**Graduate Tutorial Notes 2004**

# Theory of Electromagnetic Nondestructive Evaluation

## Chapter 3: Integral Methods for Scalar Boundary Value Problems

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# Chapter 4

# Integral Methods for Scalar Boundary Value Problems

## 4.1 Integral Formulations for Scalar Problems

### 4.1.1 The Delta Function

A *function* $f(x)$ say has by definition a numerical value for each value of $x$. The so called delta function, $\delta(x)$ is not actually function but is properly called a *generalized function* or a *distribution* because it does not have a value for $x = 0$. Its defining property is as follows:

$$f(x) = \int f(x')\delta(x - x')\, dx' \tag{4.1}$$

See the first appendix of [3] for further details. One can consider $\delta(x)$ as the limiting case of a pulse function:

$$\delta_a(x) = \begin{cases} \frac{1}{a} & -\frac{a}{2} < x < \frac{a}{2} \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

which has height $1/a$ and width $a$. Hence the area is 1. Taking the limit

$$\lim_{a \to 0} \delta_a(x) = \delta(x) \tag{4.3}$$

Note from (4.1) with $f(x) = 1$,

$$\int \delta(x - x')\, dx' = 1 \tag{4.4}$$

**Exercise 1.** Distribution theory allows us to treat a delta function like an ordinary function when integrating by parts. This implies that the derivative of the delta function

$$\delta'(x) = \frac{d\delta(x)}{dx}$$

has a meaning in an integral. To illustrate the properties of the derivative, evaluate

$$\int_a^b f(x')\delta'(x - x')\, dx'$$

for $a < x < b$.

**Exercise 2.**
(a) Evaluate the Heaviside function $H(x)$ given that

$$H(x) = \frac{1}{2}\left(\frac{d|x|}{dx} + 1\right)$$

(b) Evaluate

$$\int \frac{dH(x-x')}{dx} f(x')\, dx'$$

where $f(x)$ and its derivatives vanish as $x \to \pm\infty$. Compare the result with (4.1) to determine the relationship between the Heaviside function and the delta function.

In three dimensions, define the delta function as

$$F(\mathbf{r}) = \int F(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')\, d\mathbf{r}' \tag{4.5}$$

Note that with $F(\mathbf{r}) = 1$,

$$\int \delta(\mathbf{r}-\mathbf{r}')\, d\mathbf{r}' = 1 \tag{4.6}$$

Also note that we can write

$$\delta(\mathbf{r}-\mathbf{r}') = \delta(x-x')\delta(y-y')\delta(z-z') \tag{4.7}$$

### 4.1.2 Scalar Field in Three Dimensions

Consider a general problem in which a potential $\Phi$ is sought satisfying the Poisson equation:

$$\nabla^2\Phi(\mathbf{r}) = -\rho(\mathbf{r}), \tag{4.8}$$

where $\rho(\mathbf{r})$ is a prescribe source. The solution can be expressed using a Green's function satisfying

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'), \tag{4.9}$$

whose free space solution is

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{4.10}$$

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Derivation of (4.10):

As a simple derivation write $\nabla^2 = \nabla\cdot\nabla$ in (4.9), integrate over a sphere of radius $R = |\mathbf{r}-\mathbf{r}'|$ centered at $\mathbf{r}'$ and apply the Gauss divergence theorem and (4.5) with $F(\mathbf{r}) = 1$ to give

$$\int_{S_0} \nabla G(\mathbf{r},\mathbf{r}')\cdot d\mathbf{S} = -1 \tag{4.11}$$

where $S_0$ is the surface of the sphere. Consider a solution that vanishes as $R \to \infty$, that is therefore spherically symmetric about the point whose coordinate is $\mathbf{r}'$. Note that the outward normal to the spherical surface of which $d\mathbf{S}$ is an elementary portion is in the direction of the radial unit vector $\hat{R}$. Then we find

$$4\pi R^2 \frac{\partial G}{\partial R} = -1 \tag{4.12}$$

and (4.10) follows by integration.

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A formal equation for $\Phi$ is found using Green's second theorem applied to a suitably chosen region $\Omega$, bounded by a surface $S$. Green's second theorem is established using a vector identity,

$$\nabla \cdot (A\nabla B) = \nabla A \cdot \nabla B + A\nabla^2 B,$$

from which it follows immediately that

$$A\nabla^2 B - B\nabla^2 A = \nabla \cdot [A\nabla B - B\nabla A].$$

Applying this identity with $G \equiv A$ and $\Phi \equiv B$ and integrating the result over a volume $\Omega$ gives

$$\int_\Omega G(\mathbf{r}',\mathbf{r})\nabla'^2\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla'^2 G(\mathbf{r}',\mathbf{r})\, d\mathbf{r}' = \int_\Omega \nabla'.[G(\mathbf{r}',\mathbf{r})\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r}',\mathbf{r})]\, d\mathbf{r}'. \tag{4.13}$$

