Theory of eddy current inversion

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The inverse eddy current problem can be described as the task of reconstructing an unknown distribution of electrical conductivity from eddy-current probe impedance measurements recorded as a function of probe position, excitation frequency, or both. In eddy current nondestructive evaluation, this is widely recognized as a central theoretical problem whose solution is likely to have a significant impact on the characterization of flaws in conducting materials. Because the inverse problem is nonlinear, we propose using an iterative least-squares algorithm for recovering the conductivity. In this algorithm, the conductivity distribution sought minimizes the mean-square difference between the predicted and measured impedance values. The gradient of the impedance plays a fundamental role since it tells us how to update the conductivity in such a way as to guarantee a reduction in the mean-square difference. The impedance gradient is obtained in analytic form using function-space methods. The resulting expression is independent of the type of discretization ultimately chosen to approximate the flaw, and thus has greater generality than an approach in which discretization is performed first. The gradient is derived from the solution to two forward problems: an ordinary and an "adjoint" problem. In contrast, a finite difference computation of the gradient requires the solution of multiple forward problems, one for each unknown parameter used in modeling the flaw. Two general types of inverse problems are considered: the reconstruction of a conductivity distribution, and the reconstruction of the shape of an inclusion or crack whose conductivity is known or assumed to be zero. A layered conductor with unknown layer conductivities is treated as an example of the first type of inversion problem. An ellipsoidal crack is presented as an example of the second type of inversion problem.

I. INTRODUCTION

In this article, we develop a general theory for the reconstruction of flaws from eddy current impedance data. Here, the term "flaw" or flaw function is used to signify an arbitrary conductivity distribution, or the departure of the conductivity from a known background. It is well known that the reconstruction of an unknown conductivity variation from eddy current data is a nonlinear problem, since both the conductivity itself and the electromagnetic field at the flaw are simultaneously unknown. If the conductivity variations are small, the inverse problem can be linearized by replacing the fields inside the conductor with the fields in the absence of the flaw (the Born approximation).[1] In many cases, however, the weakly varying conductivity assumption is poor—a crack is an obvious example—and any realistic approach must take account of the inherent nonlinearity of the problem. Here we propose using the least-squares criterion as the basis of an iterative scheme for reconstructing the flaw function.[2]

In the least-squares approach, a flaw function is sought that minimizes the mean-square difference between the actual measurements and the predicted measurements derived by solving the forward problem using the current

estimate of the flaw conductivity. Thus, an iterative scheme proceeds by updating the flaw function at each iteration in such a way that the mean-square difference or error between the measured and predicted data is driven to zero, or to a minimum in the presence of noise and model inaccuracies.

The mean-square error may be regarded as a functional over the function space comprised of all possible flaw functions and our task is to select the flaw function that minimizes this functional using some suitable descent algorithm. Possible descent algorithms include the method of steepest descent, the conjugate gradient algorithm, and the Levenberg–Marquardt algorithm. A key requirement of any such algorithm is the ability to compute the gradient of the mean-square error at each iteration. The gradient tells us in what "direction" to update the current flaw estimate in order to guarantee a reduction in the mean-square error. A straightforward but tedious approach to computing the gradient is to first discretize the flaw function in some convenient way, for example by dividing it into small blocks or other basis elements, and then to compute the partial derivatives of the measurements with respect to each basis element by taking finite differences. This procedure is computationally expensive since a flaw function broken into N blocks would require the solution to N separate forward problems to obtain the N components of

the impedance gradient. Moreover, N solutions need to be recomputed at every iteration.

An alternative to finite differences is used by some geophysicists to develop nonlinear inversion schemes for electromagnetic survey data. In their approach, the conductivity is modeled using a number of undetermined parameters (say N). The field equations are then differentiated explicitly with respect to these model parameters. This general procedure has been applied both to two-dimensional problems[3] and in three dimensions.[4] The resulting equations for the field derivatives are of similar form to the forward problem field equations and can be solved in the same way, but we now have N additional equations that must be solved for each observation at every iteration.

Below we use function space methods to derive an explicit expression for the gradient in terms of the solutions of *two* forward problems instead of N forward problems. The two forward problems correspond to an "ordinary" forward problem and an "adjoint" problem defined below. In certain cases, for example a planar layered structure of unknown conductivity, the ordinary and adjoint problems are identical, in which case only one forward solution need be computed per iteration. The analysis does not require discretization, and thus is independent of the type of discretization chosen at a later stage. In addition, since the gradient is obtained analytically, it is exact, in contrast to the gradient obtained through a finite-difference computation.

The article is organized as follows: We begin with a brief review of the forward problem (Sec. II). A more complete discussion of the forward problem and its numerical implementation can be found in Ref. 5. The inverse problem is then formulated in continuous space using the least-square error criterion and an explicit expression for the gradient of the mean-square error is derived (Sec. III). The expression for the gradient is a central result of the paper. The role the gradient plays in the formulation of two types of flaw reconstruction problems is then described: in the first problem, the flaw function to be reconstructed is assumed to be an arbitrary volume distribution of conductivity (Sec. III); in the second, the flaw is assumed to be a void or crack inside of which the conductivity vanishes (Sec. IV). In the latter problem, the shape of the flaw boundary is to be determined. Since the continuous-space formulation is both more compact and more general, the analysis is carried out without discretization up to this point. In a numerical implementation of the theory, however, a discrete representation of the flaw is ultimately required, and we show how the continuous-space results can be employed in deriving inversion algorithms for a flaw (or flaw surface) approximated by a finite number of undetermined parameters (Sec. V).

To illustrate how the theory can be used in solving specific problems, the gradient of the mean-square error is derived explicitly for a layered conductor and a surface-breaking hemispherical flaw. The latter is an example of a parametric flaw model where a small number of free parameters are used to define the conductivity distribution or the flaw shape. For the case of the layered conductor (Sec.

VI), we consider two possible parameterizations. In the first, the layer thicknesses are assumed known, and the conductivities of the individual layers are to be determined. In the second parameterization, the layer thicknesses are to be determined, but the layer conductivities are assumed known. As a final example of Sec. VI, we consider a single layer overlying a substrate of known conductivity. Here we wish to determine the thickness and conductivity of the layer. We conclude with a simple illustration of an explicit impedance gradient calculation in which the radius of a hemispherical surface indentation is varied and the derivative of the impedance determined in the low frequency limit (Sec. VII).

In the above problems, it is assumed that impedance data may be recorded as a function of frequency or probe position, or both, whichever is appropriate. In fact the present formulation is sufficiently general to accommodate other probe variables or a combination of such variables.

II. THE FORWARD PROBLEM

Let the flaw function be defined by $v(\mathbf{r}) = [\sigma(\mathbf{r}) - \sigma_0]/\sigma_0$, where $\sigma(\mathbf{r})$ is a spatially varying conductivity and σ_0 is the host conductivity. If the flaw is assumed to be a void or crack, $\sigma(\mathbf{r})$ vanishes inside the flaw, in which case $v(\mathbf{r}) = -1$ inside the flaw and is zero outside. If the flaw represents an arbitrary conductivity distribution, we assume for convenience that $\sigma(\mathbf{r}) \equiv \sigma_0$ outside of some bounded domain V, so that $v(\mathbf{r}) \equiv 0$ outside of V. Thus, volume integrations involving the flaw function vanish outside of V. Whether the conductivity is continuous or discrete, we refer to V as the volume of the flaw region.

