Graduate Tutorial Notes 2004

Theory of Electromagnetic Nondestructive Evaluation

Chapter 5: Vector Problems in Electromagnetic Theory

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Chapter 5

Dyadic Green's Functions in Electromagnetic NDE

5.1 Magnetic Vector Potential

Initially we shall review the way the magnetic vector potential is introduced in wave theory because is familiar from basic courses in electromagnetism. Particularly for those who have studied antenna theory and the Hertzian dipole. Then later we revert to the quasi-static limit and ignore the displacement current.

By using a magnetic vector potential formulation we can get a simple integral express for a field solution in free space that can be used as a starting point for the introduction of dyadic kernels [27].

5.1.1 Wave Equation for Lossless Media

First recall the form of Maxwell's equations for a linear lossless medium. Putting $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{D} = \varepsilon\mathbf{E}$ gives

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \tag{5.1}$$

and

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \tag{5.2}$$

Take the curl of (5.1) and eliminate $\mathbf{H}$ using (5.2) to give

$$\nabla \times \nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t} \nabla \times \mathbf{H} = -\mu \frac{\partial}{\partial t}\left(\epsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}\right),$$

hence

$$\nabla \times \nabla \times \mathbf{E} + \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu \frac{\partial \mathbf{J}}{\partial t} \tag{5.3}$$

Similarly

$$\nabla \times \nabla \times \mathbf{H} + \epsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2} = \nabla \times \mathbf{J}. \tag{5.4}$$

Then introduce magnetic vector potential such that

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{5.5}$$

This form for $\mathbf{B}$ guarantees that it has zero divergence since $\nabla . \nabla \times \mathbf{A} = 0$. But if we add to $\mathbf{A}$ the gradient of a scalar field, ∇V to give $\mathbf{B} = \nabla \times [\mathbf{A} + \nabla V]$, it doesn't affect the value of $\mathbf{B}$.

So $\mathbf{A}$ isn't defined completely yet. The gradient of a scalar can be added without changing the magnetic flux density. To complete the definition of $\mathbf{A}$ it is necessary to specify the divergence of $\mathbf{A}$ as well as its curl. This step is taken later. First sub into Faraday's induction law

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} = -\frac{\partial \mathbf{B}}{\partial t}$$

to give

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \nabla \times \mathbf{A}$$

or

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0.$$

"Uncurl" this relationship to give

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \tag{5.6}$$

where V is, as yet, an undefined scalar potential. Substitute (5.5) and (5.6) into the Maxwell-Ampére law (5.2), to give

$$\nabla \times \nabla \times \mathbf{A} = -\epsilon\mu \frac{\partial}{\partial t} \left\{ \nabla V + \frac{\partial \mathbf{A}}{\partial t} \right\} + \mu \mathbf{J}.$$

Putting $\nabla \times \nabla \times = \nabla\nabla. - \nabla^2$ gives

$$(\nabla\nabla. - \nabla^2)\mathbf{A} = -\epsilon\mu \frac{\partial}{\partial t} \left\{ \nabla V + \frac{\partial \mathbf{A}}{\partial t} \right\} + \mu \mathbf{J}.$$

Now fix the divergence of $\mathbf{A}$ by choosing the Lorentz gauge

$$\nabla.\mathbf{A} = -\epsilon\mu \frac{\partial V}{\partial t}. \tag{5.7}$$

Then the equation for $\mathbf{A}$ separates from the equation for V:

$$\left\{ \nabla^2 - \epsilon\mu \frac{\partial^2}{\partial t^2} \right\} \mathbf{A} = -\mu \mathbf{J}. \tag{5.8}$$

Taking the divergence of (5.6), using $\nabla.\mathbf{D} = \rho$ and (5.7) gives

$$\left\{ \nabla^2 - \epsilon\mu \frac{\partial^2}{\partial t^2} \right\} V = -\frac{1}{\epsilon}\rho. \tag{5.9}$$