The left hand side is transformed using (4.8) and (4.9) with primed and unprimed co-ordinate reversed, while the divergence theorem is applied to the right hand side. This gives

$$\Phi(\mathbf{r}) - \int_\Omega G(\mathbf{r}',\mathbf{r})\rho(\mathbf{r}')\, d\mathbf{r}' = \int_S G(\mathbf{r}',\mathbf{r})\frac{\partial\Phi(\mathbf{r}')}{\partial n'} - \Phi(\mathbf{r}')\frac{\partial G(\mathbf{r}',\mathbf{r})}{\partial n'}\, dS', \tag{4.14}$$

where the surface $S$ encloses $\Omega$ and the coordinate $n'$ is in the direction of the outward normal to the surface $S$ as seen from the region $\Omega$. Thus, in general, $\Phi$ is given by a volume integral plus a surface integral but the basic strategy in problem solving is to choose the region $\Omega$ such that one or other of these integrals vanishes. We consider first examples in which the volume integral vanishes.

> **Exercise 2.** Deduce the Green's function $g(\mathbf{r}|\mathbf{r}')$, for a two dimensional Laplace or Poisson problems using the method given above for the three dimensional problem.

## 4.2 Single Layer Potential

### 4.2.1 Charged Conducting Plate

As an example of the application of Green's theorem we consider the problem of finding the electric field due to a charged conducting plate in air. The plate is of negligible thickness and is defined on an open surface $S_0$. The surface can be planar, such as a circular disk or non-planar, such as a spherical cap. In electrostatics the electric field is irrotational ($\nabla \times \mathbf{E} = 0$), and can therefore be written as

$$\mathbf{E} = -\nabla V, \tag{4.15}$$

where $V$ is the electric potential (potential energy per unit charge). Because $\nabla \cdot \mathbf{D} = \rho$ where $\rho$ is the charge density and in air, $\mathbf{D} = \varepsilon_0\mathbf{E}$, taking the divergence of (4.15) shows that the potential satisfies the Poisson equation

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}. \tag{4.16}$$

The Poisson equation is to be solved subject boundary conditions at infinity and on the plate. In the problem considered here, the potential at infinity is zero and the potential on the plate is a constant $V_0$. A boundary value problem such as this, in which the potential on the boundary is specified is called a *Dirichlet problem* and the boundary condition, a *Dirichlet boundary condition.*

### 4.2.2 Charge density on the Plate

The charge density on the plate is expressed as the charge per unit area $\sigma$. The source density is related to a jump in the gradient of the potential at the plate. To establish the relationship between the charge density and $\nabla V$, write the Poisson equation as $\nabla \cdot \nabla V = -\rho/\varepsilon_0$ then apply the divergence theorem to an infinitesimally small cylindrical region which encloses part of the plate such that the ends of the cylinder are parallel to plate surface. The divergence theorem then gives

$$\int_{S_\mathrm{c}} \nabla V \cdot d\mathbf{S} = -\int_{\Omega_\mathrm{c}} \rho/\varepsilon_0 d\mathbf{r} \tag{4.17}$$

where $S_\mathrm{c}$ is the entire surface of the cylinder and $\Omega_\mathrm{c}$ the region it encloses. Evaluation of the right hand side is carried out by noting that in the limiting case of an infinitesimally thin plate, the volume charge density has the form $\rho = \sigma\delta(n - n')$, $n'$ being the normal coordinate of the plate surface and $\sigma$ being the surface charge density. To evaluate the left hand side, note that because the potential on the plate is constant, the normal gradient at the cylindrical surface is zero, hence, in the limit as the ends of the cylinder approach the faces of the plate (denoted as positive and negative) we get

$$\left.\frac{\partial V}{\partial n}\right|_+ - \left.\frac{\partial V}{\partial n}\right|_- = -\frac{\sigma}{\varepsilon_0} \tag{4.18}$$

where $n$ is a coordinate representing the distance from the plate in the direction of the normal on the side labelled $+$. Note that a potential such as $V$ which arises from a discontinuity in its normal gradient is known as a *single layer potential.*

### 4.2.3 Integral Equation for the Surface Charge Density

An integral equation for the charge distribution on the plate is found by applying Green's second theorem to a region $\Omega$ external to the plate bounded by a surface which is the union of a surface enclosing the plate $S$ and another at infinity. This surface does not enclose any charge because it excludes the plate itself hence the term corresponding the the volume integral in (4.14) is zero.

Let the surface $S$ collapse onto the surface $S_0$, then in the limit it is found that

$$V(\mathbf{r}) = -\int_{S_0} G(\mathbf{r}', \mathbf{r}) \left[ \left.\frac{\partial V}{\partial n'}\right|_+ - \left.\frac{\partial V}{\partial n'}\right|_- \right] dS' - \int_{S_0} \frac{\partial G(\mathbf{r}', \mathbf{r})}{\partial n'} [V|_+ - V|_-] \, dS' \tag{4.19}$$

where the coordinate $n'$ is in the direction of the normal to the surface $S_0$ on the $+$ side. The potential is continuous at the crack, therefore $V|_+ = V|_- = V_0$ and the second integral vanishes. Using (4.18) gives

$$V(\mathbf{r}) = \frac{1}{\varepsilon_0} \int_{S_0} G(\mathbf{r}, \mathbf{r}')\sigma(\mathbf{r}')\, d\mathbf{r}' \tag{4.20}$$

which gives the potential at an arbitrary point $\mathbf{r}$ in terms of the surface charge density on the plate.

