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Time domain half-space dyadic Green's functions for eddy-current calculations

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The field due to an impulsive current dipole embedded in a half-space conductor adjoining a nonconducting half space is given by an exact solution of the quasistatic field equations. This solution has been used to construct a half-space dyadic Green's function containing a term for an unbounded conductor plus terms representing the field reflected at the interface between conducting and nonconducting regions. The resulting kernel can be used in the formulation of time-dependent scattering problems to express the electric field in a conductor as an integral over an electric dipole distribution. [S0021-8979(99)08422-4]

I. TIME DOMAIN INTERACTION

The calculation of eddy-current interaction with an inhomogeneity in an otherwise uniform conductor has applications in a number of areas including nondestructive evaluation (NDE) and geophysics. In common with other electromagnetic scattering problems, the aim is to determine the scattered field when the unperturbed field and scatterer are predefined. In NDE, the perturbation is typically due to a cavity or a crack in a metal and in geophysics it may be due to a body of ore in the earth.[1] In general, the problem can be formulated by representing the effect of the scatterer as equivalent to an induced source distribution. The source density is then found by solving an integral equation. Here a dyadic integral kernel is derived for solving time-domain eddy-current scattering problems for cases in which a scatterer is embedded in an otherwise homogeneous isotropic half-space conductor adjoining a nonconducting half space.

Numerical solutions of time-harmonic eddy-current problems can be found with the aid of volume-element and boundary-element schemes. In the volume and boundary integral formulations, a dyadic integral kernel transforms an electric source in the conductor into the corresponding electric field.[2,3] The success of these calculations depends in part on an ability to express the dyadic Green's function[4] conveniently in terms of standard analytical functions[5] which can be computer coded using polynomial approximations.

Here an integral kernel is derived for carrying out similar calculations for transient excitations. The derivation makes use of the inverse Laplace transform to obtain the time-domain solution from the relevant frequency-domain expression. In contrast to the general treatments of the unbounded domain solution,[6,7] the quasistatic approximation is assumed throughout, which means that the displacement current is neglected. This approximation simplifies the transformation to the time domain, particularly where terms that account for reflection from the interface are involved, because the integrands of the inverse Laplace transform contain only one branch point in the quasistatic limit. As a consequence, the transformation can be carried out analytically.

In eddy-current problems, the unbounded domain dyadic kernel for a uniform isotropic conductor can be expressed in terms of a scalar Green's function satisfying the diffusion equation, as shown in Sec. II. For completeness, a derivation of the scalar Green's function is given in Appendix A. This function is simply a Gaussian distribution in three dimensions as in the analogous problem in heat conduction in which the temperature distribution arises from an impulsive singular heat source.[8] The half-space dyadic Green's function in Sec. III, consists of the unbounded domain dyad plus two terms representing the partial reflection at the air–conductor interface. All three contributions are expressed here explicitly in terms of standard functions and their derivatives.

II. FIELD IN INTEGRAL FORM

A. Electric source in a conductor

Fundamental solutions of the quasistatic Maxwell equations can be obtained with either a singular impulsive electric current dipole source or a magnetic dipole source in a homogeneous conductor. Here the field due to a current dipole is considered in order to find an integral kernel that transforms a time-dependent distributed electric source $\mathbf{P}(\mathbf{r},t)$ into the electric field. Assuming the material permeability is equal to that of free space, the required electromagnetic field satisfies

$$\nabla\times\mathbf{E}(\mathbf{r},t)=-\mu_0\frac{\partial\mathbf{H}(\mathbf{r},t)}{\partial t} \tag{1a}$$

and

$$\nabla\times\mathbf{H}(\mathbf{r},t)=\sigma\mathbf{E}(\mathbf{r},t)+\mathbf{P}(\mathbf{r},t). \tag{1b}$$

The electric field, vanishing for $t<0$ and vanishing as $|\mathbf{r}|\to\infty$ for all t, can be expressed in terms of the electric–electric dyadic Green's function, $\mathcal{G}^{(\mathbf{ee})}(\mathbf{r},\mathbf{r}',t,t')$, representing the electric field arising from an impulsive current dipole at $\mathbf{r}'$. With $\mathcal{G}^{(\mathbf{me})}(\mathbf{r},\mathbf{r}',t,t')$ as the corresponding magnetic field due to the singular electric source, these dyads satisfy

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$$\nabla\times\mathcal{G}^{(\mathbf{ee})}(\mathbf{r},\mathbf{r}',t,t')=-\mu_0\frac{\partial}{\partial t}\mathcal{G}^{(\mathbf{me})}(\mathbf{r},\mathbf{r}',t,t') \tag{2}$$

and

$$\nabla\times\mathcal{G}^{(\mathbf{me})}(\mathbf{r},\mathbf{r}',t,t')=\sigma\mathcal{G}^{(\mathbf{ee})}(\mathbf{r},\mathbf{r}',t,t')+\mathcal{I}\delta(\mathbf{r}-\mathbf{r}')\times\delta(t-t'), \tag{3}$$

where $\mathcal{I}=\hat{x}\hat{x}+\hat{y}\hat{y}+\hat{z}\hat{z}$ is the unit tensor expressed here in terms of unit vectors.