5.1.2 Quasi-static equations for a homogeneous conductor

Consider a homogeneous conductor containing an electric source term $\mathbf{P}$. In the quasi-static limit, $\sigma\mathbf{E}(\mathbf{r})$ replaces the displacement current term in the free space lossless media problem. Then it is found that

$$\nabla \times \nabla \times \mathbf{A} = -\mu\sigma \left\{ \nabla V + \frac{\partial \mathbf{A}}{\partial t} \right\} + \mu \mathbf{P}.$$

and the gauge condition becomes

$$\nabla.\mathbf{A} = -\mu\sigma V. \tag{5.10}$$

Then the equation for $\mathbf{A}$ separates from the equation for V to give

$$\left\{ \nabla^2 - \mu\sigma \frac{\partial}{\partial t} \right\} \mathbf{A} = -\mu \mathbf{P} \tag{5.11}$$

which is a diffusion equation.

5.1.3 Helmholtz Equation for a Time harmonic Vector Potential

If the field is time-harmonic, the time dependent magnetic vector potential $\mathbf{A}(\mathbf{r}, t)$ is will be written in terms of a phasor $\mathbf{A}(\mathbf{r})$ as

$$\mathbf{A}(\mathbf{r}, t) = \Re\{\mathbf{A}(\mathbf{r})e^{j\omega t}\}$$

therefore

$$\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} = \Re\{j\omega\mathbf{A}(\mathbf{r})e^{j\omega t}\}$$

and

$$\frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2} = -\Re\{\omega^2\mathbf{A}(\mathbf{r})e^{j\omega t}\}$$

In the phasor representation of a time-harmonic field, $j\omega$ replaces the time derivative and $-\omega^2$ replaces the second order time derivative. Hence (5.11) becomes

$$(\nabla^2 + k^2)\mathbf{A} = -\mu\mathbf{P} \tag{5.12}$$

where $k^2 = -j\omega\mu\sigma$. A solution in the unbounded domain can be expressed in terms of the Green's function

$$G_0(\mathbf{r}|\mathbf{r}') = \frac{e^{jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{5.13}$$

Satisfying

$$(\nabla^2 + k^2)G_0(\mathbf{r}|\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') \tag{5.14}$$

This solution is written

$$\mathbf{A}(\mathbf{r}) = \mu\int_\Omega G_0(\mathbf{r}|\mathbf{r}')\mathbf{P}(\mathbf{r}')d\mathbf{r}' \tag{5.15}$$

The scalar Green function used here is limited because it implies that one can determine the field at a point due to a source at another point if they are in the same direction by multiplying by a scalar quantity. This works for the magnetic vector potential in an unbounded domain but in general the source and the resulting field are not in the same direction. This means we need a Green's function that is not a scalar but a tensor. In fact a dyadic Green's function is needed, which is a special kind of tensor.

5.1.4 Notes on Dyads and Dyadics

A dyad is an operator represented by a pair of vectors which acts on a vector. In general we write a dyad as $\mathcal{D} = \mathbf{AB}$. Then the scalar product of $\mathcal{D}$ with a vector is another vector:

$$\mathcal{D}\cdot\mathbf{X} = \mathbf{A}(\mathbf{B}\cdot\mathbf{X}) = (\mathbf{B}\cdot\mathbf{X})\mathbf{A} = \alpha\mathbf{A}$$

where $\alpha = \mathbf{B}\cdot\mathbf{X}$. $\mathcal{D}$ can be represented as,

$$\mathcal{D} = \begin{bmatrix} A_xB_x\hat{x}\hat{x} & A_xB_y\hat{x}\hat{y} & A_xB_z\hat{x}\hat{z} \\ A_yB_x\hat{y}\hat{x} & A_yB_y\hat{y}\hat{y} & A_yB_z\hat{y}\hat{z} \\ A_zB_x\hat{z}\hat{x} & A_zB_y\hat{z}\hat{y} & A_zB_z\hat{z}\hat{z} \end{bmatrix}$$

hence

$$\mathcal{D}\cdot\mathbf{X} = \begin{bmatrix} A_xB_x\hat{x}\hat{x} & A_xB_y\hat{x}\hat{y} & A_xB_z\hat{x}\hat{z} \\ A_yB_x\hat{y}\hat{x} & A_yB_y\hat{y}\hat{y} & A_yB_z\hat{y}\hat{z} \\ A_zB_x\hat{z}\hat{x} & A_zB_y\hat{z}\hat{y} & A_zB_z\hat{z}\hat{z} \end{bmatrix}\cdot\begin{bmatrix} X_x\hat{x} \\ X_y\hat{y} \\ X_z\hat{z} \end{bmatrix} = (B_xX_x+B_yX_y+B_zX_z)(A_x\hat{x}+A_y\hat{y}+A_z\hat{z})$$