Assuming time-harmonic excitation $[\exp(-i\omega t)]$ and neglecting the displacement current, Maxwell's equations read

$$\nabla \times \mathbf{E}(\mathbf{r}) = i\omega\mu_0 \mathbf{H}(\mathbf{r})$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = \sigma(\mathbf{r})\mathbf{E}(\mathbf{r}) = \sigma_0\mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r}), \qquad (1)$$

where

$$\mathbf{P}(\mathbf{r}) \equiv [\sigma(\mathbf{r}) - \sigma_0]\mathbf{E}(\mathbf{r}) = \sigma_0 v(\mathbf{r})\mathbf{E}(\mathbf{r}). \qquad (2)$$

In (1), $\mathbf{P}(\mathbf{r})$ may be interpreted as an effective current dipole density at the flaw due to the variation $\sigma(\mathbf{r}) - \sigma_0$ of the conductivity from that of the host σ_0. With the aid of Green's theorem, the required solution of Maxwell's equations may be expressed in integral form as

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{(i)}(\mathbf{r}) + i\omega\mu_0 \int_V \mathbf{G}(\mathbf{r}|\mathbf{r}') \cdot \mathbf{P}(\mathbf{r}') d\mathbf{r}', \qquad (3)$$

where the integration is over the volume of the flaw. In (3), $\mathbf{E}^{(i)}(\mathbf{r})$ is the "incident field" produced by a primary source in the absence of the flaw, and $\mathbf{G}(\mathbf{r}|\mathbf{r}')$ is the dyadic Green's function obeying[5]

$$\nabla \times \nabla \times \mathbf{G}(\mathbf{r}|\mathbf{r}') - k^2 \mathbf{G}(\mathbf{r}|\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')\mathbf{I}.$$

In this equation, $k^2 \equiv i\omega\mu_0\sigma_0$, $\delta(\mathbf{r} - \mathbf{r}')$ is the three-dimensional (3D) Dirac delta function and $\mathbf{I}$ is the unit dyad. Multiplying (3) by $\sigma_0 v(\mathbf{r})$, we obtain in view of (2),

$$P(r) = P^{(i)}(r) + v(r)k^2 \int_V G(r|r') \cdot P(r')dr', \quad (4)$$

where

$$P^{(i)}(r) \equiv \sigma_0 v(r)E^{(i)}(r). \quad (5)$$

For unit source current, the probe impedance Z due to the flaw is given by

$$Z = -\int_{coil} E^{(s)}(r') \cdot J(r')dr',$$

where the integration is over the sensing coil, $J(r)$ is the current density in the source coil and $E^{(s)}(r) \equiv E(r) - E^{(i)}(r)$ is the "scattered field" due to the flaw given by the integral on the right of (3). The application of a reciprocity theorem relating the scattered field at the coil to the incident field at the flaw shows that the impedance may be expressed as[5]

$$Z = -\int_V E^{(i)}(r') \cdot P(r')dr', \quad (6)$$

where the integration is over the volume of the flaw.

The forward problem may be defined as follows: given the flaw function $v(r)$ and the incident field $E^{(i)}(r)$ produced by the primary source, compute the impedance Z. To accomplish this, the linear integral Eq. (4) is solved for the polarization $P(r)$, which in turn is substituted into (6). Further details on the above formulation of the forward problem and its numerical solution are contained in Ref. 5. A discussion of reciprocity theorems can be found in Ref. 6.

III. RECONSTRUCTION OF AN ARBITRARY CONDUCTIVITY DISTRIBUTION

Suppose $v(r)$ is an estimate of the true flaw function $v_{true}(r)$. Then, on the basis of the estimate $v(r)$, we solve the forward problem to obtain the predicted measurement. Here the measurement is the impedance change due to the flaw Z detected by a probe sensor at the probe position r_0 at an excitation frequency ω. For brevity, the predictions and measurements are expressed as a function of the vector $m = (r_0, \omega)$, where m varies over some observation domain M.

Denote the predicted impedance by $Z[m,v]$, which is a functional on the space of functions $v(r)$. Let $Z_{obs}(m)$ denote the observed impedance, so that $Z[m,v_{true}] = Z_{obs}(m)$ in the absence of noise and modeling uncertainties. (In the following, we generally use square brackets, [ ], to signify a functional relationship as opposed to a function.) Now define the mean-square difference between the predicted and measured impedances as

$$\mathscr{E}[v] = \int_M W(m)|Z[m,v] - Z_{obs}(m)|^2 dm, \quad (7)$$

where $W(m)$ is a real, positive weighting function that could reflect probabilistic information about the measurements. If the measurement space is discrete, then (7) is replaced by a sum.

We pause to point out that it may be useful at this stage to regularize a potentially ill-conditioned problem by imposing an *a priori* constraint of some form, in addition to minimizing the mean-square error (7). In the case of a discrete flaw, one could, for example, attempt to select the flaw that simultaneously minimizes (7) and minimizes the flaw surface area or the flaw volume. One strategy is to add a penalty functional to (7), for example to minimize a functional of the form $\mathscr{E}[v] + \lambda \operatorname{Vol}[v]$ or $\mathscr{E}[v] + \lambda \operatorname{Sur}[v]$, where Vol $[v]$ and Sur$[v]$ are, respectively, expressions for the flaw volume and the flaw surface area, and λ is a weighting parameter. Various types of smoothing constraints are also possible.

Now consider an iterative algorithm which, given the $(n-1)$th estimate, $v_{n-1}(r)$, of $v_{true}(r)$, generates the nth estimate by means of

$$v_n(r) = v_{n-1}(r) + \alpha_n \delta v_n(r), \quad (8)$$

where α_n is a constant depending on n and the function $\delta v_n(r)$ may be regarded as an incremental change in $v_{n-1}(r)$. To find the variation $\delta v(r)$ (dropping temporarily the subscript n) that maximizes the reduction in mean-square error when the current estimate is $v(r)$, we write the Gateaux differential of $\mathscr{E}[v]$, denoted by $d\mathscr{E}[v,\delta v]$, and defined as the change in $\mathscr{E}$ resulting from the change δv in v, as follows[7,8]

$$d\mathscr{E}[v,\delta v] \equiv \lim_{\beta \to 0} \frac{\mathscr{E}[v+\beta\delta v] - \mathscr{E}[v]}{\beta}$$

$$= \frac{d}{d\beta}\mathscr{E}[v+\beta\delta v]\Big|_{\beta=0}. \quad (9)$$

Here, and henceforth, the boldface d is used to denote a functional differential resulting from the variation of some function, where the variation is denoted by the symbol δ. In the present section, the function to be varied is the (continuous) flaw function $v(r)$; in the next section, we vary the boundary of the flaw function.

The functional gradient of $\mathscr{E}[v]$ with respect to v, denoted by $\nabla\mathscr{E}[v]$, is related to the differential $d\mathscr{E}[v,\delta v]$ by

$$d\mathscr{E}[v,\delta v] \equiv \int_V \nabla\mathscr{E}[v(r)]\delta v(r)dr. \quad (10)$$

Note that $d\mathscr{E}$ may be thought of as the continuous-space analogue of the discrete-space directional derivative, where (10) is analogous to the dot product of the gradient $\nabla\mathscr{E}$ and the "vector" $\delta v(r)$.

Equation (10) shows that the largest decrease in the mean-square error is obtained by selecting the direction $\delta v(r) = -\alpha\nabla\mathscr{E}[v]$, α being a constant; thus, (8) becomes

$$v_n(r) = v_{n-1}(r) - \alpha_n\nabla\mathscr{E}[v_{n-1}(r)]. \quad (11)$$

This is the steepest descent algorithm. Here α_n is a "stepsize" parameter which tells us how far to advance in the direction $\nabla\mathscr{E}$ to obtain the next update v_n. For nonlinear problems, the parameter α_n is typically determined by means of a one-dimensional (1D) line search that minimizes $\mathscr{E}[v_{n-1} - \alpha_n\nabla\mathscr{E}_{n-1}]$ with respect to α_n while holding v_{n-1} and $\nabla\mathscr{E}_{n-1}$ fixed. Such a search would normally re-

quire several more solutions to the forward problem. However, if the shape of the surface $\mathscr{S}[v]$ is approximately quadratic, an approximate, but explicit, formula for α_n can be derived that requires no additional solutions to the forward problem, resulting in a significant savings in computing time. A derivation of the formula for α_n is given in Appendix A.

The conjugate gradient algorithm is similar in form to (11), and also employs the gradient $\nabla\mathscr{S}$, but can be shown to accelerate convergence, often significantly, with relatively little additional computational expense.[9–12] This algorithm updates the previous flaw estimate $v_{n-1}(\mathbf{r})$ as follows:

$$v_n(\mathbf{r}) = v_{n-1}(\mathbf{r}) + \alpha_n f_n(\mathbf{r}), \tag{12}$$

where

$$f_n(\mathbf{r}) = -\nabla\mathscr{S}[v_{n-1}(\mathbf{r})] + \beta_n f_{n-1}(\mathbf{r}), \tag{13}$$

with the initial condition $f_1(\mathbf{r}) = -\nabla\mathscr{S}[v_0(\mathbf{r})]$. Here α_n plays a role similar to the step-size parameter in the steepest-descent algorithm. Explicit formulas for the parameters α_n and β_n are given in Appendix A.