From the solution of Eqs. (2) and (3), the electric field due to the current dipole distribution is given by the superposition principle as

$$\mathbf{E}(\mathbf{r},t)=\int_0^t\int_\Omega\mathcal{G}^{(\mathbf{ee})}(\mathbf{r},\mathbf{r}',t,t')\cdot\mathbf{P}(\mathbf{r}',t')d\mathbf{r}'dt, \tag{4}$$

where $\mathbf{P}(\mathbf{r},t)$ occupies a source region Ω and has the property $\mathbf{P}(\mathbf{r},t)=0$ for $t<0$. Thus $\mathcal{G}^{(\mathbf{ee})}(\mathbf{r},\mathbf{r}',t',t')$ transforms a time-dependent electric source into the corresponding electric field. Similarly, the magnetic field is given by

$$\mathbf{H}(\mathbf{r},t)=\int_0^t\int_\Omega\mathcal{G}^{(\mathbf{me})}(\mathbf{r},\mathbf{r}',t,t')\cdot\mathbf{P}(\mathbf{r}',t')d\mathbf{r}'dt'. \tag{5}$$

In order to find an equation for the dyadic kernel $\mathcal{G}^{(\mathbf{ee})}(\mathbf{r},\mathbf{r}',t,t')$, the magnetic–electric dyad is eliminated from Eqs. (2) and (3) to give

$$\nabla\times\nabla\times\mathcal{G}^{(\mathbf{ee})}(\mathbf{r},\mathbf{r}',t,t')+\mu_0\sigma\frac{\partial}{\partial t}\mathcal{G}^{(\mathbf{ee})}(\mathbf{r},\mathbf{r}',t,t')=-\mu_0\frac{\partial}{\partial t}\mathcal{I}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{6}$$

The solution vanishing for $t<t'$ and vanishing as $|\mathbf{r}-\mathbf{r}'|\to\infty$ for all t will be derived first for an unbounded homogeneous conductor and second for a half-space conductor. The approach taken here parallels that in the text by Felsen and Marcuvitz who consider radiating fields.[9] As a first step, the relationship between the electric–electric dyadic Green's function and the scalar Green's function for the three-dimensional diffusion equation is established.

B. Green's dyad for an unbounded domain

Note that by taking the divergence of Eq. (3), it is found that

$$\sigma_0\nabla\cdot\mathcal{G}^{(\mathbf{ee})}(\mathbf{r},\mathbf{r}',t,t')=-\nabla\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{7}$$

for an unbounded domain in which $\sigma=\sigma_0$. From Eq. (7) and the identity $\nabla\times\nabla\times\equiv\nabla\nabla\cdot-\nabla^2$, Eq. (6) becomes

$$\left(\nabla^2-\mu_0\sigma_0\frac{\partial}{\partial t}\right)\mathcal{G}^{(\mathbf{ee})}(\mathbf{r},\mathbf{r}',t,t')=\left(\mu_0\frac{\partial}{\partial t}\mathcal{I}-\frac{1}{\sigma_0}\nabla\nabla\right)\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{8}$$

It is evident by direct substitution into Eq. (8) that a solution vanishing as $|\mathbf{r}-\mathbf{r}'|\to\infty$ in an unbounded domain is given by

$$\mathcal{G}_0^{(\mathrm{ee})}(\mathbf{r},\mathbf{r}',t,t')=-\left(\mu_0\frac{\partial}{\partial t}\mathcal{I}-\frac{1}{\sigma_0}\nabla\nabla\right)\phi(\mathbf{r},\mathbf{r}',t,t'), \tag{9}$$

where $\phi(\mathbf{r},\mathbf{r}',t,t')$ vanishes for $t<t'$, vanishes as $|\mathbf{r}-\mathbf{r}'|\to\infty$ for all t and satisfies the scalar diffusion equation

$$\left(\nabla^2-\mu_0\sigma_0\frac{\partial}{\partial t}\right)\phi(\mathbf{r},\mathbf{r}',t,t')=-\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{10}$$

The subscript 0 denotes the unbounded domain solution. Equation (10) can be solved using the Fourier–Laplace transform to give (see Appendix A)

$$\phi(\mathbf{r},\mathbf{r}',t,t')=\frac{1}{8}\sqrt{\frac{\mu_0\sigma_0}{\pi^3(t-t')^3}}\exp[-\mu_0\sigma_0|\mathbf{r}-\mathbf{r}'|^2/4(t-t')]H(t-t'), \tag{11}$$

where $H(t)$ is the Heavyside step function: $H(t)=1.0$ for a non-negative argument and is otherwise zero. Equation (11) is a Gaussian distribution in three dimensions with the quantity $\alpha(t-t')=2\sqrt{(t-t')/\mu_0\sigma_0}$ representing a length parameter characterizing the spread of the function in space. From here on, the fact that α is time dependent will not be shown explicitly.