The sum of dyads is called a dyadic. The simplest example is the identity dyadic

$$\mathcal{I} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$$

which has the property

$$\mathcal{I} \cdot \mathbf{A} = \mathbf{A}$$

5.2 Solution of Helmholtz Equation for the Magnetic Vector Potential Using a Dyadic Green's function

5.2.1 Free space Dyadic Green's Function via the Vector Potential Method

The solution of (5.12) can be written as

$$\mathbf{A}(\mathbf{r}) = \mu \int_{\Omega} G_0(\mathbf{r}|\mathbf{r}')\mathcal{I} \cdot \mathbf{P}(\mathbf{r}')d\mathbf{r}' \tag{5.16}$$

where, $\mathcal{I}$ represents unit dyad. It can be written as $(\hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z})$, has the property that when is forms a dot product with a vector, it leaves the vector unchange: $\mathcal{I} \cdot \mathbf{X} = \mathbf{X}$ and this dot product commutes $\mathbf{X} \cdot \mathcal{I} = \mathbf{X}$. Since we have the solution for $\mathbf{A}(\mathbf{r})$, then we can find $\mathbf{H}(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})$ using

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu}\nabla \times \mathbf{A}(\mathbf{r}) \tag{5.17}$$

and

$$\mathbf{E}(\mathbf{r}) = -\jmath\omega\mathbf{A}(\mathbf{r}) - \nabla V \tag{5.18}$$

respectively. From (5.18) and the gauge condition $\nabla \cdot \mathbf{A} = \mu_0\sigma V$, we have

$$\mathbf{E}(\mathbf{r}) = -\jmath\omega\left[1 + \frac{1}{k^2}\nabla\nabla\right]\mathbf{A}(\mathbf{r}) \tag{5.19}$$

where $k^2 = -\jmath\omega\mu\sigma$ and $\nabla\nabla$ is a dyadic differential operator. Rather that finding the vector potential first and the calculating the electric field, the electric field can be found directly from an integral formulae as follows. From (5.16) we can write (5.19) as,

$$\mathbf{E}(\mathbf{r}) = -\jmath\omega\mu \int_{\Omega} \mathcal{G}_0(\mathbf{r}|\mathbf{r}') \cdot \mathbf{P}(\mathbf{r}')d\mathbf{r}' \tag{5.20}$$

where

$$\mathcal{G}_0(\mathbf{r}|\mathbf{r}') = \left[\mathcal{I} + \frac{1}{k^2}\nabla\nabla\right]G_0(\mathbf{r}|\mathbf{r}') \tag{5.21}$$

In getting (5.20) the differential operator has been moved inside the integral which is not always possible and cannot be carried out at the singular point of the Green's function. Hence the evaluation of $\mathbf{E}$ requires special consideration for points inside the source region.

5.3 Volume Integral Formulation

For time harmonic quasi-static field inside a conductor, we can express Maxwell's equations as,

$$\nabla \times \mathbf{E}(\mathbf{r}) = -\jmath\omega\mu\mathbf{H}(\mathbf{r}) \tag{5.22}$$

$$
\begin{aligned}
\nabla \times \mathbf{E}(\mathbf{r}) &= \sigma(\mathbf{r})\mathbf{E}(\mathbf{r}) \\
&= \sigma_0\mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})
\end{aligned} \tag{5.23}
$$

where σ_0 is the host conductivity and P is dipole density given by,

$$
\mathbf{P}(\mathbf{r}) = [\sigma(\mathbf{r}) - \sigma_0]\mathbf{E}(\mathbf{r}) \tag{5.24}
$$