The Levenberg–Marquardt algorithm[11,12] is a modification of the Gauss–Newton method and as a quadratic scheme, has the potential for very rapid convergence, but the price paid is the need to invert a system of linear equations at each iteration. For a large number of unknowns, this may be impractical, in which case the recommended method is the conjugate gradient algorithm. The point to appreciate, however, is that all of the above algorithms—steepest descent, conjugate gradient, or Levenberg–Marquardt—require the ability to compute the mean-square error gradient $\nabla\mathscr{S}[v(\mathbf{r})]$ at each iteration. We now turn to the derivation of this gradient.

Substituting (7) into (9) and taking the limit, gives

$$d\mathscr{S}[v,\delta v] = 2\,\mathrm{Re}\int_M W(\mathbf{m})\{Z[\mathbf{m},v] - Z_{\mathrm{obs}}(\mathbf{m})\}^*$$

$$\times dZ[\mathbf{m};v,\delta v]dm, \tag{14}$$

where Re means real part, * denotes complex conjugate, and dZ is the Gateaux differential of the impedance Z. In the equations of the previous section, the dependence on the observation parameter $\mathbf{m}$ is not explicitly shown. Except in equations like (14) where there is an integral over the observation domain, we will usually continue to suppress this dependence for brevity, but one must keep in mind that the impedance Z and all field quantities $\mathbf{E}^{(i)}$, $\mathbf{E}$, $\mathbf{P}^{(i)}$, and $\mathbf{P}$ depend on $\mathbf{m}$.

The impedance $Z[v]$, defined by (6), is a functional of $v(\mathbf{r})$. The functional differential of $Z[v]$ is defined as in (9) with $Z[v]$ replacing $\mathscr{S}[v]$. Thus by definition

$$dZ[v,\delta v] \equiv \int \nabla Z(\mathbf{r})\delta v(\mathbf{r})d\mathbf{r}. \tag{15}$$

To relate the functional gradient of the mean-square error $\nabla\mathscr{S}[v]$ to the functional gradient of the impedance $\nabla Z[v]$ (15) is substituted into (14) and the orders of integration interchanged to yield

$$d\mathscr{S}[v,\delta v] = \int\left(2\,\mathrm{Re}\int_M W(\mathbf{m})\{Z[\mathbf{m},v] - Z_{\mathrm{obs}}(\mathbf{m})\}^*\right.$$

$$\left.\times\nabla Z(\mathbf{m},\mathbf{r})dm\right)\delta v(\mathbf{r})d\mathbf{r}.$$

Comparing this to Eq. (10) shows that

$$\nabla\mathscr{S}(r) = 2\,\mathrm{Re}\int_M W(\mathbf{m})\{Z[\mathbf{m},v] - Z_{\mathrm{obs}}(m)\}^*$$

$$\times\nabla Z(\mathbf{m},\mathbf{r})dm, \tag{16}$$

where we have indicated explicitly the dependence of ∇Z on $\mathbf{m}$.

To compute $\nabla\mathscr{S}(\mathbf{r})$ from (16), a formula for the functional gradient of the impedance, $\nabla Z(\mathbf{r})$ is needed and is here derived from the equations of the forward problem. Below, all integrations are over the flaw region, unless noted otherwise. Taking the differential of (6) gives

$$dZ = -\int \mathbf{E}^{(i)}(\mathbf{r}')\cdot\mathbf{dP}(\mathbf{r}')d\mathbf{r}'. \tag{17}$$

Similarly, the differential of Eq. (4), after substituting Eq. (5) for $\mathbf{P}^{(i)}(\mathbf{r})$, is

$$\mathbf{dP}(\mathbf{r}) = \delta v(\mathbf{r})\sigma_0\mathbf{E}^{(i)}(\mathbf{r}) + \delta v(\mathbf{r})k^2\int \mathbf{G}(\mathbf{r}|\mathbf{r}')\cdot\mathbf{P}(\mathbf{r}')d\mathbf{r}'$$

$$+ v(\mathbf{r})k^2\int \mathbf{G}(\mathbf{r}|\mathbf{r}')\cdot\mathbf{dP}(\mathbf{r}')d\mathbf{r}'$$

$$= \delta v(\mathbf{r})\sigma_0\mathbf{E}(\mathbf{r}) + v(\mathbf{r})k^2\int \mathbf{G}(\mathbf{r}|\mathbf{r}')\cdot\mathbf{dP}(\mathbf{r}')d\mathbf{r}', \tag{18}$$

where the second line follows on substituting $\mathbf{E}(\mathbf{r})$ from (3). To obtain an equation for dZ in which the differential $\mathbf{dP}$ does not appear, we proceed as follows. Define $\tilde{\mathbf{E}}(\mathbf{r})$ as the solution to

$$\mathbf{E}^{(i)}(\mathbf{r}') = \tilde{\mathbf{E}}(\mathbf{r}') - k^2\int \tilde{\mathbf{G}}(\mathbf{r}'|\mathbf{r})\cdot\tilde{\mathbf{E}}(\mathbf{r})v(\mathbf{r})d\mathbf{r}, \tag{19}$$

where $\tilde{\mathbf{G}}$ is the adjoint of $\mathbf{G}$; that is

$$\tilde{\mathbf{G}}(\bar{\mathbf{r}}'|\mathbf{r}) \equiv \mathbf{G}^T(\mathbf{r}|\mathbf{r}'), \tag{20}$$

with T denoting transpose of the dyad.

Note that (19) is similar in structure to (4) as can be seen by defining $\tilde{\mathbf{P}}(\mathbf{r}) = \sigma_0\tilde{\mathbf{E}}(\mathbf{r})v(\mathbf{r})$, multiplying (19) by $\sigma_0 v(\mathbf{r}')$ and interchanging $\mathbf{r}$ and $\mathbf{r}'$ to give

$$\tilde{\mathbf{P}}(\mathbf{r}) = \mathbf{P}^{(i)}(\mathbf{r}) + v(\mathbf{r})k^2\int \tilde{\mathbf{G}}(\mathbf{r}|\mathbf{r}')\cdot\tilde{\mathbf{P}}(\mathbf{r}')d\mathbf{r}'. \tag{21}$$

(21) differs from (4) only in that it contains the adjoint dyad.

Now substituting (19) into (17), interchanging orders of integration and using (20) gives

$$dZ = -\int \widetilde{\mathbf{E}}(\mathbf{r}') \cdot d\mathbf{P}(\mathbf{r}') d\mathbf{r}'$$

$$+ k^2 \int v(\mathbf{r}) \widetilde{\mathbf{E}}(\mathbf{r}) \cdot \left[\int \mathbf{G}(\mathbf{r}|\mathbf{r}') \cdot d\mathbf{P}(\mathbf{r}') d\mathbf{r}' \right] d\mathbf{r}$$

$$= -\int \widetilde{\mathbf{E}}(\mathbf{r}) \cdot \left[d\mathbf{P}(\mathbf{r}) \right.$$

$$\left. - v(\mathbf{r}) k^2 \int \mathbf{G}(\mathbf{r}|\mathbf{r}') \cdot d\mathbf{P}(\mathbf{r}') d\mathbf{r}' \right] d\mathbf{r}. \qquad (22)$$

The quantity in brackets is seen from (18) to be $\delta v(\mathbf{r}) \sigma_0 \mathbf{E}(\mathbf{r})$. Thus (22) reduces to

$$dZ[v, \delta v] = -\sigma_0 \int \widetilde{\mathbf{E}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) \delta v(\mathbf{r}) d\mathbf{r}. \qquad (23)$$

Comparing (23) and (15) shows that the functional gradient of the impedance is given by

$$\nabla Z(\mathbf{r}) = -\sigma_0 \widetilde{\mathbf{E}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}). \qquad (24)$$

This is a key result. $\nabla Z(\mathbf{r})$ gives the differential change in the impedance due to a differential change in the flaw function v at the point $\mathbf{r}$ in the flaw. For an arbitrary variation in the flaw $\delta v(\mathbf{r})$, the total differential change in the impedance dZ is given by the integral of $\delta v(\mathbf{r})$ over the gradient, as shown by (23).