The charge density is found by applying the boundary condition. For $\mathbf{r_0} \in S_0$, the potential is specified as $V(\mathbf{r}_0) = V_0$. Thus

$$V_0(\mathbf{r_0}) = \frac{1}{\varepsilon_0} \int_{S_0} G(\mathbf{r}_0, \mathbf{r}')\sigma(\mathbf{r}')\, dS' \tag{4.21}$$

An approximate solution of equation (4.21) can be found using the moment method. Exact solutions may be found for simple geometries. An example follows.

### 4.2.4 Charged Circular Disk

**Problem 1.** Determine the potential $V$ which satisfies the Laplace equation subject to the boundary conditions in the plane $z = 0$ of a circular disk of unit radius:

$$V = 1 \qquad 0 \le \rho < 1 \tag{4.22}$$

$$\frac{\partial V}{\partial z} = 0 \qquad \rho > 1 \tag{4.23}$$

and vanishes as the distance from the center of the disk tends to infinity.

Note that the problems has axial symmetry and therefore $V$ is independent of the azimuthal angle. In cylindrical polar coordinates, we have

$$\left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \right) V(\rho, z) = 0 \tag{4.24}$$

In applying the method of separation of variables we seek a solution of the form

$$V(\rho, z) = R(\rho)Z(z) \tag{4.25}$$

Sub into (4.24), divide by $R(\rho)Z(z)$ to give

$$\left\{ \frac{1}{R(\rho)} \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[ \rho \frac{\partial}{\partial \rho} R(\rho) \right] + \frac{1}{Z(z)} \frac{\partial^2}{\partial z^2} Z(z) \right\} = 0 \tag{4.26}$$

Note that in the above equation the sum of two terms is a constant, zero. The first term can only depend on the variable $\rho$. The second can only depend on the variable $z$ but because their sum is a constant then they are both equal to a quantity that does not depend on $\rho$ or $z$. Write

$$\frac{1}{Z(z)} \frac{\partial^2}{\partial z^2} Z(z) = \kappa^2 \tag{4.27}$$

where $\kappa$ is independent of $\rho$ or $z$. Then

$$\frac{\partial^2 Z(z)}{\partial z^2} - \kappa^2 Z(z) = 0 \tag{4.28}$$

which has solutions $e^{-\kappa z}$ and $e^{\kappa z}$. Similarly

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial R(\rho)}{\partial\rho}\right] + \kappa^2 R(\rho) = 0 \tag{4.29}$$

which has solutions $J_0(\kappa\rho)$ and $Y_0(\kappa\rho)$ [2]. We shall construct a solution that vanishes as $|z| \to \infty$ and $\rho \to \infty$ of the most general form. Noting that the equations are satisfied for any real value of $\kappa$ and that nothing is gained by allowing $\kappa$ to be negative we consider all positive values for $\kappa$ and write our general solution as

$$V(\rho, z) = \int_0^\infty \frac{1}{\kappa} A(\kappa) e^{-\kappa|z|} J_0(\kappa\rho)\, d\kappa \tag{4.30}$$

With this formulation of the problem, a solution amounts to the problem of finding $A(\kappa)$.

The solution can be found through Weber's method in which one needs to first find certain integral forms as follows:

$$\int_0^\infty \frac{1}{\kappa} \sin\kappa J_0(\kappa\rho)\, d\kappa = \begin{cases} \frac{\pi}{2} & \rho < 1 \\ \sin^{-1}(1/\rho) & \rho > 1 \end{cases} \tag{4.31}$$

$$\int_0^\infty \sin\kappa J_0(\kappa\rho)\, d\kappa = \begin{cases} (1-\rho^2)^{-1/2} & \rho < 1 \\ 0 & \rho > 1 \end{cases} \tag{4.32}$$

Apply the boundary conditions to give

$$\int_0^\infty \frac{1}{\kappa} A(\kappa) J_0(\kappa\rho)\, d\kappa = 1 \qquad \rho < 1 \tag{4.33}$$

$$\int_0^\infty A(\kappa) J_0(\kappa\rho)\, d\kappa = 0 \qquad \rho > 1 \tag{4.34}$$

and compare with the integrals above to reveal that

$$A(\kappa) = \frac{2}{\pi}\sin\kappa \tag{4.35}$$

The charge density can be deduced also from (4.18) and (4.32):

$$\sigma(\rho) = \frac{4\epsilon_0}{\pi}\frac{1}{(1-\rho^2)^{1/2}} \tag{4.36}$$

Note that the charge density approached $\infty$ as $\rho$ approaches the edge of the plate. For a point on the plate close to the edge, $\rho = 1-\xi$ where $\xi$ is small. Substituting into (4.36) and neglecting terms of second order in $\xi$ one finds that

$$\sigma \approx \frac{4\epsilon_0}{\pi}\xi^{-1/2}.$$

Hence the singular edge behavior of the charge density characterized by a half-power law.