For a cavity $\sigma(\mathbf{r}) = 0$. Hence, for volume integral formulation we can write total electric field as incident + scattered field. Thus

$$
\mathbf{E}(\mathbf{r}) = \mathbf{E}^0(\mathbf{r}) - \jmath\omega\mu \int_\Omega \mathcal{G}(\mathbf{r}|\mathbf{r}') \cdot \mathbf{P}(\mathbf{r}')d\mathbf{r}' \tag{5.25}
$$

Multiply (11) by $\sigma(r) - \sigma_0$ to give

$$
\mathbf{P}(\mathbf{r}) = \mathbf{P}^{(0)}(\mathbf{r}) - k^2\upsilon(\mathbf{r}) \int_\Omega \mathcal{G}(\mathbf{r}|\mathbf{r}').\mathbf{P}(\mathbf{r}')d\mathbf{r}' \tag{5.26}
$$

where $k^2 = -j\omega\mu\sigma_0$ and the flaw function $\upsilon(\mathbf{r})$ is defined $\upsilon(\mathbf{r}) = [\sigma(\mathbf{r}) - \sigma_0]/\sigma_0$ while $\mathbf{P^{(0)}}(\mathbf{r}) = [\sigma(\mathbf{r}) - \sigma_0]\mathbf{E^{(0)}}(\mathbf{r})$. An approximate solution of equation (5.26) may be found by converting it to a matrix equation using the moment method.

5.4 Scalar Decomposition

The dyadic Green's function described up until this point is suitable for an unbounded domain but the problems of interest in NDE involver a scatterer (i.e. a flaw) in one region; the conductor, and a probe in another region (air) with interface conditions coupling the field between the two regions. The simplest way to represent these two regions is as semi-spaces. Then to formulate the scattering problem, a half-space Green's function is used with a singular source (a dipole) in the conductor. This dyadic Green's function will automatically account for the the interface conditions.

We shall derive the half-space, plate and other fundamental solutions using a scalar potential representation of the field but first the scalar potentials will be used to determine the unbounded domain Green's function which we have previously derived via the magnetic vector potential.

5.4.1 Unbounded Conductive Region

Note that the electric field in a conductor containing a flaw is a solution of

$$
\nabla \times \nabla \times \mathbf{E} - k^2\mathbf{E} = -j\omega\mu\mathbf{P} \tag{5.27}
$$

where the flaw polarization is $\mathbf{P}(\mathbf{r}) = [\sigma(\mathbf{r}) - \sigma_0]\mathbf{E}(\mathbf{r})$. Although we usually work with the electric field in eddy current calculation, it is worth noting that the magnetic field satisfies

$$
\nabla \times \nabla \times \mathbf{H} - k^2\mathbf{H} = \nabla \times \mathbf{P} \tag{5.28}
$$

We shall seek a solution in terms of a half-space dyadic Green's function which will be derived using transverse electric and transverse magnetic scalar potentials. Thus the electric field is written as

$$
\mathbf{E}(\mathbf{r}) = -j\omega\mu \left\{ \nabla \times [\hat{z}\psi'(\mathbf{r})] + \frac{1}{k^2}\nabla \times \nabla \times [\hat{z}\psi''(\mathbf{r})] \right\} \tag{5.29}
$$

which can be used where $\mathbf{E}$ has zero divergence, i.e. source free homogeneous regions.

This scalar decomposition is based on the fact that a field with zero divergence, called a solenoidal vector field, has only two independent components. Given two components, the zero divergence property will give the third. The form of (5.29) means that the electric field in the plane perpendicular to the preferred direction, the z-direction in this case, is given by

$$\mathbf{E}_t(\mathbf{r}) = -j\omega\mu \left\{ \nabla_t \times [\hat{z}\psi'(\mathbf{r})] + \frac{1}{k^2}\nabla_t \frac{\partial}{\partial z}\psi''(\mathbf{r}) \right\} \tag{5.30}$$

where the subscript t denotes the tangential components with respect to the preferred direction. This expresses the two-component tangential field as the sum of an irrotational (zero curl) and solenoidal (zero divergence) part which are orthogonal in the the Fourier space defined by a Fourier transform with respect to x and y.