From (24) we see that the computation of the functional gradient ∇Z requires the numerical solution to two "forward problems," defined, respectively, by (4) and its adjoint (21) [or equivalently (19)]. Once ∇Z is computed, the gradient of the mean-square error, $\nabla \mathscr{E}$, is obtained from (16), which in turn can be substituted into the descent algorithm of one's choice to reconstruct the flaw.

The above derivation can be expressed more abstractly, and compactly, using an operator formalism (Appendix B).

IV. RECONSTRUCTION OF A HOMOGENEOUS FLAW

In the above, the flaw function is defined by $v(\mathbf{r}) = [\sigma(\mathbf{r}) - \sigma_0]/\sigma_0$, where $\sigma(\mathbf{r})$ is regarded as an arbitrary distribution of conductivity. In dealing with potentially ill-conditioned problems, it is helpful if the number of degrees of freedom allowed by the proposed form of the flaw function can be restricted in some way, for example, by using *a priori* knowledge. In this section, we consider the problem of finding the shape of a flaw in the form of a void or crack of finite volume having a conductivity that is known to be zero. Thus $\sigma(\mathbf{r}) = 0$ inside the flaw and $\sigma(\mathbf{r}) = \sigma_0$ outside, or $v(\mathbf{r}) = -1$ inside the flaw and $v(\mathbf{r}) = 0$ outside.

Our ultimate task is to find the shape of the flaw boundary, but initially we need to determine the mean-square error gradient with respect to a variation in the position of the boundary. The gradient can then be substituted into any of the descent algorithms discussed in the Introduction to find the best estimate of the position of the flaw surface. We consider a variation over a regular surface S_0 which, for an embedded flaw, can be its entire bounding surface. In the case of a surface breaking flaw, as shown in Fig. 1, it is only necessary to consider a variation at the
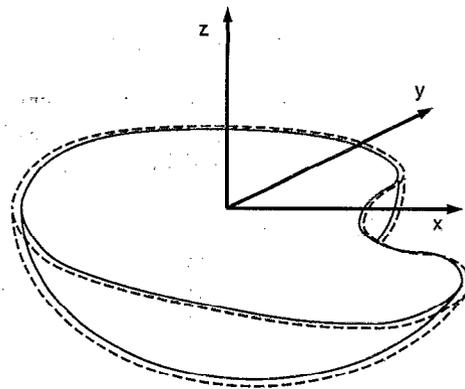
FIG. 1. Homogeneous volumetric flaw whose boundary is varied from that indicated by the dashed outline to the location shown as the shaded region.

conductor flaw interface and not the interface with air. We remark that our approach can also be adapted to the problem of reconstructing the shape of an inclusion with a constant, but known, interior conductivity.

We represent the flaw surface by the equation $s(\mathbf{r}) = 0$, in order that the flaw function may be written as

$$v(\mathbf{r}) = -H[s(\mathbf{r})],$$

where $H(u)$ is the Heaviside step function: $H(u) = 1$ for $u \geqslant 0$ and $H(u) = 0$ for $u < 0$. $s(\mathbf{r})$ is greater than zero inside the flaw and less than zero outside. For a variation of the boundary, the change in the flaw function is

$$\delta v(\mathbf{r}) = -\{H[s(\mathbf{r}) + \delta s(\mathbf{r})] - H[s(\mathbf{r})]\}.$$

Clearly $\delta v(\mathbf{r})$ is nonzero only in the region between the surface of the original flaw and the surface after the variation. Later, we shall consider an infinitesimal variation, but to understand and interpret Eq. (23) in the present circumstances, consider for the moment that the change in the position of the boundary is due to a small but *finite* contraction of the flaw region. (23) then reduces to an integration over the domain of contraction, illustrated in Fig. 1 as the region between the surface outlined by dashed lines and the shaded surface.

$\widetilde{\mathbf{E}}(\mathbf{r})$, as given by (19), is the adjoint field due to the original flaw. In the contraction region, this field is *inside* the unvaried flaw. $\mathbf{E}(\mathbf{r})$, on the other hand, is derived from the term in brackets in (22) and therefore represents the *external* field of the varied flaw. This difference between internal and external fields is important because there is a jump in the normal component of the electric field at the surface. Indeed, because the conductivity within the flaw is here taken to be zero, the normal component of the external electric field at the surface, like that of the corresponding current density, is zero. Therefore, in the limit of an infinitesimal variation, the tangential field, $\mathbf{E}_t(\mathbf{r})$ can replace $\mathbf{E}(\mathbf{r})$ in (23), because the normal component is zero.

For a homogeneous flaw of the type considered in this section, the equation analogous to (23) is

$$dZ(s,\delta s) = -\sigma_0 \int \widetilde{\mathbf{E}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) dv[s,\delta s] d\mathbf{r}, \quad (25)$$

where $\delta s(\mathbf{t})$ is an arbitrary variation of the flaw surface here written as a function of the variable $\mathbf{t}$, a two component vector representing surface coordinates on the unvaried flaw. In Eq. (25), $v(\mathbf{r})$ can itself be regarded as a functional of the flaw surface $s(\mathbf{t})$, that is, $v = v[s]$. We have indicated this by writing dv, instead of δv, in (25).

By definition, the functional gradient of the impedance with respect to the flaw surface, $\nabla_s Z(\mathbf{t})$, is related to the functional differential $dZ[s,\delta s]$ by

$$dZ[s,\delta s] \equiv \int_{S_0} \nabla_s Z(\mathbf{t}) \delta s(\mathbf{t}) d\mathbf{t}. \quad (26)$$

An infinitesimal variation of the flaw function is given by

$$dv[s,\delta s] \equiv \frac{d}{d\beta} v(s+\beta\delta s) \Big|_{\beta=0}$$

$$= -\frac{d}{d\beta} H(s+\beta\delta s) \Big|_{\beta=0}$$

$$= -\delta(s(\mathbf{r}))\delta s(\mathbf{t}),$$

where $\delta(z)$ is the 1D Dirac delta function. Substituting this result into (25) and recalling that the normal component of the external field is zero, gives

$$dZ[s,\delta s] = \sigma_0 \int \widetilde{\mathbf{E}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) \delta[s(\mathbf{r})] \delta s(\mathbf{t}) d\mathbf{r}$$

$$= \sigma_0 \int_{S_0} \widetilde{\mathbf{E}}_t(\mathbf{t},s) \cdot \mathbf{E}_t(\mathbf{t},s) \delta s(\mathbf{t}) d\mathbf{t}. \quad (27)$$

Comparing (27) and (26) shows that

$$\nabla_s Z(\mathbf{t}) = \sigma_0 \widetilde{\mathbf{E}}_t(\mathbf{t},s(\mathbf{t})) \cdot \mathbf{E}_t(\mathbf{t},s(\mathbf{t})) = \sigma_0 \widetilde{\mathbf{E}}_t \cdot \mathbf{E}_t \big|_{\text{flaw surface}}, \quad (28)$$

which is the "flaw-surface" analogue of (24).

Finally, if we replace v and δv by s and δs in (14), substitute dZ from (26) and interchange orders of integration, we obtain a mean-square error gradient similar in form to (16)

$$\nabla \mathscr{E}(\mathbf{t}) = 2 \operatorname{Re} \int_M W(\mathbf{m}) \{Z[\mathbf{m},s] - Z_{\text{obs}}(\mathbf{m})\}^*$$

$$\times \nabla_s Z(\mathbf{m},\mathbf{t}) d\mathbf{m}, \quad (29)$$

where $\nabla Z(\mathbf{m},\mathbf{t})$ is defined by (28). The updated formula for the steepest descent algorithm, for example, would now read

$$s_n(\mathbf{t}) = s_{n-1}(\mathbf{t}) - \alpha_n \nabla \mathscr{E}[s_{n-1}(\mathbf{t})], \quad (30)$$

instead of (11).

V. FINITE NUMBER OF FLAW PARAMETERS

Until now, the flaw function $v(\mathbf{r})$ or its boundary $s(\mathbf{t})$ was assumed to be an arbitrary function in an infinite-dimensional function space. In a numerical implementation, however, parameterization of the flaw function at some stage is required; that is, the flaw function must ultimately be approximated, or modeled, using a finite number of variables. These variables could, for example, represent the mean conductivities within small boxes, or other basis elements, in terms of which the true conductivity is approximated; or, perhaps, they could define the geometric shape of a flaw boundary. The inverse problem then reduces to the task of determining this finite set of numbers. In this section, we derive the form of the mean-square error gradient $\nabla \mathscr{E}$, when the flaw function is parameterized in this way.