## 4.3 Double Layer Potential

### 4.3.1 Electric Current and an Ideal Crack

An ideal crack in a conductor is one in which the crack opening is negligible yet the crack is impenetrable to electric current. The crack is defined on an open surface $S_0$ in an otherwise homogeneous conductor of conductivity $\sigma_0$ and interacts with a steady direct current of current density $\mathbf{J}(\mathbf{r})$. The current density is written in terms of the scalar potential as

$$\mathbf{J}(\mathbf{r}) = -\sigma_0 \nabla V(\mathbf{r})$$

Conservation of charge requires that $\mathbf{J}$ has zero divergence in static conditions therefore

$$\nabla^2 V(\mathbf{r}) = 0$$

Applying Green's second theorem to the region surrounding the crack and letting the surface collapse onto $S_0$ gives, in the limit

$$V(\mathbf{r}) = V^{(0)}(\mathbf{r}) - \int_{S_0} \frac{\partial G(\mathbf{r}', \mathbf{r})}{\partial n'} [\, V|_+ \, - \, V|_- \,] \, dS' \tag{4.37}$$

where it has been noted that the normal gradient of the potential vanishes because $J_n = 0$ at the surface of the crack and a term $V^{(0)}(\mathbf{r})$ has been added representing the unperturbed field.

Take the normal gradient and multiply by $\sigma_0$ to give

$$\mathbf{J}(\mathbf{r}) = \mathbf{J}^{(0)}(\mathbf{r}) - \nabla \int_{S_0} \frac{\partial G(\mathbf{r}', \mathbf{r})}{\partial n'} p(\mathbf{r}') \, dS' \tag{4.38}$$

where

$$p(\mathbf{r}_0) = \sigma_0 \; V|_+ \, - \, V|_- \, . \tag{4.39}$$

Apply the boundary condition $J_n(\mathbf{r}_0) = 0$ to get an equation for $p(\mathbf{r})$. Thus

$$J_n^{(0)}(\mathbf{r}) = \int_{S_0} \frac{\partial^2 G(\mathbf{r}', \mathbf{r})}{\partial^2 n'} p(\mathbf{r}') \, dS' \tag{4.40}$$

This equation can be solved numerically using the moment or Nyström method. First examine an analytical solution.

### 4.3.2 Steady Current and the Penny Shaped Ideal Crack

For a penny shape ideal crack in a uniform current stream, the incident current density is uniform. This means that the normal gradient of the perturbed potential at the face of the crack is constant over the surface of the crack. We shall consider a more general case in which it varies with the distance from the center as $f(\rho)$.

**Problem 2.** Determine the potential $V$ which satisfies the Laplace equation subject to the boundary conditions in the plane $z = 0$ of a circular disk of unit radius:

$$\frac{\partial V}{\partial z} = f(\rho) \qquad 0 \le \rho < 1 \qquad z = 0 \tag{4.41}$$

$$V = 0 \qquad \rho > 1 \qquad z = 0 \tag{4.42}$$

and vanishes as the distance from the center of the disk tends to infinity.

Write the solution as

$$V(\rho, z) = \int_0^\infty \psi(\kappa) e^{-\kappa|z|} J_0(\kappa\rho)\, d\kappa \tag{4.43}$$

With this formulation, the problem, a solution is found by finding $\psi(\kappa)$. Apply the boundary conditions to give

$$\int_0^\infty \kappa\psi(\kappa) J_0(\kappa\rho)\, d\kappa = f(\rho) \qquad \rho < 1 \qquad z = 0 \tag{4.44}$$

$$\int_0^\infty \psi(\kappa) J_0(\kappa\rho)\, d\kappa = 0 \qquad \rho > 1 \qquad z = 0 \tag{4.45}$$

A means of finding a solution is indicated by the properties of the discontinuous integral, 6.671 7 from reference [2],

$$\int_0^\infty \sin(u\kappa) J_0(\kappa\rho)\, d\kappa = \begin{cases} (u^2 - \rho^2)^{-1/2} & \rho < u \\ 0 & \rho > u \end{cases} \tag{4.46}$$

which suggests that by limiting $u$ to a range up to 1, one can satisfy (4.45)with a solution of the form

$$\psi(\kappa) = \int_0^1 \chi(u) \sin(\kappa u)\, du \qquad \chi(0) = 0 \tag{4.47}$$

It can be shown that [8]

$$\chi(u) = \frac{2}{\pi} \int_0^u \frac{\rho f(\rho)}{\sqrt{u^2 - \rho^2}}\, d\rho \tag{4.48}$$