One notes that by using Faraday's law, the magnetic field can be written as

$$\mathbf{H}(\mathbf{r}) = \nabla \times \nabla \times [\hat{z}\psi'(\mathbf{r})] + \nabla \times [\hat{z}\psi''(\mathbf{r})], \tag{5.31}$$

assuming that ψ'' satisfies the Helmholtz equation:

$$(\nabla^2 + k^2)\psi''(\mathbf{r}) = 0$$

In an unbounded domain or a half-space, or a layered half-space problem, this can be assumed[1] outside of the source region. Substituting (5.29) into (5.27) gives

$$(\nabla^2 + k^2)\{\nabla \times [\hat{z}\psi'(\mathbf{r})] + \frac{1}{k^2}\nabla \times \nabla \times [\hat{z}\psi''(\mathbf{r})]\} = -\mathbf{P} \tag{5.34}$$

Operate with $\hat{z} \cdot \nabla\times$ to give

$$(\nabla^2 + k^2)\nabla_t^2\psi'(\mathbf{r}) = \rho' \qquad \text{where} \qquad \rho' = \hat{z} \cdot \nabla \times \mathbf{P} \tag{5.35}$$

Proceeding similarly by substituting (5.31) into (5.28) gives

$$(\nabla^2 + k^2)\nabla_t^2\psi''(\mathbf{r}) = \rho'' \qquad \text{where} \qquad \rho'' = \hat{z} \cdot \nabla \times \nabla \times \mathbf{P}$$

Integral Solution

Solutions are of the form

$$\psi(\mathbf{r}) = \int_\Omega U_0(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')\, d\mathbf{r}' \tag{5.36}$$

where the kernel can be found from the free space solution of

$$(\nabla^2 + k^2)G_0(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \tag{5.37}$$

where

$$G_0(\mathbf{r}, \mathbf{r}') = -\nabla_t^2 U_0(\mathbf{r}, \mathbf{r}') \tag{5.38}$$

[1] In fact if we substitute into the source free equation for the electric field (5.27) and take the z-component, then the results is

$$(\nabla^2 + k^2)\nabla_t^2\psi''(\mathbf{r}) = 0 \tag{5.32}$$

Formally integrating gives

$$(\nabla^2 + k^2)\psi''(\mathbf{r}) = f(\mathbf{r}) \tag{5.33}$$

where $f(\mathbf{r})$ is a 2D Laplacian. Now we must decide if nontrivial solutions of the Laplace equation are needed. For a half-space problem we are usually interested in a solutions ψ'' that vanishes as $|x| \to \infty$ and $|y| \to \infty$. It seems that this condition cannot be satisfied unless $f(\mathbf{r})$ also vanishes in these limits. If $f(\mathbf{r})$ vanishes as $|x| \to \infty$ and $|y| \to \infty$ and satisfies the Laplace equation it must be zero.

We have shown that

$$G_0(\mathbf{r},\mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{5.39}$$

and that is 2D Fourier transform is

$$\widetilde{G}_0(u,v,z,z') = \frac{1}{2\kappa}e^{-\gamma|z-z'|} \tag{5.40}$$

with $\gamma = \sqrt{u^2+v^2-k^2}$. Therefore $U_0(\mathbf{r},\mathbf{r}')$ is defined but we shall not write down or require an explicit formula for it.