Let $\mathbf{p}$ be a vector whose components are the flaw parameters. In (23) we now replace δv with

$$dv[\mathbf{p},\delta\mathbf{p}] = \sum_i \frac{\partial v(\mathbf{r})}{\partial p_i} \delta p_i, \quad (31)$$

which is the change in the flaw function due to the variation in the parameters, δp_i. Equation (23) then becomes[13]

$$dZ[\mathbf{p},\delta\mathbf{p}] = -\sigma_0 \sum_i \left[\int \widetilde{\mathbf{E}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) \frac{\partial v(\mathbf{r})}{\partial p_i} d\mathbf{r} \right] \delta p_i, \quad (32)$$

$$\equiv \sum_i (\nabla Z)_i \delta p_i, \quad (33)$$

where $(\nabla Z)_i \equiv \partial Z/\partial p_i$.

Equation (33) is the discrete analogue of (15). From (32) and (33) the ith component of the gradient of the impedance is seen to be

$$(\nabla Z)_i = -\sigma_0 \int \widetilde{\mathbf{E}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) \frac{\partial v(\mathbf{r})}{\partial p_i} d\mathbf{r}. \quad (34)$$

Now, the differential change in the mean-square error is

$$d\mathscr{E}[\mathbf{p},\delta\mathbf{p}] \equiv \sum_i (\nabla \mathscr{E})_i \delta p_i. \quad (35)$$

Substituting (33) and (35) into (14), we obtain for the ith component of the mean-square error gradient

$$(\nabla \mathscr{E})_i = 2 \operatorname{Re} \int_M W(\mathbf{m})$$

$$\times \{Z[\mathbf{m},\mathbf{p}] - Z_{\text{obs}}(\mathbf{m})\}^* (\nabla Z(\mathbf{m}))_i d\mathbf{m}, \quad (36)$$

which is the discrete analogue of (16). The update equation for the method of steepest descent now becomes

$$p_i^n = p_i^{n-1} - \alpha_n (\nabla \mathscr{E}^{n-1})_i, \quad (37)$$

where n indexes the iteration number. Equation (37) is the discrete flaw space analogue of (11).

As an example of a finite parameterization of the flaw function $v(\mathbf{r})$ let us write

$$v(\mathbf{r}) = \sum_{i=1}^N c_i \phi_i(\mathbf{r}), \quad (38)$$

where $\phi_i(\mathbf{r})$, $i=1,\ldots,N$ represent some convenient set of linearly independent basis functions and the c_i's are expansion coefficients. For instance, if $v(\mathbf{r})$ is sampled at a set of discrete points $\mathbf{r}_i$, $i=1,\ldots,N$, the basis functions may be written $\phi_i(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}_i)$, where $\delta(\mathbf{r})$ is the 3D Dirac delta

function. In the next section, we consider a layered conductor, in which case the basis functions are piece-wise constant between the layer boundaries.

If we let the coefficients in (38) play the role of the parameters, that is $p_i = c_i$, then substituting (38) into (34) yields

$$(\nabla Z)_i = -\sigma_0 \int \tilde{\mathbf{E}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) \phi_i(\mathbf{r}) d\mathbf{r}, \qquad (39)$$

for the components of the gradient of the impedance.

In the latter example, the expansion coefficients c_i were regarded as the unknowns. In some problems, the basis functions may themselves be functions of the parameters, and thus can vary. Normally, either the coefficients c_i are allowed to vary or the basis functions ϕ_i are allowed to vary, but not both (an exception is case 3 of Sec. V). Both of these situations are illustrated in the next section in the example of a layered conductor. In that section, two problems are treated. In the first, we assume the layer thicknesses are known, and the conductivities of the individual layers are to be determined. This corresponds to unknown c_i (the normalized conductivities) and known basis functions ϕ_i. In the second problem, the normalized conductivities (c_i) are assumed known, and the layer thicknesses are to be determined. In this case, the coefficients c_i are known, but the basis functions ϕ_i are allowed to vary.

One could formulate a problem in which both the conductivities and the layer thicknesses are simultaneously unknown, but for more than a single layer, this leads to a nonunique solution. This is a consequence of the fact that different combinations of thicknesses and conductivities can give rise to the same set of impedance measurements. Under these conditions, a unique reconstruction of the thicknesses and conductivities is not possible. The latter, therefore, is an example of a poor parameterization of the flaw function. The general issue of choosing an effective flaw parameterization is related to the fundamental problem of estimating the sensitivity (or lack of sensitivity) of the measurements to variations in the flaw parameters. This sensitivity can be checked by computing, for example, the singular values of a matrix symbolically denoted

$$\mathcal{M}_{\mathbf{m},\mathbf{p}} \equiv \nabla Z(\mathbf{m},\mathbf{p}),$$

where $\nabla Z(\mathbf{m},\mathbf{p})$ is the gradient of the impedance. As before, $\mathbf{p}$ signifies any vector of parameters used in modeling the flaw. Note that $\mathcal{M}_{\mathbf{m},\mathbf{p}}$ is just the matrix of partial derivatives of the measurements with respect to the unknown parameters, i.e., $\mathcal{M}_{m_j p_i} = \partial Z(m_j)/\partial p_i$. If this matrix is singular, or nearly singular, then the measurements are incapable of uniquely recovering the flaw parameters. In fact, a test of this type applied to any proposed measurement scheme—carried out by computing the singular-value decomposition of $\nabla Z(\mathbf{m},\mathbf{p})$ on a number of conceivable flaws—is an extremely effective way of evaluating the sensitivity of the generated data to variations in the flaw parameters, and hence the potential success of an inversion method in recovering the flaw parameters. Such a test would also reveal potential nonuniqueness problems in a proposed flaw parameterization, if $\nabla Z(\mathbf{m},\mathbf{p})$ is singular, or

ill-conditioning problems of an inversion scheme, if $\nabla Z(\mathbf{m},\mathbf{p})$ is nearly singular (that is, if $\nabla Z(\mathbf{m},\mathbf{p})$ has a large condition number).

VI. EXAMPLE: A LAYERED CONDUCTOR

As an illustration of the preceding theory, consider a conductor composed of N parallel layers, where the conductivity of the ith layer is σ_i. The z axis is assumed normal to the layer boundaries, and the boundaries of the ith layer are at depth coordinates z_i and z_{i+1}. The conductivity distribution $\sigma(z)$ may then be expressed

$$\sigma(z) = \sum_{i=1}^{N} \sigma_i \, \text{Layer}_i(z), \qquad (40)$$

where

$$\text{Layer}_i(z) \equiv H(z-z_i) - H(z-z_{i+1}), \qquad (41)$$

and $H(z)$ is the Heaviside step function. Thus, Layer $_i(z) = 1$ for $z_i \leq z < z_{i+1}$ and Layer $_i(z) = 0$ otherwise. The flaw function is defined as $v(z) = [\sigma(z) - \sigma_0]/\sigma_0$, and hence, from (40)

$$v(z) = \sum_{i=1}^{N} c_i \, \text{Layer}_i(z), \qquad (42)$$

where

$$c_i \equiv \frac{\sigma_i - \sigma_0}{\sigma_0}. \qquad (43)$$

Here, the constant σ_0 could represent the mean conductivity of the layers or, perhaps, the conductivity of an underlying substrate. For convenience, we will refer to c_i defined by (43) as the "normalized conductivity."

Now consider three problems: in case 1, the layer thicknesses are assumed known, and the conductivities are to be determined; in case 2, the conductivities are assumed known, and the layer thicknesses (or equivalently, the coordinates z_i of the boundaries) are to be determined; in case 3, assume a single layer of unknown conductivity and unknown thickness on a substrate of known conductivity. The layer thickness and conductivity are to be determined.

In the above problems, we propose using multifrequency impedance measurements to determine the unknown parameters, in which case the components of the measurement vector $\mathbf{m}$ in (36) are the frequencies ω; then (36) becomes

$$(\nabla \mathcal{E})_i = 2 \, \text{Re} \int_{\Omega} W(\omega) \{Z[\omega,\mathbf{p}] - Z_{\text{obs}}(\omega)\}^*$$

$$\times (\nabla Z(\omega))_i \, d\omega, \qquad (44)$$

where Ω denotes the frequency range over which the impedance is measured. In case 1, $\mathbf{p} \equiv \{c_i\}$ are the normalized conductivities; in case 2, $\mathbf{p} \equiv \{z_i\}$ are the coordinates of the layer boundaries; in case 3, $\mathbf{p} \equiv \{c,d\}$ are the normalized conductivity and thickness of a single layer on a substrate.