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**Derivation of** (4.48): Integrate (4.47) by parts:

$$\psi(\kappa) = -\chi(1)\cos(\kappa) + \int_0^1 \chi'(u) \cos(\kappa u)\, du \tag{4.49}$$

Substitute into (4.44) and use

$$\int_0^\infty \cos(u\kappa) J_0(\kappa\rho)\, d\kappa = \begin{cases} 0 & \rho < u \\ (u^2 - \rho^2)^{-1/2} & \rho > u \end{cases} \tag{4.50}$$

to give

$$f(\rho) = \int_0^\rho \frac{\chi'(u)}{\sqrt{\rho^2 - u^2}}\, du \qquad 0 \le \rho < 1 \tag{4.51}$$

Consider the integral

$$\begin{aligned} \int_0^\rho \frac{v f(v)}{\sqrt{\rho^2 - v^2}}\, dv &= \int_0^\rho \int_0^\rho \frac{v\chi'(u)}{\sqrt{v^2 - u^2}\sqrt{\rho^2 - v^2}}\, du\, dv \\ &= \int_0^\rho \chi'(u) \int_0^\rho \frac{v}{\sqrt{v^2 - u^2}\sqrt{\rho^2 - v^2}}\, dv\, du \\ &= \frac{\pi}{2}\chi(\rho) \end{aligned} \tag{4.52}$$

from which (4.48) follows.

By substituting (4.47) into (4.43), setting $z = 0$ it is found that

$$V(\rho, 0) = \frac{2}{\pi} \int_\rho^1 \frac{\chi(u)}{\sqrt{u^2 - \rho^2}} \, du \qquad \rho < 1 \tag{4.53}$$

It is then easy to show that for $f(\rho) = 1$, $\chi(u) = u$ and

$$V(\rho, 0_\pm) = \pm \frac{2}{\pi} \sqrt{1 - \rho^2} \tag{4.54}$$

hence, from (4.39),

$$p(\rho) = \frac{4}{\pi} \sqrt{1 - \rho^2}. \tag{4.55}$$

## 4.4 Moment Method

Using moment method to solve the following equation to find the surface charge density $\sigma(\mathbf{r})$ on an infinitesimally thin rectangular plate defined on the open surface $S_0$:

$$V_0(\mathbf{r}_0) = \frac{1}{\varepsilon_0} \int_{S_0} G(\mathbf{r}_0|\mathbf{r}')\sigma(\mathbf{r}')dS' \tag{4.56}$$

Divide $S_0$ into $N = N_x \times N_y$ rectangles (boundary elements). Approximate $\sigma$ as constant on each boundary element. Thus $\sigma(\mathbf{r})$ is then approximated by

$$\sigma(\mathbf{r}) \approx \sum_{j=0}^{N_x-1} \sum_{k=0}^{N_y-1} \sigma_{jk}\psi_{jk}(\mathbf{r}) \tag{4.57}$$

where the $\psi_{jk}(\mathbf{r})$, the functions used for expanding the unknown, are called basis functions. These are usually local; zero outside a limited range. They are typically a low order polynomial which represents the unknown as piecewise constant or piecewise linear or piecewise quadratic and they are defined with respect to the boundary elements. Substitute equation (4.57) into (4.56) and require that (4.56) is satisfied at points whose coordinate $\mathbf{r}_{jk}$ $j = 0, 2, ...N_x - 1$ $k = 0, 2, ...N_y - 1$ are the coordinate of the center of each boundary element. This is called point matching or co-location. The procedure gives

$$V_{jk} = \sum_{j'=0}^{N_x-1} \sum_{k'=0}^{N_y-1} M_{jk,j'k'}\sigma_{j'k'} \tag{4.58}$$

Note $V_{jk} = V_0(\mathbf{r}_{jk})$ and

$$M_{jk,j'k'} = \frac{1}{\varepsilon_0} \int_{S_{j'k'}} G(\mathbf{r}_{jk}|\mathbf{r}')\psi_{j'k'}(\mathbf{r}') \, dS' \tag{4.59}$$

is an integration over the $j'k'$-th boundary element. Equation (4.58) can also be written as:

$$\bar{V} = \overline{\overline{M}}\bar{\sigma} \tag{4.60}$$

where $\bar{V}$ and $\bar{\sigma}$ are column vectors with $N$ elements and $\overline{\overline{M}}$ is an $N \times N$ matrix.

