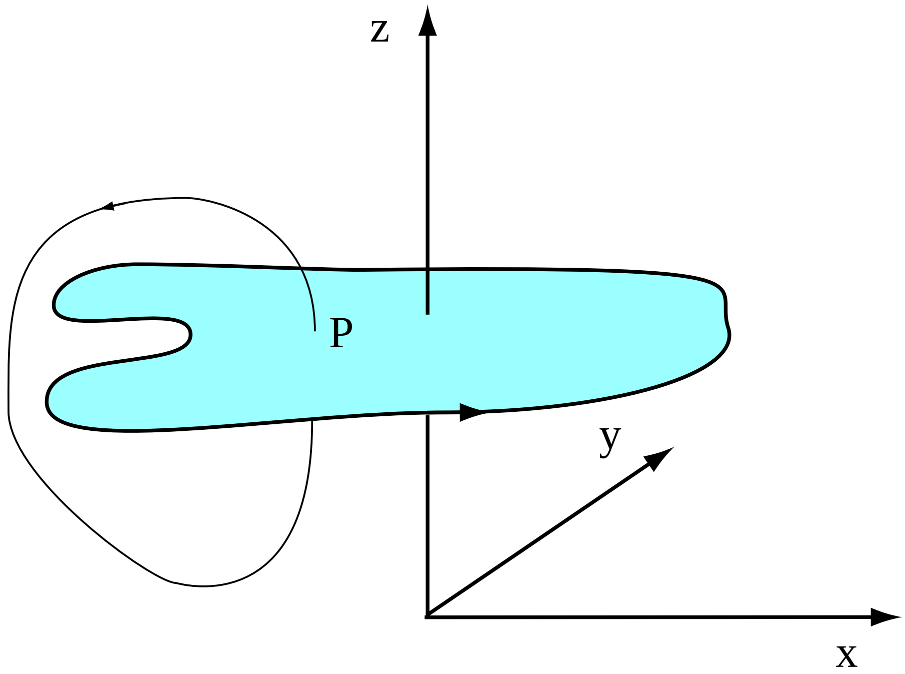

Figure 1.5: Magnetic shell due to a filament

1.5.4 Discontinuity in the Potential

Consider an arbitrary shaped closed current filament carrying a current I in the plane $z = z_0$. The magnetic field at $z = z_0$ is directed normal to the plane. This means that the potential outside the loop is constant in the plane. We shall choose this constant to be zero.

Substitute a planar magnetic shell of constant dipole density bounded by the filament for the current source. Using (1.89) it clear that Ampère's circuital theorem is satisfied by putting

$$\mathbf{M} = \mu_0 \delta(z - z_0) I \hat{z}. \tag{1.94}$$

Integrating $\mathbf{H}$ across the shell from $z = z_0 - \epsilon$ to $z = z_0 + \epsilon$ letting ϵ approach zero and using (1.89) and (1.94) shows that the potential has a jump

$$\Delta\Phi = I \tag{1.95}$$

at the open surface S_0 in the plane of the filament inside the loop.

1.5.5 Solution Using Green's Second Theorem

The solution of (1.90), rewritten as

$$\nabla^2 \Phi(\mathbf{r}) = -\rho(\mathbf{r}) \tag{1.96}$$

where $\rho(\mathbf{r})$ is the scalar source, can be expressed using a Green's function satisfying

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \tag{1.97}$$

whose solution is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|} \tag{1.98}$$

The required relationship is found using Green's second theorem applied to a region Ω, bounded by a surface S. The choice of region and surface will be decided later. The theorem is established using a vector identity,

$$\nabla.(A\nabla B) = \nabla A.\nabla B + A\nabla^2 B$$

from which it follows immediately that

$$\int_\Omega G(\mathbf{r}',\mathbf{r})\nabla'^2\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla'^2 G(\mathbf{r}',\mathbf{r})\, d\mathbf{r}' = \int_\Omega \nabla'.[G(\mathbf{r}',\mathbf{r})\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r}',\mathbf{r})]\, d\mathbf{r}' \quad (1.99)$$

The left hand side is be transformed using (1.73) and (1.96) while the divergence theorem is applied to the right hand side. This gives

$$\Phi(\mathbf{r}) - \int_\Omega G(\mathbf{r}',\mathbf{r})\rho(\mathbf{r})\, d\mathbf{r}' = \int_{S_0} G(\mathbf{r}',\mathbf{r})\frac{\partial\Phi(\mathbf{r}')}{\partial n'} - \Phi(\mathbf{r}')\frac{\partial G(\mathbf{r}',\mathbf{r})}{\partial n'}\, dS' \quad (1.100)$$

where the coordinate n' is in the direction of the outward normal to the surface S as seen from the region Ω. Thus, in general Φ is given by a volume integral and a surface integral but the usual strategy is to choose the region Ω such that one or other of these integrals vanishes.

Let the volume Ω be a source free region bounded on the inside by a surface surrounding the magnetic shell and at infinite by a second surface which together compose S. The the volume integral and the integral over the surface at infinity vanishes. Allowing the inner surface to collapse until it approaches the source, it is found that in the limit of close approach

$$\Phi(\mathbf{r}) = I\int_{S_0} \frac{\partial G(\mathbf{r}',\mathbf{r})}{\partial z'}\, dS' \quad (1.101)$$

where (1.95) has been used. It has also been noted that the magnetic field in the vertical direction, z-direction, is continuous across the plane of the filament.