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A. Case 1

Here the wish to find the unknown normalized conductivities c_i assuming the boundary coordinates z_i are given. This problem fits the scheme of the last section. Letting **c** denote the vector of coefficients c_i, the differential of (42) is

$$dv[\mathbf{c},\delta\mathbf{c}] = \sum_{i=1}^{N} \delta c_i \, \text{Layer}_i(z), \qquad (45)$$

and comparing this to (31) shows that $\partial v(\mathbf{r})/\partial c_i = \text{Layer}_i(z)$. Then, from (34), the components of the gradient of the impedance are

$$(\boldsymbol{\nabla}Z)_i = -\sigma_0 \int \int \int \widetilde{\mathbf{E}}(x,y,z)$$

$$\cdot \mathbf{E}(x,y,z) \text{Layer}_i(z)dxdydz$$

$$= -\sigma_0 \int_{z_i}^{z_{i+1}} dz \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{\mathbf{E}}(x,y,z)$$

$$\cdot \mathbf{E}(x,y,z)dxdy. \qquad (46)$$

[We could have obtained the same result by substituting the basis function $\phi_i(z) = \text{Layer}_i(z)$ directly into (39).] The gradient of the mean-square error $(\boldsymbol{\nabla}\mathscr{E})_i$ follows on substituting (46) into (36), and the steepest-descent update is then given by (37).

B. Case 2

Here the normalized conductivities c_i are given and the boundary coordinates z_i are to be determined. The approach for deriving the gradient of the impedance is similar to that of Sec. IV for the reconstruction of a flaw surface. Letting **z** signify a vector whose components are the numbers z_i, the differential of (40) is

$$dv[\mathbf{z},\delta\mathbf{z}] = \sum_{i=1}^{N} c_i \, d\text{Layer}_i(z). \qquad (47)$$

From (41)

$$d\text{Layer}_i(z) = -\delta(z-z_i)\delta z_i + \delta(z-z_{i+1})\delta z_{i+1}, \qquad (48)$$

and substituting (48) into (47) yields

$$dv[\mathbf{z},\delta\mathbf{z}] = \sum_{i=1}^{N} c_i[-\delta(z-z_i)\delta z_i + \delta(z-z_{i+1})\delta z_{i+1}].$$

By defining $c_0 \equiv 0$ and $c_{N+1} \equiv 0$ and relabeling indices, this may be written

$$dv[\mathbf{z},\delta\mathbf{z}] = \sum_{i=1}^{N+1} (c_{i-1}-c_i)\delta(z-z_i)\delta z_i. \qquad (49)$$

Substituting (49) into (25) now gives the incremental change in impedance due to the incremental change $\delta\mathbf{z}$ in the boundary coordinates **z**

$$dZ[\mathbf{z},\delta\mathbf{z}] = -\sigma_0 \sum_{i=1}^{N+1} (c_{i-1}-c_i)$$

$$\times \left(\int \int \widetilde{\mathbf{E}}(x,y,z_i) \cdot \mathbf{E}(x,y,z_i)dxdy \right) \delta z_i$$

$$\equiv \sum_{i=1}^{N+1} (\boldsymbol{\nabla}Z)_i \, \delta z_i,$$

which shows that the ith component of the gradient of the impedance is

$$(\boldsymbol{\nabla}Z)_i = -(\sigma_{i-1}-\sigma_i)$$

$$\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{\mathbf{E}}(x,y,z_i) \cdot \mathbf{E}(x,y,z_i)dxdy, \qquad (50)$$

where, from (43), $\sigma_{i-1}-\sigma_i$ was substituted for $\sigma_0(c_{i-1}-c_i)$. Substituting (50) into (44) gives $(\boldsymbol{\nabla}\mathscr{E})_i$, and the steepest-descent update becomes

$$z_i^n = z_i^{n-1} - \alpha_n(\boldsymbol{\nabla}\mathscr{E}^{n-1})_i. \qquad (51)$$

C. Case 3

Here we consider a single layer of unknown thickness d and unknown normalized conductivity c on a substrate of known normalized conductivity c_s. Equation (42) then becomes

$$v(z) = c \, \text{Layer}(z) + c_s \, \text{Layer}_s(z), \qquad (52)$$

where $\text{Layer}(z)$ and $\text{Layer}_s(z)$ represent, respectively, the overlying layer and the substrate

$$\text{Layer}(z) = H(z) - H(z-d), \qquad (53)$$

$$\text{Layer}_s(z) = H(z-d). \qquad (54)$$

Assuming c_s is known, the variation of (52) is

$$dv[c,d;\delta c,\delta d] = \delta c \, \text{Layer}(z) + c \, d\text{Layer}(z)$$

$$+ c_s \, d\text{Layer}_s(z). \qquad (55)$$

From (53) and (54), $d\text{Layer}(z) = \delta(z-d)\delta d$ and $d\text{Layer}_s(z) = -\delta(z-d)\delta d$, and (55) becomes

$$dv[c,d;\delta c,\delta d] = [H(z)-H(z-d)]\delta c$$

$$+ [(c-c_s)\delta(z-d)]\delta d.$$

Now substituting this into (25), and comparing to

$$dZ[c,d;\delta c,\delta d] \equiv (\boldsymbol{\nabla}Z)_c\delta c + (\boldsymbol{\nabla}Z)_d\delta d,$$

we obtain for the components of the gradient of the impedance

$$(\boldsymbol{\nabla}Z)_c = -\sigma_0 \int_0^d dz \int\int_{-\infty}^{\infty} \widetilde{\mathbf{E}}(x,y,z) \cdot \mathbf{E}(x,y,z)dxdy,$$

$$(\boldsymbol{\nabla}Z)_d = -(\sigma-\sigma_s) \int\int_{-\infty}^{\infty} \widetilde{\mathbf{E}}(x,y,d) \cdot \mathbf{E}(x,y,d)dxdy,$$

where, from (43), the relation $\sigma_0(c-c_s) = \sigma - \sigma_s$ was used. From (44), the corresponding components of the gradient of the mean-square error are

$$(\nabla \mathscr{E})_c = 2\text{Re} \int_\Omega W(\omega)\{Z[\omega,c,d] - Z_{\text{obs}}(\omega)\}^*$$

$$\times [\nabla Z(\omega)]_c \, d\omega$$

$$(\nabla \mathscr{E})_d = 2\text{Re} \int_\Omega W(\omega)\{Z[\omega,c,d] - Z_{\text{obs}}(\omega)\}^*$$

$$\times [\nabla Z(\omega)]_d \, d\omega,$$

and the steepest descent update becomes

$$c^n = c^{n-1} - \alpha_n(\nabla \mathscr{E}^{n-1})_c,$$

$$d^n = d^{n-1} - \alpha_n(\nabla \mathscr{E}^{n-1})_d.$$

VII. EXAMPLE: A HEMISPHERICAL FLAW

In the low frequency limit the electric field in the conductor may be expressed as the gradient of a scalar potential satisfying the Laplace equation. For simple flaws such as a hemispherical indentation in the surface, this allows us to make use of known static solutions to evaluate the impedance. In fact, a solution found from the Laplace equation may be regarded as the first term in a more general solution expressed as a power series expansion in ka where a is of the order of a characteristic linear dimension of the flaw. We shall evaluate the first nonvanishing term in the impedance series for a nonconducting hemispherical flaw or radius a in a uniform field and show that (26) and (28) give the correct derivative of the impedance with respect to the radius.

Imaging theory allows us to reflect the hemisphere in the surface plane to get a complete sphere. Adapting the solution for a dielectric sphere in a uniform field[14] we find that the external field is that of a dipole and the internal electric field is a constant given by

$$\mathbf{E} = \tfrac{3}{2}\mathbf{E}_0, \tag{56}$$

where $\mathbf{E}_0$ is the unperturbed field due to unit current in the probe. From (6) the flaw impedance

$$Z = \pi a^3 \sigma_0 E_0^2, \tag{57}$$

is found by integrating over the hemisphere.