From the curl of Eqs. (9) and (2), it is found that the magnetic–electric dyadic Green's function is

$$\mathcal{G}^{(\mathbf{me})}(\mathbf{r},\mathbf{r}',t,t')=\mathcal{G}_0^{(\mathrm{me})}(\mathbf{r},\mathbf{r}',t,t')=\nabla\times[\mathcal{I}\phi(\mathbf{r},\mathbf{r}',t,t')]. \tag{12}$$

The time integral of Eq. (11) represents the solution for a current-dipole source whose time dependence is a step function in time. This solution is given by

$$\Phi(\mathbf{r},\mathbf{r}',t,t')=\int\phi(\mathbf{r},\mathbf{r}',t,t')d(t-t')=\frac{\mathrm{erfc}\left(\frac{|\mathbf{r}-\mathbf{r}'|}{\alpha}\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{13}$$

The integration of Eq. (11) with respect to time can be carried out by making the substitution $\xi=1/\alpha=\frac{1}{2}\sqrt{\mu_0\sigma_0/(t-t')}$ and integrating with respect to ξ (see Appendix A for an alternative derivation).

III. HALF–SPACE CONDUCTOR

The electric–electric half-space dyadic Green's function for a source in a conducting region is a solution of Eq. (6) and satisfies continuity conditions at the interface of the semispaces ensuring that the tangential electric and magnetic fields are continuous. A derivation of the required dyad using scalar decomposition into transverse electric (TE) and transverse magnetic (TM) field components is given in this section. It is based on an identity that involves derivatives defined with respect to a preferred direction; the direction is normal to the interface. Initially, this identity is used to express the unbounded-domain Green's function in the form of a scalar decomposition. Then the scalar representation is adapted to deal with a half-space conductor.

With the normal to the conductor–air interface as the reference direction, the transverse gradient is written as $\nabla_t = \hat{x}(\partial/\partial x)+\hat{y}(\partial/\partial y)$. This gradient is contained in an identity

$$-\left(\mu_0\mathcal{I}\frac{\partial}{\partial t}-\frac{1}{\sigma_0}\nabla\nabla\right)\nabla_t^2 =\frac{1}{\sigma_0}(\hat{z}\hat{z}\nabla_t^2+\nabla_t\nabla_t)\left(\nabla^2-\mu_0\sigma_0\frac{\partial}{\partial t}\right)-\mu_0\frac{\partial}{\partial t} \times(\nabla\times\hat{z})(\nabla\times\hat{z})-\frac{1}{\sigma_0}[\nabla\times(\nabla\times\hat{z})][\nabla\times(\nabla\times\hat{z})], \tag{14}$$

similar to one given by Felsen and Marcuvitz[9] (see page 18), and proved in the same way. Let Eq. (14) act on the function $U_0(\mathbf{r},\mathbf{r}',t,t')$ where $\phi(\mathbf{r},\mathbf{r}',t,t')=\nabla_t^2 U_0(\mathbf{r},\mathbf{r}',t,t')$. Then, from Eqs. (9) and (10),

$$\mathcal{G}_0^{(\mathbf{ee})}(\mathbf{r},\mathbf{r}',t,t')=-\frac{1}{\sigma_0}\left(\hat{z}\hat{z}+\frac{\nabla_t\nabla_t}{\nabla_t^2}\right)\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t') +\mu_0\frac{\partial}{\partial t}[\nabla\times\hat{z}][\nabla'\times\hat{z}U_0(\mathbf{r},\mathbf{r}',t,t')] -\frac{1}{\sigma_0}[\nabla\times\nabla\times\hat{z}] \times[\nabla'\times\nabla'\times\hat{z}U_0(\mathbf{r},\mathbf{r}',t,t')], \tag{15}$$

where ∇' is defined in terms of source the coordinates x', y', and z'. Equation (15) is an alternative to Eq. (9) as a representation of the unbounded domain dyadic Green's function.

For a domain divided at the plane $z=0$ into two semi-infinite regions, one conducting ($\sigma=\sigma_0$, $z<0$) and one nonconducting ($\sigma=0$, $z>0$), the electric–electric dyadic Green's function may be written in the form

$$\mathcal{G}^{(\mathbf{ee})}(\mathbf{r},\mathbf{r}',t,t')=-\frac{1}{\sigma_0}\left(\hat{z}\hat{z}+\frac{\nabla_t\nabla_t}{\nabla_t^2}\right)\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t') +\mu_0\frac{\partial}{\partial t}[\nabla\times\hat{z}][\nabla'\times\hat{z}U''(\mathbf{r},\mathbf{r}',t,t')] -\frac{1}{\sigma_0}[\nabla\times\nabla\times\hat{z}] \times[\nabla'\times\nabla'\times\hat{z}U'(\mathbf{r},\mathbf{r}',t,t')]. \tag{16}$$