Taking note of the expressions for ρ' and ρ'' in (5.34) and (5.35) we see that starting with the form (5.36) one can integrate by parts to give[2]

$$\psi'(\mathbf{r}) = \int_\Omega \nabla' \times [\hat{z}U_0(\mathbf{r},\mathbf{r}')] \cdot \mathbf{P}(\mathbf{r}')\,d\mathbf{r}' \tag{5.41}$$

and

$$\psi''(\mathbf{r}) = \int_\Omega \nabla' \times \nabla' \times [\hat{z}U_0(\mathbf{r},\mathbf{r}')] \cdot \mathbf{P}(\mathbf{r}')\,d\mathbf{r}' \tag{5.42}$$

These are the transverse electric and transverse magnetic potentials respectively. Note that in an unbounded domain, they are related to the source through the same kernel $U_0(\mathbf{r},\mathbf{r}')$. Substituting (5.41) and (5.41) into (5.29) taking the derivatives inside the integral and comparing the result with (5.21) shows that the dyadic Green's function may be expressed in the form

$$\mathcal{G}_0(\mathbf{r},\mathbf{r}') = \left[\mathcal{I} + \frac{1}{k^2}\nabla\nabla\right]G_0(\mathbf{r}|\mathbf{r}') = \left\{\nabla \times \hat{z}\nabla' \times \hat{z} + \frac{1}{k^2}[\nabla \times (\nabla \times \hat{z})][\nabla' \times (\nabla' \times \hat{z})]\right\}U_0(\mathbf{r},\mathbf{r}') \tag{5.43}$$

which is valid except at the singularity.

The above discussion has avoided the problem of deriving an integral form of the solution that is valid in the electric source region. However the difficulties with the source region and the singularity can be treated by referring back to the vector potential formulation given previously for which we have a valid source region treatment. The advantage of the scalar formulation is that it provides a convenient means of dealing with layered media. It does no deal with the source region or the singularity of the Green's function correctly (although this could possibly done with a more careful analysis).

5.4.2 Source in a Conductive Half-space

The conductor is the source region $z < 0$ adjacent to a non-conductive region $z > 0$ (air for instance). Denote the source region by subscript 0 and the region above it by a subscript 1. Use the index $i = 0, 1$ to indicate the region. The electric field is written

$$\mathbf{E}_i(\mathbf{r}) = -j\omega\mu_i\left\{\nabla \times [\hat{z}\psi'(\mathbf{r})] + \frac{\mu_0}{\mu_i k_0^2}\nabla \times \nabla \times [\hat{z}\psi''(\mathbf{r})]\right\} \tag{5.44}$$

[2]The derivation of the expressions for ψ' and ψ'' make use of the vector identity

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}.$$

Applying Gauss's Divergence Theorem to the resultant $\nabla' \cdot \{\mathbf{P}(\mathbf{r}') \times [\hat{z}U_0(\mathbf{r},\mathbf{r}')]\}$ term gives

$$\int_\Omega \nabla' \cdot \{\mathbf{P}(\mathbf{r}') \times [\hat{z}U_0(\mathbf{r},\mathbf{r}')]\}\,d\mathbf{r}' = \int_S \{\mathbf{P}(\mathbf{r}') \times [\hat{z}U_0(\mathbf{r},\mathbf{r}')]\} \cdot d\mathbf{S}'.$$

The region of the volume integration Ω is that of the source but this can be extended outside the source region without changing the result because in the outer region $\mathbf{P}(\mathbf{r}')$ is zero. Then $\mathbf{P}(\mathbf{r}')$ is zero over the surface S and hence the surface integral vanishes.

which is similar to (5.29) except that the coefficient of the TM term has been modified for the half-space problem. It would seem logical to use $1/k_i^2$ in the coefficient but this will not work because region 1 is air and hence $k_1 = 0$. Therefore we have used $k_1 = 0$. The magnetic field is

$$\mathbf{H}_i(\mathbf{r}) = \nabla \times \nabla \times [\hat{z}\psi'(\mathbf{r})] + \frac{\mu_0 k_i^2}{\mu_i k_0^2} \nabla \times [\hat{z}\psi''(\mathbf{r})], \tag{5.45}$$

Clearly the TM term is zero for the air region. Solutions are to be found for a half-space problem subject to interface conditions between the conductor and air.

$$\begin{aligned} \left[\mu\psi'\right] &= 0 & \qquad \left[\sigma\psi''\right] &= 0 \\ \left[\frac{\partial \psi'}{\partial z}\right] &= 0 & \qquad \left[\frac{\partial \psi''}{\partial z}\right] &= 0 \end{aligned} \tag{5.46}$$

which can be deduced from the continuity of the tangential electric and magnetic fields at the interface. All of this leads us to define a basic half space problem as follows.