From (26) and (28)

$$\frac{\partial Z}{\partial a} = \sigma_0 \int_{S_0} E_t^2(t)dt, \tag{58}$$

there being no difference between the ordinary and the adjoint problem in the low frequency limit. θ is a polar angle defined with respect to an axis in the direction of the incident field through the center of the sphere. Then $E_t = 3/2E_0 \sin \theta$ and

$$\frac{\partial Z}{\partial a} = \frac{9}{4}\pi a^2 \sigma_0 E_0^2 \int_0^\pi \sin^3 \theta d \theta = 3\pi a^2 \sigma_0 E_0^2. \tag{59}$$

Differentiating (57) with respect to a gives the same result and provides a simple check on (26) and (28).

VIII. CONCLUSIONS

Because of the inherently nonlinear nature of the eddy-current inverse problem, an iterative algorithm of some kind seems to be the most promising approach for general problems. We have shown how to develop such an algorithm based on the mean-square error criterion. A crucial part of this development is the derivation of the gradient of the mean-square error, which in turn is expressed in terms of the gradient of the impedance. The latter gradient not only plays an essential role in the least-squares algorithm, but also provides information about the sensitivity of the measurements to variations in the flaw parameters. This gradient has been derived in continuous space before the choice of discretization is made. A straightforward, but very costly, approach to computing the impedance gradient is to discretize first and then compute numerically the partial derivatives of the flaw parameters by means of finite differences. This method would require the solution to a separate forward problem for each parameter. The approach here is far less costly since the gradient is obtained from the solution to a single forward problem and its adjoint problem. Moreover, because the gradient is derived analytically via infinitesimal differences, it is exact, in contrast to a finite difference computation. The essential trick used in deriving the gradient of the impedance is to exploit the symmetry of the adjoint problem, thereby eliminating the contribution due to the incremental change in the internal fields. This results in an expression for the variation of the impedance solely in terms of the variation of the flaw—the variation of the fields do not appear. We emphasize that this result, though based on a continuous-space formulation, can be exploited in deriving the gradient of the impedance for any desired discretization or parameterization of the flaw. Several simple flaw examples have been given illustrating methods of reducing the problem to one whose solution lies in a finite dimensional parameter space.

It is finally worth remarking that other cost criteria besides the mean-square deviation are possible. An alternative criterion is the mean-absolute deviation, which may be less sensitive to large or anomalous errors in the data. Whatever cost criterion is used, however, the gradient of the impedance still needs to be computed. In this sense, the usefulness of our results, such as Eq. (24) for ∇Z, is not confined to the particular choice of the mean-square error criterion.

A future objective will be to test the inversion methodology described above on experimental impedance data derived from artificial flaws, initially of very simple geometry (e.g., a layered system) and then of more complex shape.

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APPENDIX A

We derive here an explicit formula for the step-size parameter α_n in the steepest descent update Eq. (11). An essentially identical derivation can be used to obtain the conjugate-gradient step-size parameter α_n in (12). This formula is also given below. We conclude the appendix with review of the conjugate-gradient algorithm.

Begin by writing (11) as follows

$$v(\alpha) = v_{n-1} - \alpha \nabla \mathscr{E}[v_{n-1}], \tag{A1}$$

where here $v(\alpha)$ should be regarded as a function of the scalar α with v_{n-1} and $\nabla \mathscr{E}[v_{n-1}]$ held fixed. The optimum α, denoted α_n, is then the value that minimizes the mean-square error $\mathscr{E}[v(\alpha)]$. The next update is then $v_n = v(\alpha_n)$, while the previous update was $v_{n-1} = v(0)$. Thus, holding v_{n-1} and $\nabla \mathscr{E}$ constant, the variation in $v(\alpha)$ due to a change in α is from Eq. (A1)

$$\delta v = -\delta \alpha \nabla \mathscr{E}. \tag{A2}$$

By substituting (A2) into (15), we obtain the incremental change in the impedance due to the change $\delta \alpha$

$$dZ[\alpha, \delta \alpha] = -\delta \alpha \int \nabla Z(\mathbf{r}) \nabla \mathscr{E}(\mathbf{r}) d\mathbf{r} \equiv \frac{\partial Z}{\partial \alpha} \delta \alpha,$$

which implies

$$\frac{\partial Z}{\partial \alpha} = -\int \nabla Z(\mathbf{r}) \nabla \mathscr{E}(\mathbf{r}) d\mathbf{r}, \tag{A3}$$

where $\nabla Z(\mathbf{r})$ and $\nabla \mathscr{E}(\mathbf{r})$ are given by (24) and (16), respectively. Similarly, for the discrete case

$$\frac{\partial Z}{\partial \alpha} = -\sum_i (\nabla Z)_i (\nabla \mathscr{E})_i, \tag{A4}$$

where $(\nabla Z)_i$ and $(\nabla \mathscr{E})_i$ are given by (34) and (36), respectively.

Thus far, we have made no approximations. We now wish to find an estimate of the α that gives the greatest reduction in the mean-square error in the direction of $\nabla \mathscr{E}$. Write $Z(\alpha) = Z[v(\alpha)]$, so from (A1), $Z(0) = Z[v_{n-1}] \equiv Z_{n-1}$, and expand $Z(\alpha)$ in a Taylor series about $\alpha = 0$

$$Z(\alpha) = Z_{n-1} + \alpha \frac{\partial Z_{n-1}}{\partial \alpha} + \mathscr{O}(\alpha^2). \tag{A5}$$

Although α may itself not be small, we shall neglect the second-order term $\mathscr{O}(\alpha^2)$ in (A5) because this term can be shown also to be second order in the scattered field due to the flaw, whereas the term first order in α in (A5) is first order in the scattered field. If the second-order term were kept, it would contribute a term of one higher order in the scattered field (i.e., third order) than any of the terms in the equations that follow. (In fact, if we were dealing with a linear least-squares problem, i.e., if Z were a linear function of v, then the second-order term in (A5) would vanish, and the expression (A9) below for α would be exact.) Thus, dropping $\mathscr{O}(\alpha^2)$ in (A5) and substituting into (7) gives

$$\mathscr{E}(\alpha) = \int_M W(\mathbf{m}) |Z(\alpha) - Z_{\text{obs}}|^2 d\mathbf{m}$$

$$= \int_M W(\mathbf{m}) \left| Z_{n-1} - Z_{\text{obs}} + \alpha \frac{\partial Z_{n-1}}{\partial \alpha} \right|^2 d\mathbf{m}$$

$$= \mathscr{E}_{n-1} - 2\alpha P + \alpha^2 Q, \tag{A6}$$

where

$$P \equiv \text{Re} \int_M W(\mathbf{m}) (Z_{n-1} - Z_{\text{obs}})^* \frac{\partial Z_{n-1}}{\partial \alpha} d\mathbf{m} \tag{A7}$$

$$Q \equiv \int_M W(\mathbf{m}) \left| \frac{\partial Z_{n-1}}{\partial \alpha} \right|^2 d\mathbf{m}. \tag{A8}$$

The value of α that minimizes $\mathscr{E}(\alpha)$ is found by differentiating (A6) with respect to α and setting the result to zero, giving

$$\alpha = P/Q. \tag{A9}$$

This is our approximate formula for α_n [where the approximation arose by neglecting the term of order $\mathscr{O}(\alpha^2)$ in Eq. (A5)]. Some further insight may be gained by expanding the mean-square error $\mathscr{E}(\alpha)$ in a Taylor series about $\alpha = 0$

$$\mathscr{E}(\alpha) = \mathscr{E}_{n-1} + \alpha \frac{\partial \mathscr{E}_{n-1}}{\partial \alpha} + \frac{1}{2} \alpha^2 \frac{\partial^2 \mathscr{E}_{n-1}}{\partial \alpha^2}. \tag{A10}$$

Comparing this to (A6) shows that $\partial \mathscr{E}_{n-1}/\partial \alpha = -2P$ and $\partial^2 \mathscr{E}_{n-1}/\partial \alpha^2 = 2Q$. The expression for the first derivative $\partial \mathscr{E}_{n-1}/\partial \alpha$ is exact, but the second derivative is not exact since the second-order term in (A5) was dropped. However, as noted, the neglected term is of one higher order in the scattered field than any of the other terms and ignoring it should normally lead to a small error. Under conditions where the error is not small, a better approximation could in theory be obtained at the expense of computing a second forward solution corresponding to a slightly different value of α. This new solution can then be used to compute a new value of P. With two values of P, the first difference of $-2P = \partial \mathscr{E}_{n-1}/\partial \alpha$ could be computed numerically to give an estimate of the second derivative $\partial^2 \mathscr{E}_{n-1}/\partial \alpha^2 = 2Q$ directly, which in turn could be used in (A9).