It will be assumed that the singular source is in the conductor and therefore $z'<0$. The function U' represents the transverse electric field and U'' the transverse magnetic component. Note that Eq. (16) applies for all z and that the factor $1/\sigma_0$ refers to the conductivity of the conducting region. The scalar Green's functions

$$G'(\mathbf{r},\mathbf{r}',t,t')=\nabla_t^2 U'(\mathbf{r},\mathbf{r}',t,t') \tag{17a}$$

and

$$G''(\mathbf{r},\mathbf{r}',t,t')=\nabla_t^2 U''(\mathbf{r},\mathbf{r}',t,t') \tag{17b}$$

are introduced. By substituting Eq. (16) into $\nabla\times\nabla\times\mathcal{G}^{(\mathbf{ee})}(\mathbf{r},\mathbf{r}',t,t')=0$ for $z>0$ and Eq. (8) for $z<0$, it can be shown that

$$\nabla^2 G(\mathbf{r},\mathbf{r}',t,t')=0,\quad z>0, \tag{18}$$

for the air region and

$$\left(\nabla^2-\mu_0\sigma_0\frac{\partial}{\partial t}\right)G(\mathbf{r},\mathbf{r}',t,t')=-\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t'),\quad z<0, \tag{19}$$

for the conducting region, where $G(\mathbf{r},\mathbf{r}',t,t')$ of Eqs. (18) and (19) can represent either TE or TM scalar Green's functions.

In order to ensure continuity of the tangential electric and tangential magnetic fields at the interface, it is required that $\hat{z}\times\mathcal{G}^{(\mathbf{ee})}(\mathbf{r},\mathbf{r}',t,t')$ and $\hat{z}\times\nabla\times\mathcal{G}^{(\mathbf{ee})}(\mathbf{r},\mathbf{r}',t,t')$ be continuous there. Hence, using Eq. (16), one concludes that

$$\sigma G',\quad \frac{\partial G'}{\partial z},\quad G''\quad\text{and}\quad \frac{\partial G''}{\partial z} \tag{20}$$

are also continuous at the interface.

Following a similar procedure to that used in deriving the unbounded domain solution, in Appendix A, the half-space dyadic Green's function is found by taking a Laplace transform with respect to time and Fourier transforms with respect to the x and y coordinates. By changing to cylindrical coordinates and integrating with respect to the azimuthal angle, the Fourier transforms reduce to a Hankel transform. This procedure, applied to Eqs. (18) and (19) gives

$$\left(\frac{\partial^2}{\partial z^2}-\kappa^2\right)\tilde{G}(\kappa,z,z',s)=0,\quad z>0, \tag{21}$$

$$\left[\frac{\partial^2}{\partial z^2}-(\kappa^2+\mu_0\sigma_0 s)\right]\tilde{G}(\kappa,z,z',s)=-\delta(z-z'),\quad z<0, \tag{22}$$

where

$$\tilde{G}(\kappa,z,z',s)=\frac{1}{2\pi}\int_0^\infty\int_0^\infty G(\mathbf{r},\mathbf{r}',t,t') \times\exp[-s(t-t')]J_0(\kappa\rho)\rho\,d\rho\,d(t-t') \tag{23}$$

with $\rho^2=(x-x')^2+(y-y')^2$. With the Bromwich integration contour denoted by Br, the inverse transformation is written

$$G(\mathbf{r},\mathbf{r}',t,t')=\frac{1}{(2\pi)^2 i}\int_0^\infty\int_{\mathrm{Br}}\tilde{G}(\kappa,z,z',s) \times\exp[s(t-t')]J_0(\kappa\rho)\kappa\,d\kappa\,ds. \tag{24}$$

The solution of Eqs. (21) and (22), vanishing as $|z|\to\infty$, has the form

$$\tilde{G}(\kappa,z,z',s)=\frac{1}{2\gamma}T(\kappa)\exp[-\kappa z+\gamma z'],\quad z>0, \tag{25}$$

$$\tilde{G}(\kappa,z,z',s)=\frac{1}{2\gamma}\{\exp[-\gamma|z-z'|] + \Gamma(\kappa)\exp[\gamma(z+z')]\}, \quad z<0, \tag{26}$$

where $\gamma=\sqrt{\kappa^2+\mu_0\sigma_0 s}$, T in Eq. (25) is the transmission coefficient and Γ in Eq. (26) is the reflection coefficient. By using the continuity conditions, Eq. (20), the reflection and transmission coefficients are found to be

$$T'=2, \quad T''=\frac{2\gamma}{\gamma+\kappa}, \tag{27}$$

$$\Gamma'=-1, \quad \text{and} \quad \Gamma''=\frac{\gamma-\kappa}{\gamma+\kappa}. \tag{28}$$

Equations (27) and (28) complete the statement of the half-space singular solution in the form given by Eq. (16).