Fundamental Solution for Adjoining Half-Spaces

Find the solution of

$$(\nabla^2 + k^2)G(\mathbf{r}, \mathbf{r}') = \begin{cases} 0 & z > 0 \\ -\delta(\mathbf{r} - \mathbf{r}') & z < 0 \end{cases} \tag{5.47}$$

Subject to the interface conditions

$$[\alpha G] = 0 \qquad \text{and} \qquad \left[\frac{\partial G}{\partial z}\right] = 0. \tag{5.48}$$

The two dimensional Fourier transform of the Green's functions are given by

$$\tilde{G}(\kappa, z, z') = \frac{1}{2\gamma}\left[e^{-\gamma|z-z'|} + \Gamma e^{\gamma(z+z')}\right] \qquad (z < 0) \tag{5.49}$$

and

$$\tilde{G}(\kappa, z, z') = \frac{1}{2\gamma} T e^{-\kappa z + \gamma z'} \qquad (z > 0) \tag{5.50}$$

where $\kappa = \sqrt{u^2 + v^2}$ and $\gamma = \sqrt{\kappa^2 - k^2}$, taking roots with positive real parts. The transmission and reflection coefficients are given by

$$\Gamma = \frac{\alpha_1\gamma - \alpha_0\kappa}{\alpha_1\gamma + \alpha_0\kappa} \quad \text{and} \quad T = \frac{2\alpha_1\gamma}{\alpha_1\gamma + \alpha_0\kappa}. \tag{5.51}$$

TE and TM Solutions for Adjoining Half-Spaces

For the TE mode

$$\Gamma' = \frac{\mu_1\gamma - \mu_0\kappa}{\mu_1\gamma + \mu_0\kappa} \quad \text{and} \quad T' = \frac{2\mu_1\gamma}{\mu_1\gamma + \mu_0\kappa}; \tag{5.52}$$

and for the TM mode

$$\Gamma' = -1 \quad \text{and,} \quad T' = 0. \tag{5.53}$$

Hence with

$$G(\mathbf{r}, \mathbf{r}') = -\nabla_t^2 U(\mathbf{r}, \mathbf{r}') \tag{5.54}$$

we can construct Green's function of the form

$$U'(\mathbf{r},\mathbf{r}') = U_0(\mathbf{r},\mathbf{r}') + U_0(\mathbf{r},\mathbf{r}'') + V(\mathbf{r},\mathbf{r}') \tag{5.55}$$

and

$$U''(\mathbf{r},\mathbf{r}') = U_0(\mathbf{r},\mathbf{r}') - U_0(\mathbf{r},\mathbf{r}'') \tag{5.56}$$

where $\mathbf{r}'' = \mathbf{r}' - 2\hat{z}z'$ is the image co-ordinate. Also we define

$$\tilde{V}(z,z') = \frac{1}{2\gamma\kappa^2}(\Gamma' - 1)e^{\gamma(z+z')} = -\frac{\mu_0}{\gamma\kappa(\mu_1\gamma + \mu_0\kappa)}e^{\gamma(z+z')} \tag{5.57}$$

Now we have the scalar form of the fundamental solution we can construct the dyadic form using (5.43) and an similar form for an image Green's dyadic. In the next section we re-derive (5.43) and a similar relationship for an image dyadic Green's function.

5.4.3 Dyadic Representation

The dyadic Green's function is assembled from scalar components using the following identities [7]

$$\nabla_t^2\mathcal{I} = \hat{a}\hat{a}\nabla_t^2 + \nabla_t\nabla_t + \nabla\times\hat{a}\nabla\times\hat{a} \tag{5.58}$$

and

$$\nabla_t^2\nabla\nabla = (\hat{a}\hat{a}\nabla_t^2 + \nabla_t\nabla_t)\nabla^2 - [\nabla\times(\nabla\times\hat{a})][\nabla\times(\nabla\times\hat{a})] \tag{5.59}$$