For the conjugate-gradient algorithm, the update parameter α_n is also computed using formula (A9) with one change in the expression for $\partial Z/\partial \alpha$ given by (A3). In this case, the gradient $\nabla \mathscr{E}$ in (A3) is replaced by $-f_n$, where f_n is the search direction in the conjugate-gradient algorithm defined by the update relations (12) and (13) (see below also).

We conclude with a review of the conjugate-gradient algorithm. For brevity, let $g_n(\mathbf{r}) \equiv \nabla \mathscr{E}[v_n]$ denote the mean-square error gradient, where n is the iteration number. In addition, $Z_n \equiv Z[\mathbf{m}, v_n]$ represents the solution to the forward problem computed on the basis of the nth estimate v_n of the flaw function, and $Z_{\text{obs}}(\mathbf{m})$ denotes the measured data.

Similarly, let $\mathbf{E}_n$ and $\widetilde{\mathbf{E}}_n$ denote the predicted electric field and its adjoint, respectively. The nth estimate of the gradient of the impedance is then given by $\nabla Z_n =$

$-\sigma_0 \tilde{\mathbf{E}}_n(\mathbf{r}) \cdot \mathbf{E}_n(\mathbf{r})$, from Eq. (24). Assume that $v_0(\mathbf{r})$ is the initial estimate of the flaw function.

The Conjugate-Gradient Algorithm.

[$n=0$] (Initialization)

$$\epsilon_0(\mathbf{m}) \equiv Z[\mathbf{m},v_0] - Z_{\text{obs}}(\mathbf{m})$$

$$g_0(\mathbf{r}) = 2\text{Re} \int_M W(\mathbf{m})e_0(\mathbf{m})^* \nabla Z_0(\mathbf{m},\mathbf{r})d\mathbf{m}$$

$$\mathscr{S}[v_0] = \int_M W(\mathbf{m})|e_0(\mathbf{m})|^2 d\mathbf{m}$$

$$1 \leftarrow n,$$

[$n>0$]

$$G_n = \int g_{n-1}(\mathbf{r})^2 d\mathbf{r}, \qquad (a1)$$

$$\beta_n = \begin{cases} 0 & \text{if } n=1 \\ G_n/G_{n-1} & \text{if } n>1, \end{cases} \qquad (a2)$$

$$f_n(\mathbf{r}) = -g_{n-1}(\mathbf{r}) + \beta_n f_{n-1}(\mathbf{r}), \qquad (a3)$$

$$\frac{\partial Z_{n-1}}{\partial \alpha} = \int \nabla Z_{n-1}(\mathbf{r})f_n(\mathbf{r})d\mathbf{r}, \qquad (a4)$$

$$P_n = \text{Re} \int_M W(\mathbf{m})e_{n-1}^*(\mathbf{m})\frac{\partial Z_{n-1}}{\partial \alpha} d\mathbf{m}, \qquad (a5)$$

$$Q_n = \int_M W(\mathbf{m})\left|\frac{\partial Z_{n-1}}{\partial \alpha}\right|^2 d\mathbf{m}, \qquad (a6)$$

$$v_n(\mathbf{r}) = v_{n-1}(\mathbf{r}) + (P_n/Q_n)f_n(\mathbf{r}), \qquad (a7)$$

$$e_n(\mathbf{m}) = Z[\mathbf{m},v_n] - Z_{\text{obs}}(\mathbf{m}), \qquad (a8)$$

$$g_n(\mathbf{r}) = 2\text{Re} \int_M W(\mathbf{m})e_n(\mathbf{m})^* \nabla Z_n(\mathbf{m},\mathbf{r})d\mathbf{m}, \qquad (a9)$$

$$\mathscr{S}[v_n] = \int_M W(\mathbf{m})|e_n(\mathbf{m})|^2 d\mathbf{m}, \qquad (a10)$$

$$n \leftarrow n+1; \quad \text{go to a 1.} \qquad (a11)$$

The algorithm terminates when $\mathscr{S}[v_n]$ is judged sufficiently small.

APPENDIX B

Here we derive the fundamental result of the second section, the functional derivative of the impedance given by (24), using an operator formalism. Given two vector fields $\mathbf{A}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$, define the innerproduct

$$\langle \mathbf{A},\mathbf{B} \rangle \equiv \int_V \mathbf{A}(\mathbf{r}) \cdot \mathbf{B}(\mathbf{r})d\mathbf{r}.$$

Let $\mathscr{G}$ represent a dyadic integral operator that operates on a vector field $\mathbf{A}$

$$\mathscr{G} \cdot \mathbf{A} \equiv k^2 \int_V \mathbf{G}(\mathbf{r}|\mathbf{r}') \cdot \mathbf{A}(\mathbf{r}')d\mathbf{r}',$$

where $\mathbf{G}(\mathbf{r}|\mathbf{r}')$ is the dyadic Green's function. Then Eqs. (6) and (4) are

$$Z = -\langle \mathbf{E}^{(i)},\mathbf{P} \rangle, \qquad (B1)$$

and

$$\mathbf{P}^{(i)} = \mathscr{L} \cdot \mathbf{P}, \qquad (B2)$$

where

$$\mathscr{L} \equiv \mathscr{I} - v\mathscr{G}. \qquad (B3)$$

Here $\mathscr{I}$ is the identity operator. The adjoint Eq. (19) is

$$\mathbf{E}^{(i)} = \tilde{\mathscr{L}} \cdot \tilde{\mathbf{E}}, \qquad (B4)$$

where

$$\tilde{\mathscr{L}} = \mathscr{I} - \mathscr{G}v.$$

The differential of (B1) is

$$dZ = -\langle \mathbf{E}^{(i)},d\mathbf{P} \rangle. \qquad (B5)$$

The differential of (B2) is

$$d\mathbf{P}^{(i)} = d\mathscr{L} \cdot \mathbf{P} + \mathscr{L} \cdot d\mathbf{P}. \qquad (B6)$$

Now substitute (B4) into (B5)

$$dZ = -\langle \tilde{\mathscr{L}} \cdot \tilde{\mathbf{E}},d\mathbf{P} \rangle = -\langle \tilde{\mathbf{E}},\mathscr{L} \cdot d\mathbf{P} \rangle.$$

From (B6)

$$dZ = -\langle \tilde{\mathbf{E}},d\mathbf{P}^{(i)} - d\mathscr{L} \cdot \mathbf{P} \rangle. \qquad (B7)$$

But from (5), $d\mathbf{P}^{(i)} = \sigma_0 \mathbf{E}^{(i)}\delta v$, and from (B3), $d\mathscr{L} = -\delta v\mathscr{G}$. Then (B8) is

$$dZ = -\langle \tilde{\mathbf{E}},(\sigma_0 \mathbf{E}^{(i)} + \mathscr{G} \cdot \mathbf{P})\delta v \rangle. \qquad (B8)$$

But $\sigma_0 \mathbf{E} = \sigma_0 \mathbf{E}^{(i)} + \mathscr{G} \cdot \mathbf{P}$, so Eq. (B8) reduces to

$$dZ = -\sigma_0\langle \tilde{\mathbf{E}},\mathbf{E}\delta v \rangle = -\sigma_0 \int \tilde{\mathbf{E}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})\delta v(\mathbf{r})d\mathbf{r}.$$

From this it follows that

$$\nabla Z = -\sigma_0 \tilde{\mathbf{E}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}),$$

which is (24).

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[2] The theory can, however, be easily modified to accommodate other minimization criteria besides the mean-square deviation, such as the mean absolute deviation. See the discussion in Sec. VIII.

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[13] Note that in (32) we no longer employ the boldface **d** for the differential since strictly speaking Z is no longer a functional but a function, i.e., the varied quantity is not a function, but a set of numbers (the components of the vector **p**).

[14] J. A. Stratton *Electromagnetic Theory* (McGraw–Hill, New York, 1941).