### 4.4.1 Taking Advantage of the Convolutional Properties

One can choose to divide the region of the plate into a regular rectangular lattice and use the same shape function for all the basis functions[1]. In which case, the basis functions will have the shift property

$$\psi_{jk}(\mathbf{r} + \mathbf{r}_{jk}) = \psi_{nm}(\mathbf{r} + \mathbf{r}_{nm}) \tag{4.61}$$

By changing the coordinates in (4.59), putting $\mathbf{r}' = \mathbf{r} + \mathbf{r}_{j'k'}$ and applying the shift property of the basis functions we get

$$M_{jk,j'k'} = \frac{1}{\varepsilon_0} \int_{S_{00}} G(\mathbf{r}_{jk} - \mathbf{r}_{j'k'} - \mathbf{r})\psi(\mathbf{r})\, dS \tag{4.62}$$

where we have put $\psi(\mathbf{r}) = \psi_{00}(\mathbf{r} + \mathbf{r}_{00})$. It can be seen from (4.62) that $M_{jk,j'k'}$ is a Toeplitz matrix having the form

$$M_{jk,j'k'} = \mathcal{M}_{|j-j'|,|k-k'|} \tag{4.63}$$

with only $N$ distinct elements.

### 4.4.2 Testing Functions

Rather than using point matching, one can, adopt a more generally procedure and take moments of (4.56) to establish the matrix equation. Taking moments mean multiplying (4.56) by local testing functions, $\phi_{jk}$ say, and integrating over the domain of the testing function. This gives

$$V_{jk} = \int_{S_{jk}} \phi_{jk}(\mathbf{r}) V_0(\mathbf{r})\, dS \tag{4.64}$$

and

$$M_{jk,j'k'} = \frac{1}{\varepsilon_0} \int_{S_{jk}} \phi_{jk}(\mathbf{r}) \int_{S_{j'k'}} G(\mathbf{r} - \mathbf{r}')\psi_{j'k'}(\mathbf{r}')\, d\mathbf{r}'\, dS \tag{4.65}$$

**Exercise 3.** Show that for a regular grid of boundary elements, the matrix formed by testing has a Toeplitz structure and that the matrix elements are given by an equation of the form

$$\mathcal{M}_{|j-j'|,|k-k'|} = \frac{1}{\varepsilon_0} \int_{S_\beta} G(\mathbf{r}_{jk} - \mathbf{r}_{j'k'} - \mathbf{r})\beta(\mathbf{r})\, dS. \tag{4.66}$$

where $S_\beta$ is the domain of $\beta$. Give an expression for $\beta(\mathbf{r})$ in terms of $\phi(\mathbf{r})$ and $\psi(\mathbf{r})$

### 4.4.3 Singular Matrix Element

When $j = j'$ and $k = k'$, the integration involving $G(\mathbf{r}_{jk}|\mathbf{r}')$ will need to deal with the singularity in the Green's function, equation (4.65). The value of the singular Matrix element is assigned to the diagonal of $\overline{\overline{M}}$ and will be evaluated for a pulse function expansion and point matching. In which case, we have

$$M_{jk,jk} = \frac{1}{\varepsilon_0} \int_{S_{jk}} G(\mathbf{r}_{jk}|\mathbf{r}') d\mathbf{r}' \tag{4.67}$$

[1] Alternatively, one can used a higher density of rectangles near the edge to cope with the singular behavior of the charge near the edge. Also one could have special elements to describe this edge behavior.

The integration is done in two parts, by a numerical method for a region where the singularity is excluded and analytically over the exclusion zone. With $S_X$ as a $2a \times 2a$ square exclusion zone at the center of the boundary element, we have

$$M_{jk,jk} = \frac{1}{\varepsilon_0} \int_{S_{jk}-S_X} G(\mathbf{r}_{jk}|\mathbf{r}')d\mathbf{r}' + \frac{a}{\pi\varepsilon_0} I^{(1)} \tag{4.68}$$

where

$$I^{(1)} = \frac{1}{a} \int_{S_q} \frac{1}{R} \, dS \tag{4.69}$$

and $S_q$ is one quadrant of the exclusion zone and $R^2 = x^2 + y^2$. Re-scale the variables of integration with respect to $a$ and write

$$\begin{aligned} I^{(1)} &= \int_0^1 \int_0^1 \frac{x^2+y^2}{R^3} \, dx \, dy \\ &= 2 \int_0^1 \frac{1}{\sqrt{1+x^2}} \, dx \\ &= 2 \log(1+\sqrt{2}) \end{aligned} \tag{4.70}$$

This result make use of standard integrals 2.271 5 and 2.271 4 from Gradshteyn and Ryzhik [1].

**Exercise 4.** Proceeding as above with the equivalent double layer potential problem, equation (4.40), it is found that we must deal with an exclusion zone integral

$$I^{(2)} = \lim_{z \to 0} \int_0^1 \int_0^1 \frac{\partial^2}{\partial z^2} \left( \frac{1}{R} \right) \, dx \, dy. \tag{4.71}$$

where $R^2 = x^2 + y^2 + z^2$. To evaluate $I^{(2)}$ perform the integration before taking the limit.