The half-space dyadic Green's function can be expressed in an alternative form derived from Eqs. (28) and (26). In order to obtain this expression, note that the TM potential may be written as

$$\tilde{G}'(\kappa,z,z',s)=\tilde{\phi}(\kappa,z,z',s)-\tilde{\phi}(\kappa,z,-z',s), \quad z<0, \tag{29}$$

and the TE potential as

$$\tilde{G}''(\kappa,z,z',s)=\tilde{\phi}(\kappa,z,z',s)+\tilde{\phi}(\kappa,z,-z',s)+\frac{1}{2\gamma}(\Gamma''-1) \times\exp[\gamma(z+z')], \quad z<0, \tag{30}$$

where $\tilde{\phi}(\kappa,z,z',s)$ is given by Eq. (A4) of Appendix A. Equation (30) expresses $\tilde{G}''(\kappa,z,z',s)$ as the sum of three terms: a free space term, an image term and a term that accounts for the fact that the reflection at the interface is partial rather than total. The electric–electric dyadic kernel can be expressed similarly as the sum of three terms. It is constructed by taking the inverse Hankel–Laplace transform of Eqs. (29) and (30), substituting the result into Eq. (16) and using the identity, Eq. (14). This gives

$$\mathcal{G}^{(\text{ee})}(\mathbf{r},\mathbf{r}',t,t')=\mathcal{G}_0(\mathbf{r},\mathbf{r}',t,t')+\mathcal{G}_i(\mathbf{r},\mathbf{r}',t,t')+\frac{1}{\sigma_0}\frac{\partial}{\partial t}\nabla \times\hat{z}[\nabla'\times\hat{z}V(\mathbf{r},\mathbf{r}',t,t')], \tag{31}$$

where $\mathcal{G}_0(\mathbf{r},\mathbf{r}',t,t')$ is the free space dyadic Green's function and the two remaining terms in Eq. (31) are due to reflection at the surface of the material. The first of these is the image term,[4]

$$\mathcal{G}_i(\mathbf{r},\mathbf{r}',t,t')=-\left(\mu_0\frac{\partial}{\partial t}\mathcal{I}'+\frac{1}{\sigma_0}\nabla\nabla'\right)\phi(\mathbf{r},\mathbf{r}'',t,t'), \tag{32}$$

where $\mathbf{r}''=\mathbf{r}'-2\hat{z}z'$ is the image point and $\mathcal{I}'=\hat{x}\hat{x}+\hat{y}\hat{y}-\hat{z}\hat{z}$. The image term in the form given in Eq. (32) is derived using an identity found by operating with Eq. (14) on $\mathcal{I}'\phi(\mathbf{r},\mathbf{r}'',t,t')$.

The function $V(\mathbf{r},\mathbf{r}',t,t')$, which appears in Eq. (31), is defined as the inverse Hankel–Laplace transform of the third term in Eq. (30) and is given by

$$V(\mathbf{r},\mathbf{r}',t,t')=-\frac{\mu_0\sigma_0}{(2\pi)^2 i}\int_0^\infty\int_{\text{Br}}\frac{\Gamma''-1}{2\gamma\kappa}\exp[s(t-t') + \gamma(z+z')]J_0(\kappa\rho)d\kappa\, ds. \tag{33}$$

From the transverse electric reflection coefficient, given by Eq. (28), and Eqs. (A11) and (A7), Eq. (33) can be written

$$\begin{aligned} V(\mathbf{r},\mathbf{r}',t,t') &= \frac{1}{(2\pi)^2 i}\int_0^\infty\int_{\text{Br}}\frac{1}{s}\left(\frac{1}{\kappa}-\frac{1}{\gamma}\right)\exp[s(t-t') - \gamma|z+z'|]J_0(\kappa\rho)\kappa\, d\kappa\, ds \\ &= \frac{1}{2\pi}\frac{\partial}{\partial\zeta}\Lambda(\alpha,\rho,\zeta)-2\Phi(\mathbf{r},\mathbf{r}'',t,t'), \end{aligned} \tag{34}$$

where $\zeta=|z+z'|$ and the function $\Lambda(\alpha,\rho,\zeta)$ introduced here has the time derivative, $\lambda(\alpha,\rho,\zeta)$, given by

$$\begin{aligned} \lambda(\alpha,\rho,\zeta) &= \frac{\partial\Lambda(\alpha,\rho,\zeta)}{\partial t} \\ &= -\frac{1}{2\pi i}\int_0^\infty\int_{\text{Br}}\frac{1}{\gamma} \times\exp[s(t-t')-\gamma\zeta]J_0(\kappa\rho)d\kappa\, ds \\ &= -\frac{1}{2\pi i}\int_0^\infty\int_{\text{Br}}\frac{1}{\sqrt{\kappa^2+\mu_0\sigma_0 s}}\exp[s(t-t') - \sqrt{\kappa^2+\mu_0\sigma_0 s}\,\zeta]J_0(\kappa\rho)d\kappa\ \ ds \\ &= -\frac{1}{2\pi i}\int_0^\infty\exp[-\kappa^2(t-t')/\mu_0\sigma_0]J_0(\kappa\rho)\ \ d\kappa \times\int_{\text{Br}}\frac{1}{\sqrt{\mu_0\sigma_0 s}}\exp[s(t-t')-\sqrt{\mu_0\sigma_0 s}\,\zeta]\, ds. \end{aligned}$$