These can be used to show that

$$\nabla_t^2\left[\mathcal{I} + \frac{1}{k^2}\nabla\nabla\right] = \frac{1}{k^2}(\hat{a}\hat{a}\nabla_t^2 + \nabla_t\nabla_t)(\nabla^2 + k^2) + \nabla\times\hat{a}\nabla\times\hat{a} - \frac{1}{k^2}[\nabla\times(\nabla\times\hat{a})][\nabla\times(\nabla\times\hat{a})] \tag{5.60}$$

hence that

$$\begin{aligned}\mathcal{G}_0(\mathbf{r},\mathbf{r}') \quad &= \quad \frac{1}{k^2}\left(\hat{a}\hat{a} + \frac{\nabla_t\nabla_t}{\nabla_t^2}\right)\delta(\mathbf{r}-\mathbf{r}') + \\ &\left\{\nabla\times\hat{a}\nabla'\times\hat{a} + \frac{1}{k^2}[\nabla\times(\nabla\times\hat{a})][\nabla'\times(\nabla'\times\hat{a})]\right\}U_0(\mathbf{r},\mathbf{r}')\end{aligned} \tag{5.61}$$

where

$$\mathcal{G}_0(\mathbf{r},\mathbf{r}') = \left[\mathcal{I} + \frac{1}{k^2}\nabla\nabla\right]G_0(\mathbf{r},\mathbf{r}') \qquad \text{with} \qquad G_0(\mathbf{r},\mathbf{r}') = -\nabla_t^2 U_0(\mathbf{r},\mathbf{r}') \tag{5.62}$$

Note that if we act an any function, $W(\mathbf{r},\mathbf{r}')$ say, of $x-x'$ $y-y'$ and $z-z'$ that satisfies the Helmholtz equation we get

$$\mathcal{W}_0(\mathbf{r},\mathbf{r}') = \left\{\nabla\times\hat{a}\nabla'\times\hat{a} + \frac{1}{k^2}[\nabla\times(\nabla\times\hat{a})][\nabla'\times(\nabla'\times\hat{a})]\right\}W(\mathbf{r},\mathbf{r}') \tag{5.63}$$

where

$$\mathcal{W}_0(\mathbf{r},\mathbf{r}') = -\left[\mathcal{I} + \frac{1}{k^2}\nabla\nabla\right]\nabla_t^2 W(\mathbf{r},\mathbf{r}') \tag{5.64}$$

Next use (5.60) to operate on a function $\mathcal{I}'Y(\mathbf{r},\mathbf{r}'')$, of $x-x'$ $y-y'$ and $z+z'$ that satisfies the Helmholtz equation. Recall that $\mathcal{I}' = \hat{x}\hat{x} + \hat{y}\hat{y} - \hat{z}\hat{z}$ and $\mathbf{r}'' = \mathbf{r}' - 2\hat{z}z'$. Then we get

$$\nabla_t^2\left[\mathcal{I} + \frac{1}{k^2}\nabla\nabla\right]\mathcal{I}'Y(\mathbf{r}|\mathbf{r}'') = \nabla\times\hat{a}[\nabla\times\hat{a}\mathcal{I}'U(\mathbf{r}|\mathbf{r}'')] - \frac{1}{k^2}[\nabla\times(\nabla\times\hat{a})][\nabla\times(\nabla\times\hat{a})\mathcal{I}'U(\mathbf{r}|\mathbf{r}'')]$$

Hence

$$\mathcal{Y}(\mathbf{r},\mathbf{r}') = \nabla \times \hat{a}[\nabla' \times \hat{a}Y(\mathbf{r}|\mathbf{r}'')] - \frac{1}{k^2}[\nabla \times (\nabla \times \hat{a})][\nabla' \times (\nabla' \times \hat{a})Y(\mathbf{r}|\mathbf{r}'')] \tag{5.65}$$

where

$$\mathcal{Y}(\mathbf{r},\mathbf{r}') = -\left[\mathcal{I}' - \frac{1}{k^2}\nabla\nabla'\right]\nabla_t^2 Y(\mathbf{r}|\mathbf{r}'') \tag{5.66}$$