**Exercise 4: Solution.**

(a) Evaluate the integrand.

$$\frac{\partial}{\partial z} \left( \frac{1}{R} \right) = \frac{\partial R}{\partial z} \frac{\partial}{\partial R} \left( \frac{1}{R} \right) = -\frac{z}{R^3} \tag{4.72}$$

$$\frac{\partial^2}{\partial z^2} \left( \frac{1}{R} \right) = -\frac{\partial}{\partial z} \left( \frac{z}{R^3} \right) = \frac{z^2}{3R^3} - \frac{1}{R^3} \tag{4.73}$$

(b) Evaluate

$$Q_3 = \int\int \frac{1}{R^3}\,dx\,dy \tag{4.74}$$

Integrating with respect to $x$ gives

$$Q_3 = \int \frac{1}{y^2+z^2}\frac{x}{R}\,dy \tag{4.75}$$

Let $v = y/R$, then $dv = (1/R)(1 - y^2/R^2)dy$

$$Q_3 = x\int \frac{1}{z^2+x^2v^2}\,dv = \frac{1}{z}\arctan\left(\frac{x}{z}v\right)$$

Thus

$$Q_3 = \frac{1}{z}\arctan\left(\frac{xy}{zR}\right)$$

(c) Evaluate

$$Q_5 = \int\int \frac{1}{R^5}\,dx\,dy \tag{4.76}$$

Integrating with respect to $x$ gives

$$Q_5 = \frac{x}{3}\int \frac{1}{R}\left[\frac{1}{(y^2+z^2)R^2} + \frac{2}{(y^2+z^2)^2}\right]dy$$

Split the first term in bracket using partial fractions to give

$$Q_5 = \frac{x}{3}\int \frac{1}{R}\left[\frac{1}{x^2(y^2+z^2)} - \frac{1}{x^2R^2} + \frac{2}{(y^2+z^2)^2}\right]dy$$

The middle term can be integrated immediately, and the other terms dealt with using the substitution $v = y/R$ as before.

$$Q_5 = -\frac{y}{3x(x^2+z^2)R} + \frac{x}{3}\int \frac{1}{R}\left[\frac{1}{x^2(y^2+z^2)} + \frac{2}{(y^2+z^2)^2}\right]dy,$$

and the other terms dealt with using the substitution $v = y/R$ as before and thereby reduced to integration of rational functions. The first of these has been dealt with above

$$Q_5 = -\frac{y}{3x(x^2+z^2)R} + \frac{1}{3x^2z}\arctan\left(\frac{xy}{zR}\right) + \frac{2x}{3}\int \frac{1-v^2}{(z^2+x^2v^2)^2}\,dv,$$

Putting $u = zv/x$ gives

$$Q_5 = -\frac{y}{3x(x^2+z^2)R} + \frac{1}{3x^2z}\arctan\left(\frac{xy}{zR}\right) + \frac{2}{3z^3}\int \frac{1}{(1+u^2)^2}\,dv - \frac{2}{3zx^2}\int \frac{v^2}{(1+u^2)^2}\,dv,$$

## 4.5 Fourier Representation of the Green's Function

We can show that (4.10) is indeed the desired solution of equation (4.9) using Fourier transform techniques [4]. The Fourier transform with respect to $x$ and $y$ is written as

$$\tilde{f}(u,v) = \int\int_{-\infty}^{\infty} f(x,y)e^{-(iux+ivy)}\ dx\ dy \tag{4.77}$$

and the corresponding inverse Fourier transform as

$$f(\mathbf{r}) = \frac{1}{(2\pi)^2} \int\int_{-\infty}^{\infty} \tilde{f}(u, v, z) e^{iux+ivy} \, du \, dv, \tag{4.78}$$

In taking the two dimensional Fourier transform of (4.9) we have

$$\begin{aligned}
\mathcal{F}\{\nabla^2 G(\mathbf{r}|\mathbf{r}'), x \to u, y \to v\} &= \int\int_{-\infty}^{\infty} \nabla^2 G(\mathbf{r}|\mathbf{r}') e^{-(iux+ivy)} \, dx \, dy \\
&= \int\int_{-\infty}^{\infty} G(\mathbf{r}|\mathbf{r}') \nabla^2 e^{-(iux+ivy)} \, dx \, dy \\
&= \left[\frac{\partial^2}{\partial z^2} - (u^2 + v^2)\right] \int\int_{-\infty}^{\infty} G(\mathbf{r}|\mathbf{r}') e^{-(iux+ivy)} \, dx \, dy \\
&= \left[\frac{\partial^2}{\partial z^2} - (u^2 + v^2)\right] \tilde{G}(u, v, z, z') e^{-(iux'+ivy')},
\end{aligned}$$

where we have defined

$$\tilde{G}(u, v, z, z') = \int\int_{-\infty}^{\infty} G(\mathbf{r}|\mathbf{r}') e^{-iu(x-x')-iv(y-y')} \, dx \, dy \tag{4.79}$$

Hence the transformed equation is

$$\left(\frac{\partial^2}{\partial z^2} - \kappa^2\right) \tilde{G}(u, v, z, z') = -\delta(z - z'), \tag{4.80}$$

$\kappa$ being the positive root of $\kappa = (u^2 + v^2)^{1/2}$. To get a solution vanishing as $z \to \infty$ we propose $F(\kappa)e^{-\kappa(z-z')}$ for $z > z'$. Direct substitution in (4.80) confirms that this is a valid solution. Similarly, noting that the solution must also vanish as $z \to -\infty$, we propose $F(\kappa)e^{\kappa(z-z')}$ for $z < z'$. Substitution into ( 4.80) shows that the equation is satisfied for $z \neq z'$.