Evaluating integrals with respect to κ and s by means of formulas 6.618 from Ref. 10 and 29.3.84 from Ref. 11, we obtain

$$\lambda(\alpha,\rho,\zeta)=-\frac{1}{2(t-t')}I_0(\rho^2/2\alpha^2)\exp\left(-\frac{\rho^2+2\zeta^2}{2\alpha^2}\right), \tag{35}$$

where $I_0(x)$ is the zero-order modified Bessel function. Note in passing that an alternative derivation of (35), in which the integration over κ is done as an initial step, makes use of the inverse Laplace transform[12]

$$\frac{1}{2\pi i}\int_{\text{Br}}I_0(a\sqrt{s}-b\sqrt{s})K_0(a\sqrt{s}+b\sqrt{s})e^{st}\ \ ds = \frac{1}{2}\exp[-(a^2+b^2)/2t]I_0\left(\frac{a^2-b^2}{2t}\right), \tag{36}$$

which produces the same result. Thus the time derivative of $V(\mathbf{r},\mathbf{r}',t,t')$ is given by

$$\frac{\partial V(\mathbf{r},\mathbf{r}',t,t')}{\partial t}=\frac{1}{2\pi}\frac{\partial}{\partial\zeta}\lambda(\alpha,\rho,\zeta)-2\phi(\mathbf{r},\mathbf{r}'',t,t'), \tag{37}$$

with $\lambda(\alpha,\rho,\zeta)$ given by Eq. (35).

Further comment on the function $\Lambda(\alpha,\rho,\zeta)$ is given in Appendix B. The half-space magnetic–electric Green's function, derived in Appendix C, facilitates evaluation of the magnetic field both in the conducting and the nonconducting region.

IV. DISCUSSION AND CONCLUSION

The derivation of the half-space time-domain dyadic Green's function for an electric source in a conductor is a valuable step in the development of a comprehensive theory of transient eddy-current probe–flaw interaction. Because the result can be expressed in terms of standard analytical functions, it is in a convenient form for numerical calculations.

In a typical scattering problem, formulated using integral methods, the flaw field is determined by an integral equation. Special limiting cases can be defined in the time domain by comparing the characteristic dimension of the scatterer c with the diffusion length α. In the initial epoch, the condition $\alpha \ll c$ holds [recall that $\alpha(t-t')=2\sqrt{(t-t')/\mu_0\sigma_0}$] and approximations can be made taking advantage of the fact that α/c is small. Conversely, the tail of a transient signal corresponds to the phase where $\alpha \gg c$. Approximations made accordingly in the long-time limit take advantage of the fact that c/α is small, for example the exponential in Eq. (11) may be expanded as a power series in R/α and a choice made as to how many terms are retained. These approximations are valuable in predicting limiting behavior but for intermediate times, a general numerical scheme may be necessary.

The standard numerical schemes for solving integral equations are usually based on the moment method[13] or the Nyström method.[14] In applying these procedures using boundary or volume elements, a matrix is generated by integrating the Green's kernel over elemental volumes or areas. The half-space kernel derived here is in a suitable and convenient form for carrying out these calculations.

APPENDIX A: UNBOUNDED DOMAIN SCALAR GREEN'S FUNCTION

In the main text it is shown that the unbounded domain dyadic Green's function can be expressed in terms of a scalar function satisfying the equation.

$$\left(\nabla^2-\mu_0\sigma_0\frac{\partial}{\partial t}\right)\phi(\mathbf{r},\mathbf{r}',t,t')=-\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t'). \tag{A1}$$

The solution vanishing for $t<t'$ and as $|\mathbf{r}-\mathbf{r}'|\to\infty$ for all t is obtained by taking the Fourier transform with respect to x and y and the Laplace transform with respect to time. In this way it is found that

$$\left[\frac{\partial^2}{\partial z^2}-(\kappa^2+\mu_0\sigma_0 s)\right]\tilde{\phi}(\kappa,z,z',s)=-\delta(z-z'), \tag{A2}$$

where $\kappa^2=u^2+v^2$ and

$$\tilde{\phi}(\kappa,z,z',s)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\phi(\mathbf{r},\mathbf{r}',t,t')\times\exp[-s(t-t')-iu(x-x')-iv(y-y')]dx\;dy\;d(t-t'). \tag{A3}$$