$F(\kappa)$ may be determined by integrating 4.80 from $z = z' - \epsilon$ to $z = z' + \epsilon$ and taking the limit as $\epsilon \to 0$ to give

$$\left(\frac{\partial \tilde{G}}{\partial z}\right)_+ - \left(\frac{\partial \tilde{G}}{\partial z}\right)_- = -1 \tag{4.81}$$

where the $\pm$ subscripts refer to limiting values on either side of $z'$. Thus the derivative of the Green's function has a jump of -1 at $z = z'$. From the size of the jump we find that $F(\kappa) = \frac{1}{2\kappa}$ and hence

$$\tilde{G}(u, v, z, z') = \frac{1}{2\kappa} e^{-\kappa|z-z'|}. \tag{4.82}$$

The inverse transform is written

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{(2\pi)^2} \int\int_{-\infty}^{\infty} \frac{1}{2\kappa} e^{-\kappa|z-z'|} e^{iu(x-x')+iv(y-y')} \, du \, dv \tag{4.83}$$

hence we ought to be able to show by direct integration that

$$\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} = \frac{1}{(2\pi)^2} \int\int_{-\infty}^{\infty} \frac{1}{2\kappa} e^{-\kappa|z-z'|} e^{iu(x-x')+iv(y-y')} \, du \, dv. \tag{4.84}$$

**Exercise 5.** Use the method of images to write down a Green's function satisfying (4.9), vanishing as $|\mathbf{r} - \mathbf{r}'| \to \infty|$ and vanishing on the plane $z = 0$. What is the two dimensional Fourier transform with respect to $x$ and $y$ of this half-space Green's function.

**Exercise 6.** What is the Fourier transform with respect to $x$ and $y$ of the Green's function satisfying,

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') + k^2 G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'), \tag{4.85}$$

vanishing as $|\mathbf{r} - \mathbf{r}'| \to \infty$.

### 4.5.1 Vector Potential Due to an Axial Current

Let us specialize equation 2.29 assuming that the current density $\mathbf{J}(\mathbf{r})$ is axially symmetric having the form

$$\mathbf{J}(\mathbf{r}) = \overset{\circ}{J}(\rho', z')\hat{\phi}'$$

If we convert the Green's function from the last section into cylindrical coordinates by the substitutions

$$x = \rho\cos\phi \quad y = \rho\sin\phi$$
$$u = \kappa\cos\beta \quad v = \kappa\sin\beta$$

then

$$\begin{aligned} u(x-x') + v(y-y') &= \kappa[\rho\cos(\phi-\beta) - \rho'\cos(\phi'-\beta)] \\ du\, dv &= \kappa\, d\kappa\, d\beta \\ \hat{\phi}' &= \hat{\phi}\cos(\phi-\phi') + \hat{\phi}\sin(\phi-\phi') \end{aligned}$$

Substituting these results back into equation 2.29 and noting using the Fourier representation of the Green's function 4.84, we get

$$\mathbf{A}(\mathbf{r}) = \mu_0\hat{\phi}\int \overset{\circ}{G}(\rho, z|\rho', z')\, \overset{\circ}{J}(\rho' z')\, \rho' d\rho'\, dz' \tag{4.86}$$

where

$$\begin{aligned} \overset{\circ}{G}(\rho, z|\rho', z') &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\frac{1}{2\kappa}e^{-\kappa|z-z'|} \\ &\quad \cdot e^{i\kappa[\rho\cos(\phi-\beta)-\rho'\cos(\phi'-\beta)]} \\ &\quad \cdot \cos(\phi-\beta)\cos(\phi'-\beta)\kappa\, d\kappa\, d\phi'\, d\beta \end{aligned}$$

Using the standard integral [2]

$$J_1(\alpha) = \frac{1}{2\pi i}\int_0^{2\pi} e^{i\alpha\cos\theta}\cos\theta\, d\theta \tag{4.87}$$

integration with respect to $\phi'$ and $\beta$ gives

$$\overset{\circ}{G}(\rho, z|\rho', z') = \int_0^{\infty}\tilde{G}(u, v, z, z')J_1(\kappa\rho)J_1(\kappa\rho')\kappa\, d\kappa \tag{4.88}$$

This result has been derived for a coil in free space but similar integrals arise in calculating the vector potential due to an axially symmetric coil above a half-space conductor or multilayered slab [6]. The main difference in half-space problems is that one must account for reflections from the surface of the material using a modified Green's function. These reflection terms are found by applying interface conditions which match the solution in air to the solution in the conductor. Before we do this for a coil above a half-space we shall consider a simple one-dimensional interface problem.