Equation (A2) has the solution

$$\tilde{\phi}(\kappa,z,z',s)=\frac{1}{2\gamma}\exp[-\gamma|z-z'|], \tag{A4}$$

where $\gamma=\sqrt{\kappa^2+\mu_0\sigma_0 s}$, taking the root with a positive real part. Formally carrying out the inverse transformation we have

$$\phi(\mathbf{r},\mathbf{r}',t,t')=\frac{1}{(2\pi)^3 i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\mathrm{Br}}\frac{1}{2\gamma}\exp[-\gamma|z-z'|+s(t-t')+iu(x-x')+iv(y-y')]du\,dv\,ds, \tag{A5}$$

where Br denotes the Bromwich contour for the inverse Laplace transform. The integrals over spatial frequencies may be carried out first by transforming to cylindrical polar coordinates using

$$\begin{aligned} u&=\kappa\cos\theta, \quad x-x'=\rho\cos\beta,\\ v&=\kappa\sin\theta, \quad y-y'=\rho\sin\beta. \end{aligned} \tag{A6}$$

Integration with respect to θ and application of a standard integral expression for the zero-order Bessel function of the first kind, see formula 9.1.21 in Ref. 11, gives

$$\phi(\mathbf{r},\mathbf{r}',t,t')=\frac{1}{(2\pi)^2 i}\int_0^{\infty}\int_{\mathrm{Br}}\frac{1}{2\gamma}\exp[-\gamma|z-z'|+s(t-t')]J_0(\kappa\rho)\kappa\,d\kappa\,ds. \tag{A7}$$

The relationship

$$\int_0^{\infty}\frac{1}{\sqrt{u^2+k^2}}\exp[-\alpha\sqrt{u^2+k^2}]J_0(\beta u)u\,du=\frac{\exp[-k\sqrt{\alpha^2+\beta^2}]}{\sqrt{\alpha^2+\beta^2}}, \tag{A8}$$

found from the standard form 6.612 of Ref. 10, gives

$$\phi(\mathbf{r},\mathbf{r}',t,t')=\frac{1}{2\pi i}\int_{\mathrm{Br}}\frac{\exp[-\sqrt{\mu_0\sigma_0 s}R]}{4\pi R}e^{s(t-t')}ds=\frac{1}{8}\sqrt{\frac{\mu_0\sigma_0}{\pi^3(t-t')^3}}\exp[-\mu_0\sigma_0R^2/4(t-t')]H(t-t'), \tag{A9}$$

where $R^2=(x-x')^2+(y-y')^2+(z-z')^2$. In Eq. (A9) we have used a formula 29.3.83 from Ref. 11 to obtain the result given as Eq. (11) in the main text. Let $\alpha(t-t')=2\sqrt{(t-t')/\mu_0\sigma_0}$, then

$$\phi(\mathbf{r},\mathbf{r}',t,t') = \frac{1}{\mu_0\sigma_0\pi^{3/2}\alpha^3}\exp(-R^2/\alpha^2)H(t-t'). \tag{A10}$$

The corresponding integrated solution is given by

$$\begin{aligned}\Phi(\mathbf{r},\mathbf{r}',t,t') &= \int \phi(\mathbf{r},\mathbf{r}',t,t')d(t-t') \\ &= \frac{1}{2\pi i}\int_{\mathrm{Br}}\frac{\exp(-\sqrt{\mu_0\sigma_0 s}R)}{4\pi sR}e^{s(t-t')}\,ds \\ &= \frac{\mathrm{erfc}\left(\frac{R}{\alpha}\right)}{4\pi R},\end{aligned} \tag{A11}$$

which is Eq. (13) of the main text.

APPENDIX B: A TERM IN THE HALF-SPACE GREEN'S FUNCTION

The time integral of $\lambda(\alpha,\rho,\zeta)$ to give $\Lambda(\alpha,\rho,\zeta)$ cannot be performed exactly, therefore the function must remain in a suitable integral form. This is written

$$\begin{aligned}\Lambda(\alpha,\rho,\zeta) \\ &= -\frac{1}{2\pi i}\int_0^\infty\int_{\mathrm{Br}}\frac{1}{\gamma s}\exp[s(t-t')-\gamma\zeta]J_0(\kappa\rho)d\kappa\,ds \\ &= -\frac{1}{2\pi i}\int_0^\infty \exp[-\kappa^2(t-t')/\mu_0\sigma_0]J_0(\kappa\rho) \\ &\quad\times\int_{\mathrm{Br}}\frac{1}{\sqrt{s}(s-\kappa^2)}\exp[-\sqrt{s}\zeta+(t-t')s/\mu_0\sigma_0]ds\,d\kappa \\ &= -\int_0^\infty\frac{1}{2\kappa}\left[e^{-\kappa\zeta}\mathrm{erfc}\left(\frac{\zeta}{\alpha}-\frac{\alpha\kappa}{2}\right)-e^{\kappa\zeta}\right. \\ &\quad\left.\times\mathrm{erfc}\left(\frac{\zeta}{\alpha}+\frac{\alpha\kappa}{2}\right)\right]J_0(\kappa\rho)d\kappa,\end{aligned} \tag{B1}$$

where formula 29.3.90 from Ref. 11 has been used. Differentiation with respect to ζ gives

$$\begin{aligned}\frac{\partial\Lambda(\alpha,\rho,\zeta)}{\partial\zeta} &= \frac{1}{2}\int_0^\infty\left[e^{-\kappa\zeta}\,\mathrm{erfc}\left(\frac{\zeta}{\alpha}-\frac{\alpha\kappa}{2}\right)-e^{\kappa\zeta}\right. \\ &\quad\left.\times\mathrm{erfc}\left(\frac{\zeta}{\alpha}+\frac{\alpha\kappa}{2}\right)\right]J_0(\kappa\rho)d\kappa.\end{aligned} \tag{B2}$$

After a long time, the static limit is reached. Relationships valid in this limit are found from the properties of the complementary error function

$$\lim_{x\to\infty}\mathrm{erfc}(x)=0 \tag{B3}$$

and

$$\lim_{x\to\infty}\mathrm{erfc}(-x)=2. \tag{B4}$$

It is found that

$$\lim_{\alpha\to\infty}\frac{\partial\Lambda(\alpha,\rho,\zeta)}{\partial\zeta} = \int_0^\infty e^{-\kappa\zeta}J_0(\kappa\rho)d\kappa = \frac{1}{R''} \tag{B5}$$

and

$$\lim_{\alpha\to\infty}\Phi(\mathbf{r},\mathbf{r}',t,t') = \frac{1}{4\pi R''}, \tag{B6}$$

where $R''=\sqrt{\rho^2+\zeta^2}$. A conclusion drawn from Eqs. (B6) and (34) is that $V(\mathbf{r},\mathbf{r}',t,t')$ vanishes in the static limit.

APPENDIX C: EVALUATION OF THE MAGNETIC FIELD

From the curl of Eq. (31) and Eq. (2), it is concluded that the half-space magnetic-electric dyadic Green's function is given by

$$\begin{aligned}\mathcal{G}^{(\mathrm{me})}(\mathbf{r},\mathbf{r}',t,t') &= \nabla\times[\mathcal{I}\phi(\mathbf{r},\mathbf{r}',t,t')]+\nabla \\ &\quad\times[\mathcal{I}'\phi(\mathbf{r},\mathbf{r}'',t,t')]-\frac{1}{\mu_0\sigma_0}\nabla\times\nabla \\ &\quad\times\hat{z}[\nabla'\times\hat{z}V(\mathbf{r},\mathbf{r}',t,t')],\end{aligned} \tag{C1}$$

for $z<0$.

In order to evaluate the magnetic field in the region above the conductor, the curl of Eq. (16) is taken and again Eq. (2) is used to obtain

$$\mathcal{G}^{(\mathrm{me})}(\mathbf{r},\mathbf{r}',t-t') = -[\nabla\times\nabla\times\hat{z}][\nabla'\times\hat{z}U''(\mathbf{r},\mathbf{r}',t,t')], \tag{C2}$$

for $z>0$ where

$$\begin{aligned}\nabla_t^2 U''(\mathbf{r},\mathbf{r}',t-t') &= \frac{1}{(2\pi)^2 i}\int_0^\infty\int_{\mathrm{Br}}\frac{1}{(\gamma+\kappa)}\exp[s(t-t') \\ &\quad-\kappa z+\gamma z']J_0(\kappa\rho)\kappa\,d\kappa\,ds.\end{aligned} \tag{C3}$$

It is convenient to define

$$\chi(\kappa,\zeta,t) = \frac{1}{2\pi i}\int_{\mathrm{Br}}\frac{1}{\kappa+\gamma}\exp(-\gamma\zeta+st)\,ds, \tag{C4}$$

and use formula 28.3.88 in Ref. 11, to express Eq. (C3) as

$$\begin{aligned}\nabla_t^2 U''(\mathbf{r},\mathbf{r}',t-t') \\ &= \frac{1}{2\pi}\int_0^\infty \chi(\kappa,-z',t-t')e^{-\kappa z}J_0(\kappa\rho)\kappa\,d\kappa,\end{aligned} \tag{C5}$$

where

$$\begin{aligned}\chi(\kappa,\zeta,t) &= \frac{1}{\mu_0\sigma_0}\left[\sqrt{\frac{\mu_0\sigma_0}{\pi t}}\exp\left(-\frac{\mu_0\sigma_0\zeta^2}{4t}-\frac{\kappa^2 t}{\mu_0\sigma_0}\right)\right. \\ &\quad\left.-\kappa e^{\kappa\zeta}\,\mathrm{erfc}\left(\sqrt{\frac{\kappa^2 t}{\mu_0\sigma_0}}+\frac{\zeta}{2}\sqrt{\frac{\mu_0\sigma_0}{t}}\right)\right].\end{aligned} \tag{C6}$$

The magnetic field in air for a singular electric source in the conductor has the form, Eq. (C2), where $U''(\mathbf{r},\mathbf{r}',t-t')$ is given by Eq. (C5) with Eq. (C6).

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